Plates I. & VI. are wanting

A NEW and COMPREHENSIVE

SYSTEM

OF

MATHEMATICAL INSTITUTIONS,

Agreeable to the

PRESENT STATE

OF THE

NEWTONIAN MATHESIS.

VOL. II.

Containing the Inſtitutes, or Principles of

I. The *Phyſico-Mechanical* MATHESIS.
II. Univerſal OPTICS.
III. Univerſal PERSPECTIVE.
IV. LOXODROMICS, or NAVIGATION.
V. BALLISTICS, or GUNNERY.
VI. HOROLOGY, or CLOCK-WORK.

By BENJAMIN MARTIN.

LONDON:

Printed and ſold by W. OWEN, near *Temple-Bar*, and by the AUTHOR, at his Houſe in *Fleet-ſtreet*.

MDCCLXIV.

TO THE

KING,

This NEW and COMPREHENSIVE

SYSTEM

OF

MATHEMATICAL INSTITUTIONS,

Publiſhed,

By his ROYAL PERMISSION,

Under His Moſt Gracious and Auſpicious

PATRONAGE,

Is now, with all Humility,

Inſcribed,

By his MAJESTY's

Moſt Loyal,

Dutiful, and Obedient

Subject and Servant,

BENJ. MARTIN.

P 193.4*
Math 357.59

INSTITUTIONS
OF THE
PHYSICO-MECHANICAL MATHESIS.

CHAP. I.
The MENSURATION *of* SUPERFICIES.

938. BEFORE we can give any rational Account of the *philoſophical and mathematical Sciences*, and their practical Application, it is neceſſary to premiſe thoſe *mechanical Principles* which are founded in the Nature of Things, and make the Baſis of all that Sort of Knowledge; without theſe no Man can underſtand Philoſophy, or make any Progreſs in the true Matheſis. We ſhall propoſe them in a natural Order, and in the moſt perſpicuous Method we can, and begin firſt with the practical Menſuration of Superficies and Solids, which is the firſt Doctrine of practical Science.

939. The Dimenſions or Content of all Superficies are eſtimated in that of a *Square*; as a *ſquare Inch*; a *ſquare Foot*; a *ſquare Yard*, *Rod*, *Mile*, &c. and what Number of each leſſer Denomination is contained the greater, is ſhewn in the following Table:

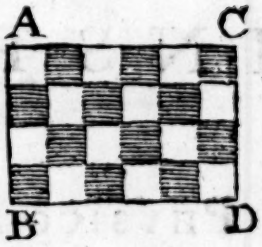
Square Inches.							
144 =	1 Foot ſquare.						
1296 =	9 =	1 Yard ſquare.					
39204 =	272¼ =	30¼ =	1 Pole ſquare.				
1568160 =	10890 =	1210 =	40 =	1 Rood ſquare.			
6272640 =	43560 =	4840 =	160 =	4 =	1 Acre ſquare.		
4014489600 =	278784 00 =	3097600 =	102400 =	2560 =	640 =	1 Mile ſquare.	

940. *To meaſure a* SQUARE A C D B.

Suppoſe the Length of each Side be = 4 *Inches*, *Feet*, *Yards*, &c. then A B = 4; and A B × A C = A B² = 4 × 4 = 16, ſquare *Inches*, *Feet*, *Yards*, &c. as is evident, by Inſpection, in the Figure.

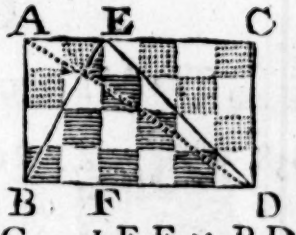


941. *To meaſure a* PARALLELOGRAM ABDC.



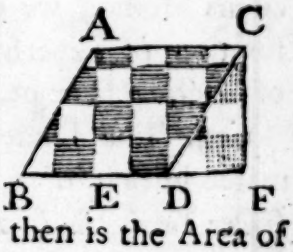
Admit the Length AC = 6, and the Breadth AB = 4; then AC × AB = 24 ſquare Inches, Feet, &c. according to the Meaſure in which the Sides were taken.

942. *To meaſure a* TRIANGLE BDE.



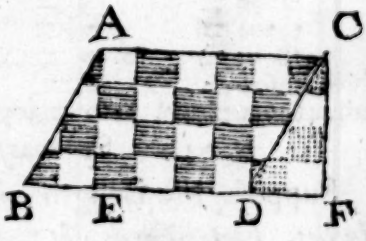
It is evident by Inſpection, the Triangle ADB is equal to half the Parallelogram ABDC; but becauſe AC ‖ BD, the Triangles ADB and BED are equal (635). Therefore the Triangle BED (= BAD) = ½ AB × AC = ½ EF × BD. Therefore if the perpendicular Altitude EF (= AB) = 4, and the Baſe BD = 6; then ½ EF × BD = 2 × 6 = 12 = the Area of the Triangle BED, as required.

943. *To meaſure the* RHOMBUS ABDC.



The Rhombus ABDC is equal to the Parallelogram AEFC, becauſe AC ‖ BF, and AC = FE (655.) Therefore the Rhombus is = AE × AC = AE × BD. Hence let the Baſe BD = 4, and the perpendicular Height AE = 3,5; then is the Area of the Rhombus ABDC = 4 × 3.5 = 14.

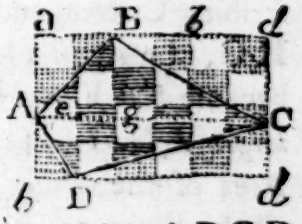
944. *To meaſure the* RHOMBOIDES ABDC.



Here again the *Rhomboides* is = to the Parallelogram AEFC (655.) Therefore let the Baſe BD = 6, the Altitude AE (= CF) = 4; then is Area of the *Rhomboides* ABDC = 4 × 6 = 24.

945.

945. *To meaſure a* TRAPEZIUM A B C D.



Draw the Diagonal A C; on which, from the Angles B, D, let fall the Perpendiculars B g, D e; then is the Triangle A C B = ½ Parallelogram A *a e* C, and the Triangle A C D = ½ Parallelogram A *b d* C. Conſequently the whole *Trapezium* A B C D = half the Parallelogram a *b c d*; = ½ *a b* × *b d* = ½ $\overline{\text{B}g+\text{D}e}$ × A C. Thus, ſuppoſe the Diagonal A C = 6, and half the Sum of the Perpendiculars = $\frac{\text{B}g+\text{D}e}{2}$ = 2; then 6 × 2. = 12, the Area of the *Trapezium*, as required.

946. *To meaſure a* CIRCLE E G F H.



We have ſhewn the Area of the Circle whoſe Diameter is 1, is 0,785398 (828) and the Areas of all Circles are as the Squares of their Diameters (840.) Let A, D, be the Area, and Diameter of the given Circle. Then, as 1^2 : 0.7854 : D D : A. Therefore the Area A = 0.7854 D D. Thus let D = 4, then D D = 16, and 0,7854 D D = 12,5664 = Area ſought.

947. Let A = Area, D = Diameter, and P = Periphery of a Circle; then any one of theſe being given, the other may be found by the following Equations, (ſee 824, 830, 840,) *viz.*

Given
- D, then 3.1416 D = P; and 0.7854 D^2 = A.
- P
  - P ÷ 3.1416 = D, or 0,3183 P = D.
  - P^2 ÷ 12.5664 = A, or 0.07957 P^2 = A.
- A
  - $\sqrt{\text{A}}$ ÷ 0,7854 = D, or $\sqrt{1.2732\ \text{A}}$ = D.
  - $\sqrt{12.5664\ \text{A}}$ = P, or $\sqrt{\text{A} \div 0{,}07957}$ = P.

948. *To meaſure any regular* POLYGON.

For Example, the *Hexagon* A B C D E F. Now ſince theſe Figures conſiſt of as many *Iſoſceles Triangles* A N F as they have



Sides; and the equal Sides of thoſe Triangles A N = N F, are the *Radii* of the circumſcribing Circles, and the perpendicular Heights N G, the *Radii* of the inſcribed Circles; and becauſe $\frac{1}{2}$ A F × N G = Area of the Triangle A N G (942); therefore the Sum of all the Areas (or Area of the whole Polygon) will be equal to G N × $\frac{1}{2}$ Sum of all the Baſes or Sides. Now in the Hexagon, the Triangle A N F is equiangular and equilateral; and therefore putting A N = A F = 1, we have $NG = \sqrt{AN^2 - AG^2} =$ 0,866; let the Sum of the Sides be S = 6, then G N × $\frac{1}{2}$ S = 0,866 × 3 + 2,598 = the Area of the Hexagon required.

949. But ſince regular Polygons are ſimilar Figures, they will be to each other as Squares of their Sides (670.) Therefore the Area of any Polygon, whoſe Side is = 1, being multiplied by the Square of the Side of any other Polygon of the ſame Sort, will give the Area of that other Polygon. Thus, let the Side of any Hexagon be 15; the Square of which is 225, then 225 × 2,598 = 583,05, the Area of the Hexagon, whoſe Side is 15, as required.

950. And that the Area of any regular Polygon may be had in the ſame Manner directly, the Areas of each, ſuppoſing the Side = 1, are computed by the above Method, as in the Table following. (Art. 847.)

Sides.	Names.	Areas.
3	Trigon,	0.433013
4	Tetragon,	1.000000
5	Pentagon,	1.720477
6	Hexagon,	2.598076
7	Heptagon,	3.633959
8	Octagon,	4.828427
9	Enneagon,	6.181827
10	Decagon,	7.694209
11	Endecagon,	9.365675
12	Dodecagon,	11.197920

CHAP.

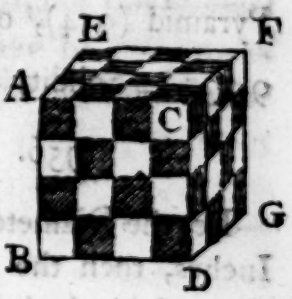
CHAP. II.

The MENSURATION *of* SOLIDS.

951. THE *Solidity*, or ſolid Content of Bodies is eſtimated in that of a *Cube*; as an *Inch Cube*, *Foot Cube*, *&c.* which are more generally called, a *cubic Inch*, *Foot*, &c. A Table of this Sort of Meaſure from the leaſt to the greateſt Denomination here follows:

Cubic Inches.						
1728 =	1 cubic Foot.					
46656 =	27 =	1 cubic Yard.				
7762329 =	4492$\frac{1}{8}$ =	166$\frac{3}{10}$ =	1 cubic Pole.			
196388176 =	1136507$\frac{6}{10}$ =	42092$\frac{8}{10}$ =	253 =	1 cubic Rood.		
157110814408 =	9092061 =	336743$\frac{10}{10}$ =	2024 =	8 =	1 c. Acre.	
254358061056000 =	147197952000 =	5451776000 =	32770584 =	129528 =	16191 =	1 c. M.

952. *To meaſure a* CUBE A G.

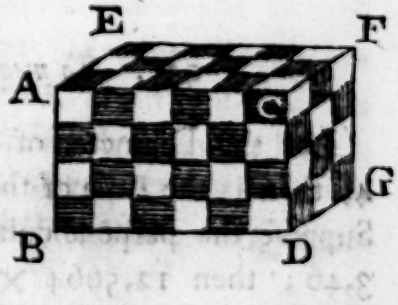


Suppoſe the Length of the Side AB = 4 Inches, then A B × A B = A B^2 = 16; and A B^2 × A B = A B^3 = 64 cubic Inches for the ſolid Content. (See 622.)

953. *To meaſure a* PARALLELOPIPEDON.

Let the Length A C = 6 Inches, the Breadth A E = 3; and the Depth A B = 4. Then

Multiply the Length	A C = 6
By the Breadth	A E = 3
And that Product	A C × A E = 18
By the Depth	A B = 4

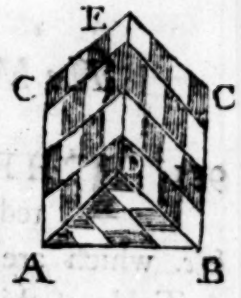


This Product is the ſold Content A C × A E × A B = 72 (by 622.)

954.

954. *To meaſure a* PRISM AC,

Suppoſe it a triangular Priſm, and let the Sides of the Baſe be equal, and A B = 4 Inches. Then $AB^2 = 16$; and 16 × 0,433 = 6,928 ſquare Inches, the Area of the Baſe A D B (942.) Then let the Height be alſo A C = 4; then 6,928 × 4 = 27,712 = $27\frac{1}{125}$ cubic Inches, the Solidity required.



955. *To meaſure a* PYRAMID AE DB.

Let it be a triangular Pyramid; and each Side of the Baſe, *viz.* A B = 4 Inches, then the Area of the Baſe A D B = 6,928; and ſuppoſing its perpendicular Height E D = 4, the Product 6,928 × 4 = Solidity of a Priſm, whoſe Baſe and Altitude are the ſame as thoſe of the Pyramid (954), one Third of which, *viz.* $\frac{4}{3}$ × 6,928 = 9,237 = Solidity of the Pyramid, (by 833.)



956. *To meaſure a* CYLINDER CF.

Let the Diameter of its circular Baſe EF = 4 Inches, then the Area will be = 12,5664, (by 946.) And ſuppoſing the Height of the Cylinder E C = 4; then 12,5664 × 4 = 50,2656 = Solidity of the Cylinder required. (See 830, 831.)



957. *To meaſure a right* CONE EDF.

Let the Diameter of its Baſe be E F = 4, then is the Area of the Baſe = 12,5664. Suppoſe the perpendicular Height D C = 3,46; then 12,5664 × 3,46 = 43,478; one Third of which is the Solidity of the Cone, *viz.* $\frac{43,478}{3}$ = 14,492. (See 834, 835.)



958.

958. *To meaſure a* SPHERE, *or* GLOBE.



Let the Diameter of the Sphere be E F = 4; then is the Area of its greateſt Circle = 12,5664; this multiplied by 4 is = 50,2656 = the circumſcribing Cylinder. Then two Thirds of this, *viz.* $\frac{2}{3} \times 50{,}2656 = 33{,}51$, *&c.* the *Solidity of the Sphere*, (by 873.)

N. B. $4 \times 12{,}5664 = 50{,}2656$ = *Superficies* of the Sphere, in *Square Meaſure*, (by 838.)

959. Or thus; becauſe *Spheres are as the Cubes of their Diameters* (841.) Then ſince the Solidity of the Sphere whoſe Diameter is 1, is 0,5236, (837, 847) if the Diameter of any other Sphere be given, as E F = 4; ſay, as $1 : 0{,}5236 :: 4^3 = 64 : 0{,}5236 \times 64 = 33{,}51$, *&c.* the Solidity of the Sphere, as above.

960. After the ſame Manner may be meaſured the *Spheroid* of (844.) Alſo the *Segment* of a Sphere may be eaſily meaſured, by the Theorem in (836,) and of the Spheroid by the Theorem in (844.)

CHAP. III.

The Philoſophical PRINCIPLES *and* LAWS *of* MOTION *and* GRAVITATION.

961. MATTER, or SUBSTANCE, of which Bodies are compoſed, in itſelf conſidered, or in reſpect of its Eſſence is unknown to us; this is acknowledged by Sir ISAAC NEWTON, and all Mankind beſides. All that we know of Matter is what relates to its various *Properties* and *Qualities* which preſent themſelves to our Senſes.

962. The firſt and principal of which is what Sir *I. Newton* calls the *vis inertiæ* of Matter, or its *natural Inactivity*. No Man ever yet obſerved any Part of Matter to have a *Principle of Action* in itſelf; but on the other Hand, it is abſolutely *paſſive*,

and

and ſubject to the Influence of every external Agent. And from hence is deduced the Firſt of thoſe, which are called the General Laws of Nature, *viz.*

963. Law I.

Every Body will perſevere in its State of Reſt or moving uniformly in a right Line, unleſs it be compelled to change its State by Forces impreſſed.

Thus a Bullet would continue at Reſt in the Gun for ever, if it was not expelled by the Force of Powder; but being thereby put into Motion, it would for ever move on in the Direction of the Axis of the Barrel, if not retarded by the Reſiſtance of the Medium, and carried downwards from that right Line by the Power of Gravity. We ſhew by Experiment, that the leſs the Friction is of the Axis of a moving Wheel, the longer its Motion continues. And we obſerve the vaſt Bodies of the Comets and Planets preſerve their Motions undiminiſhed (as to Senſe) for many 1000 Years in unreſiſting Mediums.

964. Since by Reaſon of the Inactivity of Matter (962,) there is nothing in any Body that can augment, diminiſh, or any Ways alter or vary the Action or Effect of any Force impreſſed, we thence deduce the ſecond General Law of Motion, *viz.*

Law II.

The Change of Motion is proportional to the moving Force impreſs'd, and is made according to the right Line, in which that Force is impreſs'd.

If any Force generate any Motion, a double, or triple Force will generate twice or thrice as much. But the Alteration in reſpect of the Direction of the Motion, is a complicated Affair, which we ſhall farther conſider hereafter.

965. When Bodies act upon each other, they do it by *Contact* or *Colliſion*, and ſince in this Caſe, the Action itſelf is but one and the ſame between both; the Effects which it produces muſt of Courſe be equally divided between both Bodies, and thence an equal Mutation of their State previous to the

Stroke

Stroke muſt needs follow. From this Conſideration is deduced the *third General Law of Nature*, viz.

Law III.

Re-action is always equal and contrary to Action; or the Actions of two Bodies mutually upon each other are always equal, and directed towards contrary Parts.

966. The Truth of this is abundantly confirmed by Experiments. If the Finger preſſes a Stone, the Stone re-acts and preſſes the Finger. The Hammer ſtriking the Anvil, receives the ſame Stroke from it, and is thereby made to rebound. The *Loadſtone*, if fixed, attacts the moveable Iron; and the fix'd Iron equally attracts the moveable Magnet. The Horſe draws the Stone, and the Stone equally draws the Horſe, becauſe the Action, or Force in the Rope which connects them is one and the ſame, and muſt act equally at each End, *viz.* upon the Horſe and Stone.

967. But before we can reaſon well upon the Subject of Motion, we muſt underſtand by juſt Definitions, what Ideas we are to fix to that Word. For the Word Motion is become *ambiguous*, and is uſed in a *ſimple* and complex Senſe. *Motion*, in its ſimple Acceptation, is only *a Change of Place* in Bodies. But in the *complex*, or *phyſical Senſe*, it implies all the *Change that is made in the State of a Body in regard both to its* Quantity of Matter *and* Velocity of Motion. And in this Reſpect, it is properly called the *Momentum*, or *Quantity of Motion*, and which is always equal to the *Force* which produces it by Law II. (964.)

968. That the Quantity of Motion is as the Maſs of Matter, *(cæteris paribus)* is evident from hence, that the Motion of the whole Maſs is the Sum of the Motions of all the Parts, or Particles; as the Number of Particles moved, therefore, is greater or leſs, ſo will be the Sum of all their Motions, and conſequently ſo will be the Aggregate, or whole Motion or *Momentum* of the Body.

969. Again, the *Momentum*, or *Quantity of Motion*, is *(cæteris paribus) as the Velocity* of the ſimple Motion. For ſince *Velocity* has Regard to the *Space deſcribed in the ſame Time*, (778) it is evident, that if a Body deſcribes twice the Space that an-

other equal Body deſcribes in the ſame Time, it is plain there is twice the Change of Place, and therefore twice the ſimple Motion (967) produced in the ſame Time. But the Velocity is alſo twice as great; therefore the *Motion is as the Velocity.*

970. Since then the *Quantity of Motion* is as the Maſs of Matter ſimply, and the Velocity ſimply; it will be conjointly as the Rectangle or Product of both, when no Regard is had to either ſingly. Therefore putting M = Maſs of Matter, V = the Velocity, and Q = the *Momentum* or Quantity of Motion; then in Symbols we have $Q : M \times V$; and thence

* $V : \frac{Q}{M} : Q \times \frac{1}{M}$; and $M : Q \times \frac{1}{V}$.

971. Let S = Space deſcribed by a Body in Motion, and T = the Time of deſcribing it. Now ſince in equal Times the Spaces deſcribed will be as the Celerities of Motion (969,) therefore S : V; alſo it is manifeſt, that in deſcribing the ſame Spaces, the Times will be *greater* as the Celerities of Motion are *leſs*; that is, the Time is inverſely as the Celerity or T; $\frac{1}{V}$; and ſo alſo $V : \frac{1}{T}$. Whence ſince V is ſimply as S, and $\frac{1}{T}$, it will be conjointly $V : \frac{S}{T}$, or V T : S.

972. Becauſe $V = \frac{Q}{M}$ (970) $= \frac{S}{T}$ (971,) therefore $QT = SM$; and $Q = \frac{MS}{T} = MS \times \frac{1}{T}$. That is the *Quantity of Motion is in the compound Ratio of the Space and Maſs directly, and inverſely as the Time.*

973. The Quantity of Matter (M) is (*cæteris paribus*) as the Bulk (B) of a Body; for in twice the Bulk there will be twice the Matter; therefore it is M : B. Again, the Quantity of Matter (M) will be (*cæteris paribus*) as the *Denſity* (D), that is, as the Number or Sum of the Particles in equal Bulks:

* The Reader is deſired to obſerve once for all, that $V : \frac{Q}{M}$ denotes only, that V is always proportional to $\frac{Q}{M}$; but $V = \frac{Q}{M}$ is the ſame as $\frac{V}{1} = \frac{Q}{M}$, and ſhews that the Ratio of 1 to V is the ſame as the Ratio of M to Q.

Bulks; therefore M : D. Wherefore conjointly it will be M : B × D.

974. Becauſe $M = \frac{Q\,T}{S}$ (972) = BD; therefore $Q = SBD \times \frac{I}{T}$; or the *Quantity of Motion is compounded of the direct Ratio of the Space, Bulk and Denſity, and the inverſe Ratio of the Time.*

975. Of Forces that actuate the Particles of Matter, we obſerve the following Variety. (1.) A Force or Power of one Kind cauſes the Particles to *adhere firmly to each other*; and hence it is called the Force or Power of COHESION. (2.) Another Sort of Power obliges the Particles of Matter, under ſome Circumſtances, to recede and fly from each other; which Power is therefore called the *Repellent*, or *repulſive Force*. (3.) A third Power cauſes all large Portions of Matter, or Bodies, to tend mutually towards each other, in Directions to their Centers; and hence it is called a *Centripetal Force*.

976. What thoſe Powers are in themſelves, or how they differ, I ſhall not pretend to enquire; nor alſo what is the particular *Modus agendi*, or Manner of actuating the Particles of Matter, *viz.* whether it be by *Attraction*, *Impulſion*, or otherwiſe; ſince in theſe Reſearches, there is but little Reaſon hitherto to expect any ſucceſsful Diſcoveries. It will be quite ſufficient, if we can make ourſelves thoroughly acquainted with their *Phænomena* and Effects, and apply them in a proper Manner to the various Uſes of Life.

977. The POWER OF COHESION reſpects the ſmalleſt Particles of Matter, and extends to but very ſmall Diſtances, as is plain by numberleſs Experiments; it is therefore proportional to the Surfaces in contact between two Corpuſcles, or their Coheſion is ſo much the greater, by how much the Surfaces are larger in which they touch.

978. Hence, according to the various *Figures* of Particles, they touch by different Quantities of Surface, and cohere with different Degrees of Firmneſs. Thus, for Inſtance, if all were nearly *cubical*, they would touch by a great Quantity of Surface, and conſtitute a very *hard* and firm Body. But on the other Hand, if we ſuppoſe Particles truly ſpherical, they

will touch but by a very ſmall Portion of their Surfaces, and cohere very ſlightly; and of Courſe, will conſtitute a Body whoſe Parts will be very eaſily moved among themſelves, and yield to any impreſſed Force, which Caſe we call the Fluidity of Bodies. Hence Bodies receive their various Degrees of *Hardneſs* and *Softneſs*, *Fixity* and *Fluidity*, *Firmneſs* or *Looſeneſs* of Texture, and all other Qualities depending thereon.

979. That this Power is exceeding great appears by many Experiments. Thus two Balls of Lead, having their Surfaces pared ſmooth, will cohere with a Force equal to 150 lb. tho' they touch upon no more than $\frac{1}{5}$ of an Inch ſquare. Two Braſs poliſhed Planes, 2 Inches Diameter, ſmeared over with Greaſe and put together very hot; will, when cold, cohere ſo firmly as to require 950 lb. to ſeparate them. And Wires of ſeveral Sorts of Metal $\frac{1}{10}$ of an Inch Diameter required the Weights to pull them aſunder, as in the following Table are ſpecified.

Lead	$29\frac{1}{4}$ lb.
Tin	$49\frac{1}{4}$ lb.
Copper	$299\frac{1}{4}$ lb.
Braſs	360 lb.
Silver	370 lb.
Iron	450 lb.
Gold	500 lb.

980. Though this Power acts with ſuch prodigious Force near the Surface, it decreaſes in ſuch a Manner as to become nearly inſenſible in the leaſt ſenſible Diſtance from the Surface; not only ſo, but at a certain ſmall Diſtance it is converted into another Kind of Force, or at leaſt, it acts in a Manner exactly contrary to what it did before; for it now cauſes the Parts of Matter to recede from each other, and to remain at certain equal Diſtances or Intervals among themſelves. And thus modified, it is called the *Repulſive Power* in Matter.

981. That this is Fact we are aſſured from divers Experiments, and many *Phænomena* of Bodies. Thus the Magnetic Needle will be ſtrongly attracted by either Pole of the Magnet in contact with it, but at a very ſmall Diſtance the ſame Power becomes *repulſive* in one Pole, and repels the ſaid Needle from it.

it *. So the Particles of Fluids cohere by this Power, while in a State of Contact, but when ſeparated by Heat, they repel each other, and exiſt in the Form of VAPOUR or STEAM. Again, 'tis well known by numberleſs Experiments, that in all ſolid Bodies there is a certain Matter which, while in contact with the other Parts, does firmly adhere, and is ſtrongly connected with them by this Force; but when by natural or artificial Fermentation, it is diſengaged or ſet at Liberty, it immediately (by a repellent Power among its Particles) expands into a *fine etherial Fluid*, every Way like *common Air*. But of this we ſhall ſpeak more hereafter.

982. The Conſequence of ſuch a repulſive Power among the Particles of Matter, is, that they having attained an *Equilibrium*, act mutually upon each other, and become ſuſceptible of *Compreſſion* and *Condenſation*; and of *Expanſion* and *Rarefaction*; as is well known by common Experiments on Air, Vapour, *&c.* Again, as Action and Re-action are equal (966,) it follows, that when any Force is impreſſed upon the Particles of ſuch a Fluid, they all jointly reſiſt the ſame; and when the impreſſed Force is removed, by Virtue of this Power, the Particles all retreat to their primitive equidiſtant Stations, with a Force equal to that impreſſed. And this Renitency or reſtituent Force, is what we call the SPRING, or ELASTICITY of ſuch Sort of Bodies.

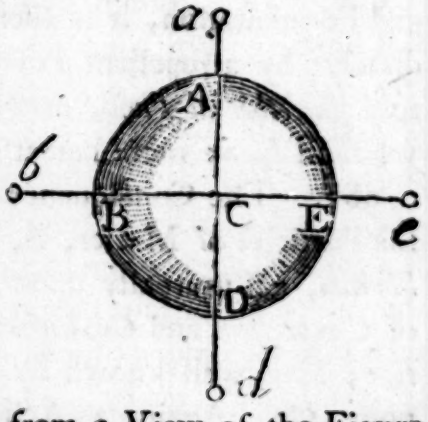
983. We alſo further learn by Experiments of *Thermometers*, &c. that Heat augments and Cold diminiſhes this elaſtic or expanſive Force in Bodies; and that their natural Dimenſions are hereby continually altering, as is evident not only in Air, but in denſer Fluids, as Spirit of Wine, Water, and even Mercury itſelf. Yea, ſolid Bodies diſcover the ſame Properties in ſeveral Degrees; *Ivory* is found to be very elaſtic; *Metals* of all Sorts expand and contract with Heat and Cold, as we ſhew by the PYROMETER, and is otherwiſe known by common Experience.

984. The third Sort of Power, or Agent, (mentioned 975,) is

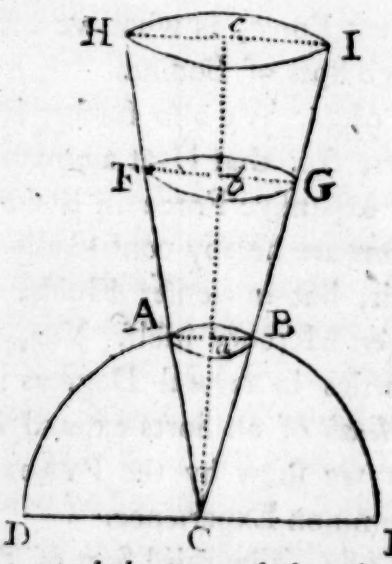
* Though it be denied by ſome, that the North Pole of the Magnet does attract the North End of the Needle juſt upon the Ends, it is certain, by Experiments, that this attracting Force appears extremely near the Ends; and there at a ſmall Diſtance it becomes repulſive.

is a *centripetal Force*, and this we obſerve to take Place among the largeſt Bodies or Syſtems of Matter. The *Phænomena* which demonſtrate ſuch a Power moſt ſenſibly to us are thoſe of *heavy Bodies falling to the Earth*. This Power we call GRAVITY, and this Tendency of Bodies to the Earth is called *Gravitation*.

985. That this is a *centripetal Force*, or that Bodies are thereby made to tend to the Center of the Earth is hence evident, that all Bodies are obſerved to fall in right Lines to the Surface of the Earth every where. But becauſe the Earth is ſpherical, thoſe Lines which are perpendicular to the Surface do all paſs through the Center of the Earth, as is evident from a View of the Figure annexed. For let A B D E be the Earth, C its Center, then ſuppoſe a Body *(a)* falling in the right Line *a* A to the Earth in A, that Line if produced muſt go through the Center C. The ſame is to be ſaid of any other Bodies at *b*, *d*, *e*.



986. But any Virtue propagated in right Lines to or from a Center, will have its Energy on Bodies every where, as the *Square of the Diſtance from the Center inverſely*. For, ſuppoſe a Cone of this gravitating Virtue be repreſented by I H C, terminating in the Center of the Earth C. If then we make C A = A F = F H; the Energy of the Virtue will be at thoſe Diſtances inverſely as the circular Areas A B, F G, and H I; for the more it is expanded and rarified, the leſs will its Effect be upon the ſame Body. But theſe circular Areas are as the Squares of their Diameters (840,) or of the Semidiameters A *a*, F *b*, H *c*; which are as the Squares of the Diſtances C A, C F, C H, (by 656.)



Therefore

Therefore *the Energy of this Power decreaſes as the Square of the Diſtance from the Center increaſes.*

987. But becauſe the Semi-diameter of the Earth is near 4000 Miles, and the greateſt Height to which we can elevate Bodies above the Earth's Surface being but a Mile or two, 'tis evident this Force in ſo ſmall a Diſtance will not ſenſibly vary; and therefore may be eſteemed *as acting uniformly* through any Spaces near the Earth's Surface.

988. The Action of this Power is *conſtant or perpetual*; this appears from hence, that the Velocity of falling Bodies is *conſtantly accelerated or increaſing*, as we know by Experiments. If a Body be put into Motion by a *ſingle* or *inſtantaneous Impulſe*, the Velocity of that Body would be *uniform* (by Law I. 963,) and its Motion rectilineal. If the Power which puts a Body in Motion be temporary, or acts only for a certain Time, the Velocity during that Time will be accelerated, and afterwards become uniform. But if the Power acts inceſſantly, the Body is every Moment impelled, and its Velocity muſt every Moment increaſe. Such therefore is the Power of Gravity.

989. Since Gravity acts *conſtantly and uniformly*, the Velocity of Bodies will be *equably accelerated* in their Deſcent to the Earth; for ſince the Impulſe communicated each Moment is the ſame, it will generate equal Velocities in the ſeveral equal Moments, which conſtantly added together, make an uniformly increaſing Sum, and therefore an *equably accelerated Velocity*. For the Velocity at any Moment is equal to the Sum of all the *momentary Increments* of Velocity from the Beginning.

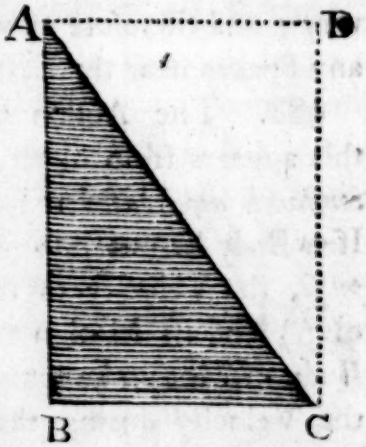
990. Hence the *Velocity* (V) of the Fall is proportional to the *Time* (T). Becauſe ſince the Action of Gravity is uniform, (987), whatever Velocity is generated in one Particle of Time, a double Velocity will be generated in twice that Time, a triple Velocity in thrice that Time; and ſo on continually, therefore it will be always T : V.

991. The Space (S) deſcribed by falling Bodies will, in equal Times, be greater as the Velocity is ſo; and therefore in this Caſe, we have S : V. Alſo the Velocity remaining the ſame, the Space will be as the Time of deſcribing it (971.) Therefore in this Caſe S : T. Conſequently, when neither

the

the Velocity nor Time is given, the Space will be in the compound Ratio of both, *viz.* $S = TV$. But it is $T : V$ (990.) Therefore $S : T^2$, or $S : V^2$; that is, *the Spaces deſcribed by falling Bodies are as the Squares of the Time or of the Velocity.* And this we prove true by Experiments.

992. To illuſtrate this Matter otherwiſe; let A B repreſent the Time of the Fall, and B C the Velocity acquired at the End of that Time; and draw A C. Then if we conceive the whole Time A B to be divided into an indefinite Number of equal Moments, the Velocity in each of thoſe Moments will be as the right Line drawn from A B parallel to B C, correſponding to the given Moment.



Now for a ſingle Moment the Velocity may be eſteemed as uniform, and ſo the Space deſcribed that Moment will be as the Velocity (by 971,) therefore the Sum of the Spaces deſcribed in all the Moments (or whole Time A B) will be as the Sum of all the momentary Velocities, or right Lines which repreſent them, but the Sum of theſe Lines make the Area of the Triangle A B C; this Area, therefore, is as the whole Space deſcribed in the Time A B. And becauſe this Area is as AB^2, or BC^2 (670,) therefore alſo the Space (S) is as T^2, or V^2, as above (991.)

993. Compleat the Parallelogram A B C D. Then if we ſuppoſe another Body (A) to commence Motion at the ſame Time with the falling Body (B,) and the Velocity of the Body (A) to be uniform and equal to the Velocity B C acquired at the End of the Fall; then it is evident the Space deſcribed by the Body (A) in the Time A B will be repreſented by the Parallelogram A B C D; *which Space is therefore double of that* (A C B) *deſcribed by the falling Body* B.

994. What has been ſaid of deſcending Bodies is in the ſame Manner applicable to aſcending Bodies. For the Motion in the latter Caſe will be *retarded* equably by the contrary Action of Gravity, as it was in the former Caſe *accelerated.* Thus let B C be the Velocity, with which any Body is projected

jected upwards from the Point B, then in every Moment of Time there will be an equal Decrement of this Velocity, ſo that at the End of the Time B A it will be all deſtroyed, and the Space it will deſcribe in that Time will be as the Area of the Triangle A B C. Therefore, *&c.*

995. Hence it follows, (1.) That the Velocity every where, at equal Intervals of Time from the Moments B and A, in the Aſcent and Deſcent is the ſame. (2.) That the Time of the Aſcent and Deſcent is the ſame, or half the whole Time of the Flight of the Projectile. (3.) That the Body by deſcending, acquired a Velocity equal to that (B C) by which it was projected. All that we have hitherto ſaid of Motion, is upon a Suppoſition that the Body moves *in Vacuo*, or in a Medium *without Reſiſtance.*

996. From what has been ſaid, it is evident, that if the Space a Body deſcribes in any given Time *in Vacuo* be known, the Space through which it will fall in any propoſed Time, will from thence be known alſo. Thus by Experiments very accurately made *, it has been found, that a Glaſs Globe filled with *Mercury*, deſcended through the Height of 220 Feet in four Seconds; and that a Ball of Lead fell thro' 272 Feet in $4\frac{1}{4}''$, and allowing for the Reſiſtance of the Air, the Motion in both theſe Caſes was at the Rate of $193\frac{1}{3}$ Inches, or 16,11 Feet in the firſt Second of Time. For the $4''$ in Air will make but $3{,}75''$ *in Vacuo*; then $\overline{3{,}75''}^2 : 220\,\text{F.} :: 1^2 : 16{,}11$ Feet, for the Space in the firſt Second, (by 991.)

997. Hence $1^2 : 16{,}11\,\text{F.} :: T^2 : S$; therefore $16{,}11\,T^2 = S$ = the Space deſcribed in any Time T expreſſed in Seconds. Hence alſo $T = \sqrt{\frac{S}{16{,}11}}$ = Time of deſcribing any given Space S.

998. The Power of Gravity (G) will be as the Velocity (V) generated in the ſame Time, becauſe that Velocity is the whole Effect of the Power, and Effects are always as their Cauſes; therefore G : V. Again, Gravity (G) will always be inverſely as the Time (T) in which the ſame Velocity is generated; for 'tis evident, a double Force (2 G) will generate

* See the *Principia*, Edit. 3. p. 346.

the ſame Velocity in half the Time ($\frac{1}{2}$ T) that the Force (G) does; or the greater the Force, the leſs will be the Time for producing the ſame Effect; therefore $G : \frac{I}{T}$: Therefore when neither the Velocity nor Time is given, we have $G = \frac{V}{T}$, or $GT = V$, and $T = V \times \frac{I}{G}$.

999. Since when the Time T is given, G is as V, and in that Caſe V is as S (991,) therefore alſo G will be as S; *or the Power of Gravity will be as the Space deſcribed in any given Time.*

1000. Since $GT = V = \frac{Q}{M}$ (972,) we have $GTM = Q$, and when the Time T is given it is $GM = Q$; that is, the *whole Quantity of Motion in falling Bodies is compounded of the accelerating Force* (G) *of Gravity, and the Maſs of Matter* (M) *in the Body.*

1001. Since $T = \frac{V}{G}$ (998,) $= \frac{SM}{Q}$ (972,) therefore $QV = GSM$. Hence whence the Space (S) is given, it is $QV = GM$. But we have always $G : V$ (998.) Therefore it is always $Q : M$; that is, *the Quantity of Motion* (Q) *is always proportional to the Quantity of Matter* (M); and ſince Gravity (G) is always the ſame near the Earth's Surface, *the Velocity* (V) *of falling Bodies will be every where the ſame too; be the Quantities of Matter in what Proportion you pleaſe.*

1002. Since $Q : M$, therefore the *Momentum*, or Force, by which Bodies tend towards the Earth's Center, is as the Quantity of Matter. But this Force or Tendency of Bodies, is what we vulgarly call their WEIGHT; hence it appears, *that the* WEIGHTS *of Bodies are always proportional to their Quantities of Matter.* And thus having premiſed the eſſential Principles of Philoſophy, we next proceed to the *Laws of Motion*, obſerved in *ſtriking Bodies*; for nothing uſeful can be known, 'till they are firſt aſcertained.

CHAP.

CHAP. IV.

The Philoſophical THEORY *of* PERCUTIENT BODIES, *and of the* COMPOSITION *and* RESOLUTION *of* FORCES.

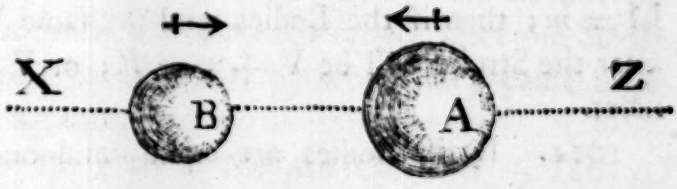
1003. TO determine the ſeveral Particulars relating to the *Motion*, *Velocities*, and *Direction* of *Percutient Bodies* A, and B; we repreſent the Quantities of Matter by M and m, and their Velocities by V and v; then will $Q = MV$ (970,) = *Momentum* of A, and $q = mv$ = *Momentum* of B.

1004. If the Body B ſtrikes the Body A in Motion, and



both move the *ſame Way*, or towards the ſame Parts, (as from X to Z) then the Sum of their Motion towards the Part Z, will be $MV + mv$, and the Velocity of both the Bodies after the Stroke towards the ſame Part, will be $\frac{MV + mv}{M + m} = V$. For the Velocity is always as the *Momentum*, divided by the Maſs of Matter (970.)

1005. If one of the Bodies, as A, has a contrary Direction, or tend towards



X, in which Caſe the Bodies will meet, then the *Momentum* of A will have a negative Sign, *viz.* $-MV$; and ſo the Sum of the Motions towards the ſame Part Z will be $mv - MV$; and the Velocity after Colliſion will be $\frac{mv - MV}{m + M} = V$.

1006. Becauſe in what we have hitherto ſaid, we ſuppoſe the Bodies A and B deſtitute of *Elaſticity*, therefore after the Stroke, there being nothing in the Bodies to cauſe a Reſilition or Separation, they will both go on together with the ſame Velocity *V*.

1007. The Sum of the Motions towards the ſame Parts is the ſame before and after the Stroke. For let them both move

the ſame Way (1004,) and let B ſtrike A; then by that Impulſe, the Motion of A, *viz.* Q, will be augmented and become $Q + x$ after the Stroke; but becauſe *Action and Reaction is equal* (965,) the Body A will re-act upon B, and produce an equal Effect by the Stroke, that is, it will diminiſh the Motion of B by the ſame Quantity x, ſo that its Motion after the Stroke will be $q - x$; but the Sum of the Motions of both Bodies after the Stroke $\overline{Q + x} + \overline{q - x} = Q + q$, the Sum of the Motions before the Stroke, ſee (1004.) And the ſame is to be ſhewn, if the Bodies meet, as (1005).

1008. The *Magnitude of the Stroke* will be proportional to the Quantity $(x,)$ becauſe that is the whole Effect or Mutation produced in the Motion of each Body. The Greatneſs of the Stroke is therefore meaſured by the Loſs (x), which the moſt powerful, or percutient Body ſuſtains in its Motion.

1009. In the above Theorems (1004, 1005,) if the Body A be ſuppoſed at Reſt, then $V = 0$, and MV vaniſhes; the Velocity then after the Stroke is $\frac{mv}{M+m} = V$; and ſo $mv = VM + mV$. Whence $V : v :: m : M + m$, and $v = \frac{M+m}{m} \times V$.

1010. If we ſuppoſe the Bodies equal, *viz.* $A = B$, or $M = m$; then if the Bodies tend the ſame Way, the Velocity after the Stroke will be $V + v = V$; or $V - v = V$, if they meet.

1011. If the Bodies are equal, and one of them at Reſt, then $\frac{mv}{M+m} = \frac{v}{2} = V$; or *the Velocity after the Stroke is equal to half that of the ſtriking Body.*

1012. If A at Reſt exceed B infinitely in Magnitude; then becauſe m is infinitely ſmall in reſpect of $M + m$; therefore ſo is V in reſpect of v (1009,) conſequently V will vaniſh, or the Body B impinging againſt any firm immoveable Object, *will after the Stroke be at Reſt.*

1013. If equal Bodies moving with equal Velocities, meet; they will mutually deſtroy each other's Motions; for in this Caſe $MV = mv$; therefore $MV - mv = 0$, conſequently $\frac{MV - mv}{M+m}$

$\frac{MV - mv}{M+m} = \frac{0}{M+m} = V = 0$; or both the Bodies *remain at Reſt after the Stroke.*

1014. The *Momentum* of the Body B, after the Stroke is $mV = \frac{MVm \pm m^2 v}{M+m}$ (1004,) therefore $mv - \frac{MVm \pm m^2 v}{M+m}$ $= \frac{MVm \mp mvM}{m+M} = \frac{Mm}{M+m} \times V \mp v =$ the Loſs of Motion in the Body B after the Stroke; but $\frac{M}{M+m}\frac{m}{}$ is a conſtant Quantity; therefore the Loſs of Motion in B is as $V \mp v$, and conſequently the Magnitude of the Stroke in Bodies tending the ſame Way, is as $V - v$; and as $V + v$, if they meet (1008.) And if A be at Reſt, then $V = 0$, the Stroke will be as v, the Velocity of the percutient Body.

1015. If the impinging Bodies A and B are *perfectly elaſtic*; then this *elaſtic Force* is ever equal in its Action to the *compreſſing Force* (982.) And whatever Action is exerted upon A by the Impulſe of B, the ſame is doubled by Virtue of this renitent, or elaſtic Force; and the Re-action of A upon B is doubled likewiſe (965;) and as the Parts of each Body are mutually compreſſed and flatted by the Stroke, ſo thoſe Parts are thrown out again by an equal Force, and by this Means the Bodies are made to recede from each other, after the Stroke with the ſame Forces, or *Momenta*, by which they came together, or ſtruck each other.

1016. But ſince the *Force of Colliſion* is as $V \mp v$, (1014,) that will alſo be as the Force of Reſilition, or that by which they ſeparate after the Stroke. Let x and y be the Velocities of the Bodies A and B after the Stroke; and then $V \mp v = x \mp y$; and $y = V \mp v \pm x$, and ſo $Mx =$ the Motion of A after the Stroke, and that of B will be $my = mV \mp mv \pm mx$. And ſince the Sums of the Motion before and after the Stroke towards the ſame Parts are equal (1007,) we have $MV \pm mv = Mx + mV \mp mv \pm mx$; and thence $Mx \pm mx = MV \pm 2mv - mV$, and ſo $\pm x = \frac{MV \pm 2mv - mV}{M+m}$.

1017.

1017. Alſo the Velocity of B, after the Stroke will be $y = V \mp v \pm x = V \mp v \pm \frac{MV - mV \pm 2mv}{M+m} = \frac{2MV \mp Mv \pm mv}{M+m}$. From whence 'tis evident, if the Bodies tend the ſame Way, the Motion of B will be always poſitive; but when the Bodies meet, the Body B will proceed or recede, according as Mv is greater or leſſer than $2MV - mv$.

1018. The *Momentum* of A after the Stroke, will be $\frac{M^2V - MmV \pm 2Mmv}{M+m}$: And that of B will be $\frac{2MVm \mp Mmv \pm m^2v}{M+m}$: Whence the Motion loſt in A will be $MV - \frac{M^2V - MmV \pm 2Mmv}{M+m} = \frac{2MmV \mp 2Mmv}{M+m}$; and that gained in B will be found the ſame; whence it is evident, *that the Effect of the Stroke on each Body is equal, and double of that in non-elaſtic Bodies* (1014.)

1019. If the Body B be at Reſt before the Stroke, then $v = 0$, and the Theorem becomes $\pm x = \frac{MV - mV}{M+m} = \frac{M-m}{M+m} \times V$. Hence it appears, the Velocity of the Body (A) after the Stroke, will be affirmative or negative, that is, forwards or backwards, as M is greater or leſſer than m, or as A is greater or leſſer than B.

1020. If B be at Reſt and equal to A; then $v = 0$, and $M = m$, and ſo $x = \frac{0}{2M} = 0$; that is, the Body A will in this Caſe be at Reſt after the Stroke; and the Body B will move on with the Velocity of A, for $y = V$ in this Caſe, (by 1016.) *

1021.

* After we have given the *Phyſical Principles* of the *Newtonian Philoſophy*, we ſhall illuſtrate and confirm every Poſition and Doctrine by Experiments, and deſcribe the Machines by which they are performed in the beſt Manner; and in that Part the Reader will ſee how exactly all the Caſes of percutient Bodies here premiſed are verified by a practical Inſtance of each of them.

1021. Let A, B, C, be three Bodies; and let A ſtrike B at Reſt; the Velocity generated in B by the Stroke, will be $y = \frac{2MV}{M+m}$ (1017, ſince $v = 0$;) and therefore the *Momentum* of B will be $\frac{2MVm}{M+m} = my$. With this *Momentum* B will ſtrike C at Reſt, and contiguous to it; the Velocity generated in C will be $\frac{2my}{m+C}$, and its *Momentum* will be $\frac{2myC}{m+C}$; and if inſtead of y, we reſtore its Value $\frac{2MV}{M+m}$; the *Momentum* of C will be $\frac{2mC}{m+C} \times \frac{2MV}{M+m} = \frac{4MVmC}{Mm+MC+Cm+m^2}$.



1022. If now we make B a variable Quantity, while A and C remain the ſame, we may determine the Proportion of B to A and C, that ſhall give the *Momentum* of C the greateſt poſſible, by making the Fluxion thereof equal to nothing, *viz.* $\frac{4M^2C^2V\dot{m} - 4MCm^2\dot{m}}{\overline{MC+Mm+mC+m^2}^2} = 0$. Whence we get $MC - mm = 0$; and therefore $MC = mm$. Whence $M : m :: m : C$, or $A : B :: B : C$; *therefore* B *is a mean Proportional between* A *and* C.

1023. Hence if there be any Number (n) of Bodies in a geometrical Ratio (r) to each other; and the Firſt be A, the Second will be rA, the Third r^2A, &c. to the laſt, which will be $r^{n-1}A$: Alſo the Velocity of the Firſt being V, that of the Second will be $\frac{2V}{1+r}$ (for $\frac{2MV}{M+m} = \frac{2AV}{A+rA} = \frac{2V}{1+r}$) that of the Third $\frac{4V}{\overline{1+r}^2} = \frac{2V}{1+r} \times \frac{2}{1+r}$, &c. to the Velocity of the laſt, which will be $\overline{\frac{2}{1+r}}^{n-1}$ V. The *Momenta*, or Motions of the Bodies would be, of the Firſt AV, of the Second $\frac{2AVr}{1+r}$, of the Third $\frac{4AVr^2}{\overline{1+r}^2}$; and to the laſt

$\overline{\frac{2r}{1+r}}^{n-1}$ A V, as they are exhibited in the following Series, *viz.*

Bodies, A, r A, r^2 A, r^3 A, r^4 A, &c. r^{n-1} A.

Velocities, V, $\frac{2V}{1+r}$, $\frac{4V}{1+r^2}$, $\frac{8V}{1+r^3}$, $\frac{16V}{1+r^4}$, &c. $\overline{\frac{2}{1+r}}^{n-1}$ V.

Motions, AV, $\frac{2AVr}{1+r}$, $\frac{4AVr^2}{1+r^2}$, $\frac{8AVr^3}{1+r^3}$, $\frac{16AVr^4}{1+r^4}$, &c. $\overline{\frac{2r}{1+r}}^{n-1}$ AV.

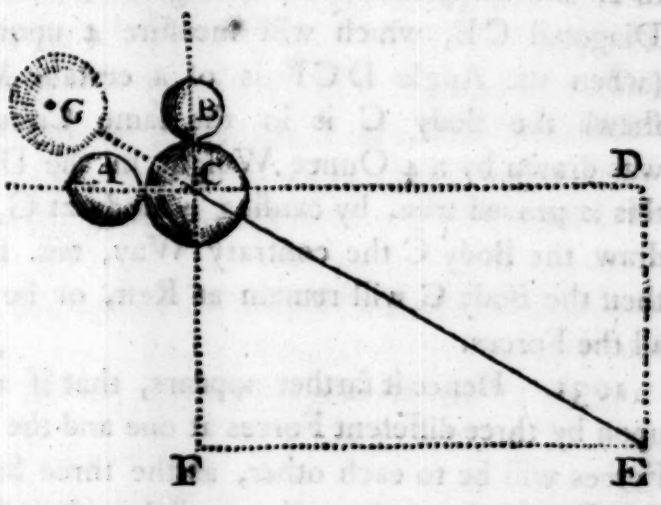
1024. If the Number of Bodies be $10 = n$, and $r = 2$; then will A : r^{n-1} A :: 1 : r^{n-1} :: 1 : 512. Alſo V : $\overline{\frac{2}{1+r}}^{n-1}$ V :: 1 : $\overline{\frac{2}{1+r}}^{n-1}$ :: 1 : 0.026 :: 38,45 : 1. Laſtly, AV : $\overline{\frac{2r}{1+r}}^{n-1}$ AV :: 1 : $\overline{\frac{2r}{1+r}}^{n-1}$ :: 1 : $13\frac{3}{10}$.

1025. If $n = 100$, and $r = 2$, then will the firſt Body A be to the laſt r^{n-1} A, as 1 to 633825300,000000000000000000000 nearly, and its Velocity to that of the laſt, as 2710220000000 00000 to 1 nearly, and the *Momentums* will be as 1 to 2338480000000, very near.

1026. Let $\frac{2r}{1+r} = R$, and let the Motion of the firſt Body be to that of the laſt, as 1 to M, that is, let $M = \overline{\frac{2r}{1+r}}^{n-1}$ $= R^{n-1}$; if L = the Logarithm, then L, $M = \overline{n-1} \times$ L, R. Alſo $\frac{L, M}{L, R} + 1 = n = \frac{L, M \times L, R}{L, R}$.

1027. By theſe Theorems, whatever relates to the Motion of Bodies *elaſtic*, or *non-elaſtic*, acted upon by a ſingle Impulſe, may eaſily be determined. I ſhall therefore now proceed to conſider the Motion of a Body, as acted upon by two or more Forces at once, and determine the Direction thereof by the foregoing Laws; as upon this Doctrine depends the whole *Rationale* of the *Mechanical Philoſophy*, (which alone can be true) and therefore it is of the higheſt Conſequence to be thoroughly and rightly underſtood.

Thus



Thus let C be a Body impelled in the Direction A D by a Body A, with ſuch a Force as ſhall cauſe it to move uniformly over the Space C D in a Second of Time. At the ſame Inſtant let it receive a Stroke by another Body B, in the Direction B F, with ſuch a Force as ſhall cauſe it to paſs over the Space C F in the ſame Time.

1028. Now it is evident the Body C cannot move in both theſe different Directions; and therefore will not move in either, but in a Direction *compounded of them both*, which is thus determined. Draw D E parallel to B F; then, though the Action of B prevents the Body from proceeding in the Right Line C D, yet it can no Ways alter its Velocity of approaching to the Line D E in the given Time, by Virtue of the Force impreſſed by A. At the End therefore of a Second of Time, the Body C will be ſomewhere in the Line D E. By the ſame Way of Reaſoning, it will at the End of the ſame Time be found ſomewhere in the Line F E, parallel to C D; and therefore in the Concourſe of both, in the Point E. Its Courſe then is the Line C E, which is a Right Line by LAW I. (963).

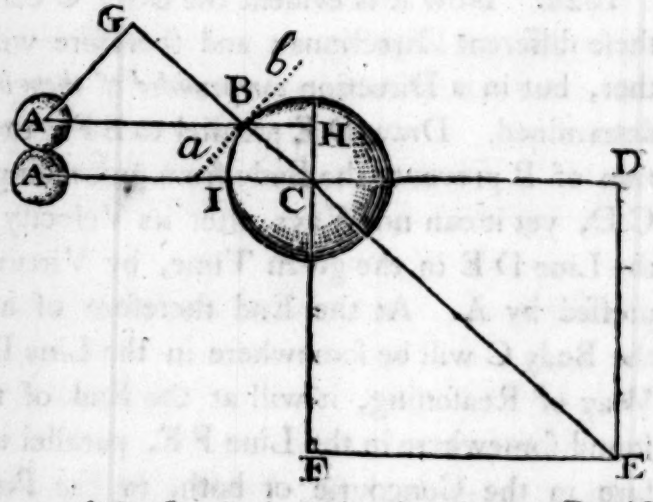
1029. Hence appears the Method of compounding a direct Force C E, out of any oblique Forces C D, and D E; and on the Contrary, of *reſolving any direct Force* C E into two other oblique Forces C D and D E. Wherefore repreſenting any two oblique Forces by the two Sides of a Parallelogram, the direct Force equivalent to them will be the Diagonal thereof. And the Truth of this Doctrine is abundantly confirmed by Experiments.

1030. Thus for Example; if the Body C be drawn with a Weight of 3 Ounces in the Direction C D, and by another of 2 Ounces in the Direction C F; then make C D to C F as 3

to 2, and compleat the Parallelogram C D E F, and draw the Diagonal C E, which will meaſure 4 upon the ſame Scale, (when the Angle D C F is of a certain Magnitude) which ſhews the Body C is in the ſame Circumſtance, as if it was drawn by a 4 Ounce Weight in the Direction C E; and this is proved true, by cauſing a Body as G, of 4 Ounces, to draw the Body C the contrary Way, *viz.* from C to G, for then the Body C will remain at Reſt, or be *in Equilibrio* with all the Forces.

1031. Hence it farther appears, that if a Body C be acted upon by three different Forces at one and the ſame Time, thoſe Forces will be to each other, as the three Sides of a Triangle C D E, which are ſeverally parallel to their Directions. This is ſo plain from what has been ſaid, that nothing more can be added.

1032. Hence we learn how to eſtimate the Quantity of any oblique Stroke. For let the Body A ſtrike the Body C in a Direction paſſing thro' its Center, as A C. Then it is certain, that it acts upon it with its whole Force; and the Stroke is ſaid to be *direct*. But if the ſame Body A ſtrikes the Body C in any Direction A B, which does not paſs through the Center C, then the Stroke is ſaid to be *oblique*; and its Force to move the Body is thus found. At the Point of impact B draw the Tangent *a b*, parallel to which draw A G to meet B C produced in G.

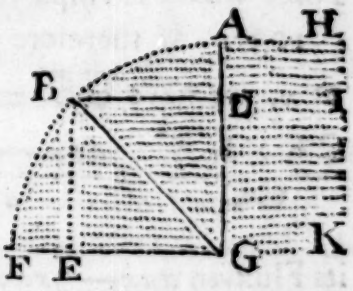


1033. Let A B repreſent the whole Force of the percutient Body A; this is reſolvable into the two Forces A G and G B (1029,) of which the Former is parallel to the Tangent *a b*, and ſo does not at all affect the Body C; but the other Force G B paſſes through the Center C, and is that alone by which the Body C is compelled to move. But A B : G B :: C B : B H (657.) That is, *The whole Force, or direct Stroke, is to the reſidual*

reſidual Force, or oblique Force, as the Radius to the Sine of the Angle of Obliquity B C H (710.)

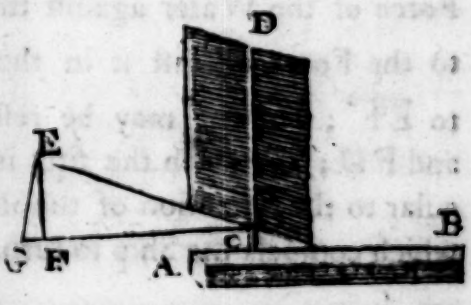
1034. But C B : B H : : C E : E F, or C D; and therefore if in any Caſe C E denote the whole Force, the diminiſhed Force or oblique Stroke will be denoted by C D; and will be greater or leſs, as the Angle of Obliquity F C E = B C H is ſo.

1035. That the extenſive Uſe of this Doctrine of the *Compoſition and Reſolution of Force* in *Mechanical Philoſophy*, may appear, I ſhall ſubjoin the following Examples thereof in the moſt intereſting Parts of the Science.



Thus let A G be the perpendicular Section of any Plane, as the SAIL of a WINDMILL, *&c.* expoſed directly to the Stream or Current of any Fluid, as Air, *&c.* repreſented by H I K; and let B G be the Section of the ſame Plane in an oblique Situation thereto. Then it is evident (1.) That the Number of fluid Particles which ſtrike upon the Plane in the *direct Poſition* A G will be to the Number of thoſe which fall on it in the *oblique Poſition* B G, as A G to D G; becauſe all the Particles between H and I will paſs by the Plane B G, and not touch it. (2.) The Force with which the Particles ſtrike the Plane in the *direct Poſition*, will be to that with which they ſtrike it in the oblique Poſition B G, as A G to D G (1033.) *Wherefore the whole Force of the Fluid upon the Plane in the direct Poſition, will be to the whole Force in the oblique one, as* $\overline{AG}^2$ *to* $\overline{DG}^2$, *or as the Square of the Radius to the Square of the Sine of the Angle of Inclination.*

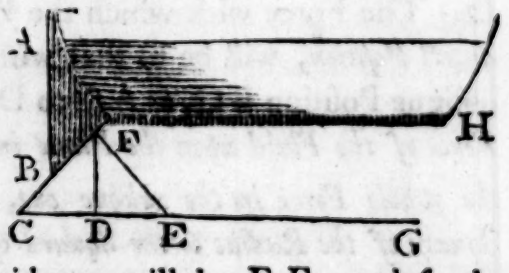
1036. Let A B be the *Axis* of the WINDMILL, C D one of the Sails placed in an oblique Poſition E C to the Direction of the Wind G C, which is parallel to the Axis A B. If then $\overline{GC}^2$ expreſs the abſolute Force of the Wind upon the Sail in a direct Poſition,



ſition, $\overline{GE}^2$ will expreſs the Quantity of the ſame Force in the oblique Poſition of the Sail, (becauſe GE is the Sine of the Angle of Incidence GCE to the Radius GC;) but the Force GE is reſolvable into the two Forces EF, and GF, of which the Latter being parallel to the Axis avails nothing in turning the Sail about it. But the Force EF being perpendicular thereto, is wholly ſpent in compelling the Sail to turn round. But the Force GE is to the Force EF as (GC to CE,) $\overline{GE}^2$ to $\frac{CE \times \overline{GE}^2}{GC}$, which therefore will expreſs the Force which is employed to turn each Sail.

1037. If therefore we put the Radius $GC = a$, and $EC = x$, we have $\overline{GE}^2 = aa - xx$, and conſequently the Force $\frac{CE \times \overline{GE}^2}{GC} = \frac{aax - xxx}{a}$; which if we make it a *Maximum*, its Fluxion $aa\dot{x} - 3xx\dot{x} = 0$, and ſo $aa = 3xx$ and $x = \sqrt{\frac{aa}{3}}$; which in Logarithms is $\frac{20,00000 - 0,477121}{2} = 9.761439$, the Logarithm Sine of 35° : 16′ equal to the Angle CGE, and therefore the Angle ECG is equal to 54° : 44′, when the Sail receives the *greateſt Force* from the Wind.

1038. If AB be the Rudder of a Ship AH, placed in the oblique Situation FC, and the Water ſtrike againſt it in the Direction GC; then making CE Radius,

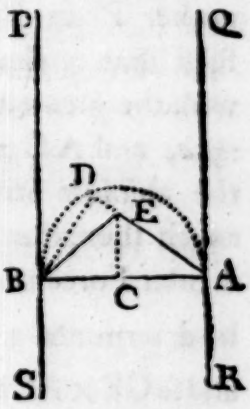


the Sine of the Angle of Incidence will be EF; and ſo the Force of the Water againſt the Rudder in a direct Poſition, is to the Force againſt it in the oblique Poſition FC, as $\overline{CE}^2$ to $\overline{EF}^2$; but EF may be reſolved into the two Forces ED, and FD; of which the firſt is parallel, and the laſt perpendicular to the Direction of the Ship. Therefore FD is the Force which compels the Ship to turn. But the Force EF is to the Force FD (as CE to CF) as $\overline{EF}^2$ to $\frac{CF \times \overline{EF}^2}{CE}$; that is,

(by

(by putting $CE = a$, $CF = x$). As $\frac{aax - x^3}{a}$; whence the Angle of Incidence $ECF = 54° : 44'$ (as before) when the Force of the Water againſt the Rudder to turn the Ship is a *Maximum*.

1039. After a like Manner is determined the Angle of Poſition $BAE = ABE$ of the GATES AE, BE of a LOCK, *or* SLUICE *upon a River* PQRS; ſuch that the ſaid Gates ſhall reſiſt the Preſſure of the Water with the greateſt poſſible Force. For ſince the Reſiſtance of the Gate AE diminiſhes in Proportion as the *Preſſure of Water*, and as the Length of the Gate increaſes; and in the ſame Depth of Water, both theſe are as

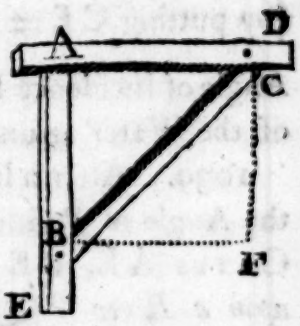


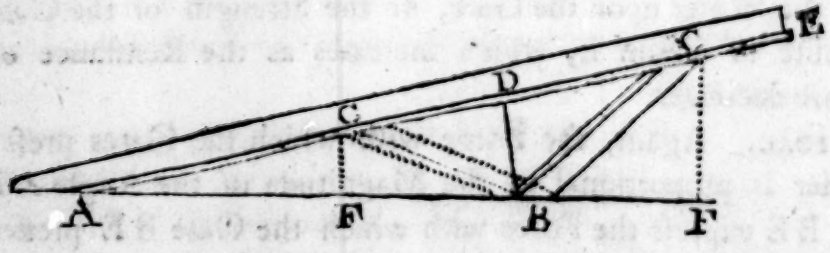
the Line AE; therefore $\overline{AE}^2$ will expreſs the whole Reſiſtance of the Gate AE. On the Diameter AB deſcribe the Semicircle ADB, and continue AE to D, and draw BD and EC; then becauſe $\overline{AE}^2 : \overline{AC}^2 :: \overline{AB}^2 : \overline{AD}^2$ by ſimilar Triangles; and ſince $\overline{AD}^2$ is inverſely as $\overline{AE}^2$, it will expreſs the Force of the Water upon the Gate, or the Strength of the Gate requiſite to ſuſtain it, which increaſes as the Reſiſtance of the Gate decreaſes.

1040. Again, the Force with which the Gates preſs each other is proportional to the Magnitude of the Angle AEB; let BE expreſs the Force with which the Gate BE preſſes the Gate AE obliquely, this is reſolvable into the two Forces DE, which is parallel to AE, and BD which is perpendicular to it; therefore that Strength of the Gate AE (equal to the Force of the Water, multiplied by the perpendicular Preſſure of the Gate BE, *viz.* $\overline{AD}^2 \times BD$,) ought to be a *Maximum*. Wherefore putting $AB = a$, and $BD = x$, we have $\overline{DA}^2 = aa - xx$, and ſo $\overline{AD}^2 \times BD = aax - x^3$, whoſe Fluxion $aa\dot{x} - 3xx\dot{x} = 0$, gives $x = \sqrt{\frac{aa}{3}}$; which ſhews the Angle $BAE = 35° : 16'$, as in the above Examples.

1041.

1041. Let A D be a Beam in the horizontal Position, ſupported at the End A by the upright Piece A E, and it is required to find the Angle of Poſition of another Piece B C, of a given Length, ſuch that it ſhall ſupport the Beam A D with the greateſt Force poſſible. Let B C $= a$, and A C $= x$, then if B C expreſs the abſolute Strength of the Piece B C; C F will expreſs ſo much thereof as ſupports the Beam A D, wherefore this perpendicular Force multiplied by the Diſtance A C, or Lever*, is to be determined a *Maximum*. Now $CF = AB = \sqrt{aa - xx}$, and ſo $CF \times AC = x \times \sqrt{aa - xx}$, whoſe Fluxion $\dot{x}\sqrt{aa - xx} - \frac{xx\dot{x}}{\sqrt{aa - xx}} = 0$; which gives $\sqrt{aa - xx} = \frac{xx}{\sqrt{aa - xx}}$; and ſo $x = \sqrt{\frac{aa}{2}}$, which ſhews the Angle A B C $= 45$ Degrees, or *Half a right one*.





1042. Let A E be a Beam, or Piece of Wood, ſo fixed in A as to make a given Angle E A B with the Horizon A B; it is required to find the Poſition of another Piece B C, of a given Length, ſuch that it ſhall ſupport the Beam A E with the greateſt poſſible Force. From C let fall the Perpendicular C F; and becauſe the Angle at A is given, the Ratio of C to A F is alſo given, which let be as n to m; and put B C $= a$, and B D $= x$; then if B C expreſs the abſolute Strength of the Piece B C, B D will expreſs ſo much thereof as ſupports the Beam A E. Now as C F : A F : : B D : A D $= \frac{m}{n}x$, and D C

=

* *N. B.* What relates to the Diſtance A C, conſidered as a Force derived from the Lever, will be explained in the next Chapter.

$= \sqrt{aa - xx}$, whence $AC = \frac{m}{n}x \pm \sqrt{aa - xx}$, which Diſtance, or Lever AC multiplied by DB muſt be a *Maximum*, *viz.* $\frac{m}{n}xx \pm x\sqrt{a^2 - x^2}$, whoſe Fluxion $\frac{2m}{n}x\dot{x} \pm \dot{x}\sqrt{aa - xx} - \frac{x^2\dot{x}}{\sqrt{aa - xx}} = 0$, whence $\frac{2m}{n}x \pm \sqrt{aa - xx} - \frac{xx}{\sqrt{aa - x}} = 0$, and by Reduction we ſhall have $x = \sqrt{\frac{1}{2}aa \pm am}$, which will determine the Angle of Poſition ABC.

Note, The Equation above requires the acute Angle ABC to be the Compliment of this obtuſe one ABC to two right Angles.

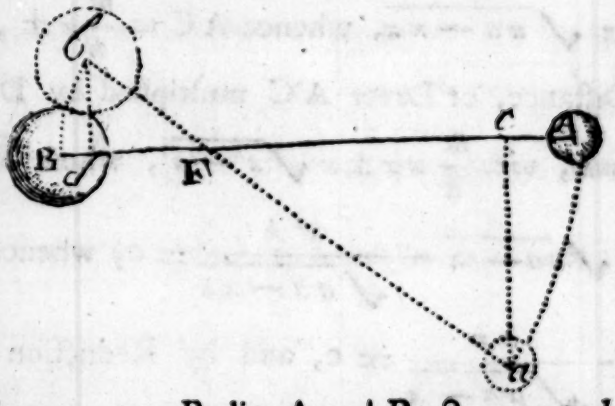
1043. If we put this Value of x into the Equation $\pm x\sqrt{aa - xx} + \frac{m}{n}xx$, we ſhall have $AC \times BD = \frac{aam + 2amm + 2ann}{2n}$, when the Angle is obtuſe; and $\frac{aam + 2amm - 2ann}{2n} = AC \times BD$, when the Angle is acute; whence 'tis evident, the Poſition of the Piece in the firſt Caſe is much more advantageous than in the laſt.

CHAP. V.

The Application of the foregoing PRINCIPLES *to ſuch* MACHINES *as are called* MECHANICAL POWERS.

1044. HAVING conſidered the Laws of Motion and moving Forces, under the Circumſtances of *Colliſion* and *Percuſſion*; we now proceed to conſider thoſe Forces which are otherwiſe applied; particularly in ſuch Caſes where they become ſubſervient to all the Purpoſes of *moving heavy Bodies*, and *overcoming Reſiſtances*. And as this is effected by Machines properly applied, ſo this Part of Philoſophy has acquired the Name of MECHANICS.

1045.



1045. The Force of Bodies acting upon each other, either immediately, or by Means of a Machine, is ſtill derived from the ſame Principles of *Matter* and *Velocity* (968, 969.) When therefore any two Bodies A and B act upon each other, by the Interpoſition of the inflexible Rod A B (the moſt ſimple of all Machines,) 'tis eaſy to aſſign the Quantity of Motion in each, while they move about any given Center of Motion F.

1046. For ſuppoſe the Rod AB to be moved out of its *horizontal Poſition* into the *oblique one a* F *b*, then will the Space deſcribed by A be the Arch A *a*, and that deſcribed by B be the Arch B *b*. Theſe Spaces then as they are deſcribed in the ſame Time, by the ſimilar Motions of A and B, will be as the Velocities of thoſe Motions (by 971.) But ſince the Sectors A F *a*, B F *b*, are ſimilar, it will be A *a* : B *b* : : A F : B F (657.) Therefore the Velocities of A and B will be denoted by their Diſtances from the Center of Motion, *viz.* A F and B F.

1047. If the Bodies be homogeneous, or of the ſame Kind, their Quantities of Matter will be as their Bulks A and B (973,) and ſince their Velocities are as A F and B F; therefore the *Momentum*, or Quantity of Motion in A will be as the Rectangle A × A F; and that of B will be as B × B F (by 970.) The Expreſſions of their Forces as required.

1048. This Theory is general, and holds good for every Sort of Motion, which Bodies ſo circumſtanced are capable of. But that Sort of Motion which reſults from the Action of Gravity, ought to have the Velocity expreſſed by the perpendicular Spaces *a c* and *b d*, by which one accedes to, and the other recedes from the Center of the Earth. Becauſe Gravity acts in thoſe Directions only (985,) and therefore its Force muſt be eſtimated thereby in given Quantities of Matter. But becauſe of ſimilar Triangles *a c* F, and *b d* F, it is *a c* : *b d* : : *a* F : *b* F : :

A F

A F : B F. Whence it appears that A × *a c*, and B × *b d*, are the ſame *Momenta* as thoſe above (1047.)

1049. Hence for Bodies in Motion there is a *threefold* Expreſſion of their Forces, as follows:

For A $\begin{cases} 1.\ Q = A \times A\,a \\ 2.\ Q = A \times AF \\ 3.\ Q = A \times a\,c \end{cases}$ For B $\begin{cases} Q = B \times B\,b \\ Q = B \times BF \\ Q = B \times b\,d. \end{cases}$

1050. If we ſuppoſe the Forces in the two Bodies equal, that is, A × A F = B × B F, or A × A *a* = B × B *b*, or A × *a c* = B × *b d*, then it is, A : B :: B F : A F :: B *b* : A *a* :: *b d* : *a c*. Therefore in caſe of an *Equilibrium*, the *Bodies are inverſely as their Diſtances from the Center of Motion*, or as the *circular Arches*, or the *perpendicular Spaces* deſcribed in the ſame Time, and this is the *fundamental Principle* of every mechanical Power, Machine, or Proceſs whatſoever.

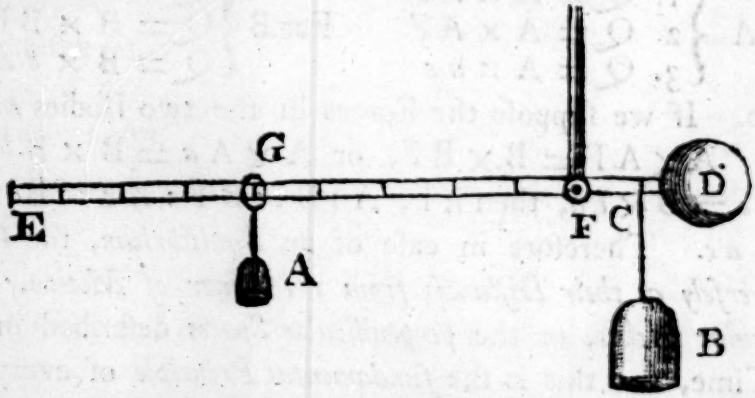
The Lever.

1051. To apply this Doctrine. Suppoſe B a very heavy Weight laid upon the End of a long Pole or Rod A B, ſuſtained by, and moveable upon the Prop, or *Fulcrum* F; and inſtead of the Weight A, ſuppoſe a Perſon's Hand were applied at the End of the Lever A, to raiſe or move the Weight B. Then the *muſcular Force* of the Arm is now to be compounded with the Velocity of Motion to conſtitute a Force equivalent to that of the Weight B. And ſince, in ſuch a Caſe, the Force of the Arm is to the Weight of the Body B, as B F to F A, it is evident, that though the Weight B encreaſes in any Proportion, and the intenſive Force of the Arm remains the ſame, yet by taking the Diſtance A F to B F, in the ſame Proportion in which B is encreaſed, the Perſon will ſtill have it in his Power to move the Body B. And this will be the Caſe in what Manner or Form ſoever the Lever be applied.



1052. From what has been ſaid, the Nature of the Balance muſt fully appear; for this is nothing more than a *Lever*, whoſe *Brachia*, or Arms A F, B F are equal; in which Caſe the Weights A and B appended at each End will have

equal Velocities; and therefore, in Caſe of an *Equilibrium*, the Weights will be equal alſo (by 1047.) But the Weights of Bodies are as the Quantities of Matter (1002,) whence *equal Quantities* of any Sorts of Matter are eaſily determined by the *Equilibrium* of the *Balance*, which is its *only Uſe*.



1053. The STEELYARD is evident nothing but a *ſuſpended Lever*, where the Weight A is applied at different Diſtances from F, the Point of Suſpenſion, to make an *Equilibrium* with B, the Body to be weighed; and when this happens, then on the Arm E F is ſhewn how many Times the Diſtance C F is contained in G F, and juſt ſo many Times is the Weight A contained in B, the Thing required to be found *.

The PULLEY.

1054. A Tackle of Pullies is another mechanical Power, which Action or Force is deriv'd from the general Principles of (1045, 1046, 1047.) For let W be a Weight to be raiſed (by Means of the Tackle of five Pullies A B) by the Power, or Weight P. Then it is evident, that when the ſaid Weight W is raiſed one Inch, each Rope belonging to the lower moveable Box of Pullies will be ſhortened one Inch, and the Rope to which the Power P is appended will be lengthened juſt ſo many Inches; conſequently, the Spaces paſſed through in the ſame Time, and therefore the Velocities of the Bodies W and P will be to each other *as Unity to the Number of Ropes belonging to*



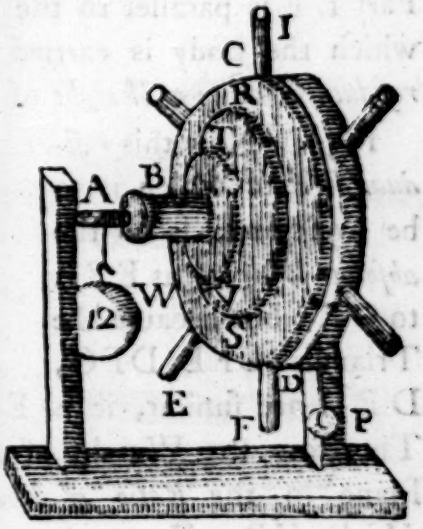
the

* *Note*, The Beam of the Steelyard is ſuppoſed to be without Weight in what has been ſaid; and in Practice, the Weight of the Steelyard is compenſated by a large Weight D; and further it is to be

the lower Sheave of Pullies, which is here as 1 to 5; therefore by ſuch a Structure of Pullies the Force of P is encreaſed five Times. And in this Manner you compute the Force of every other Form or Conſtruction of Pullies.

The WHEEL *and* AXLE.

1055. The *Axis in Peritrochio*, or WHEEL and AXLE, is another *mechanical Machine*, in which the Power P hanging from the Periphery of the Wheel CD, acts againſt the Weight W hanging from the Axis at A; and it evident when the Wheel moves, the Spaces deſcribed by the Bodies P and W, and conſequently their Velocities, will be as the Peripheries of the Wheel and of the Axle, and theſe are as the Diameters (823;) therefore in caſe of an Equilibrium, we have the Power P : the Weight W :: Diameter of the Axis : the Diameter of the Wheel; which in the Fig. is as 1 to 12; and thus the Force of P is encreaſed 12 Times. Hence as the Peripheries R S, T V, are leſs, the Effect of the Machine is alſo leſs in Proportion: And on the other Hand, the Force of the Machine is encreaſed by Means of the Spokes I F, in Proportion as the Diſtance I F is greater than CD, as is extremely obvious from (1047.)

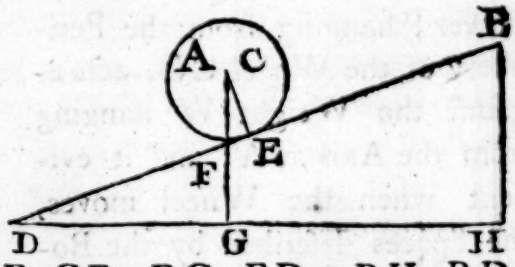


The INCLINED PLANE.

1056. The INCLINED PLANE does not (like other Machines) become a mechanical Power, by encreaſing the Velocity of the Agent, but *by diminiſhing the abſolute Weight of the Body to be moved*; which it does by its Reſiſtance or Re-action, thus; let the Body A lie on the Inclined Plane B D; and from the Center C let fall the Perpendicular C G to the Baſe of the Plane H D, cutting the Plane in F; alſo draw C E perpendicular to the Plane B D. Now ſince the Body gravitates to-

be obſerved, that in Practice we do not regard the Reſiſtance of the Air to the Weight B, by which its real Weight will be a ſmall Matter diminiſhed, as we ſhewed in our *Treatiſe on* AIR, in the YOUNG GENTLEMAN *and* LADY'S PHILOSOPHY.

wards the Earth in the Direction C G, let C F repreſent the Whole, or abſolute Gravity of the Body. This Force is reſolvable into two others, *viz.* C E and E F (1027,) of which C E is that by which it acts on the Plane directly, and is deſtroyed by the equal Re-action of the Plane (965,) the other Part E F is parallel to the Plane, and conſequently, that by which the Body is carried down the Plane, and is called the *reſidual Gravity or Weight* of the Body.



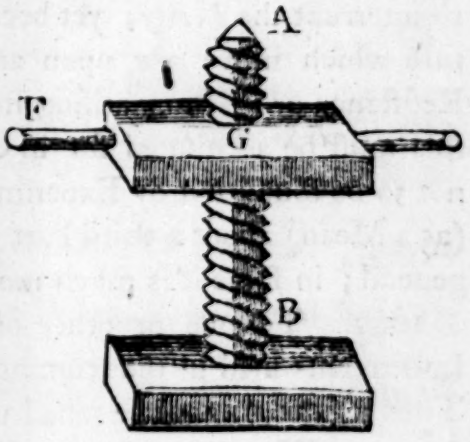
1057. Now this *reſidual Weight* which is to be overcome, is to the *abſolute Weight*, as E F is to C F; but becauſe the Triangles C F E, D F G, D B H are ſimilar, it is E F : C F :: F G : F D :: B H : B D. Therefore the Weight of the Body A is diminiſhed by the Plane in the *Ratio of the Length of the Plane* B D *to its Height* H P. Hence the more inclined the Plane is, or the leſs the Angle at D, the more eaſily will a Body be moved thereon. Therefore when B D coincides with D H, or when the Plane becomes horizontal, the whole Gravity of the Body is deſtroyed; and hence it appears that *the heavieſt Body laid upon an horizontal Plane (perfectly ſmooth) may be moved with the leaſt Force*; as having in that Caſe *no reſidual Weight to overcome.*



1058. The *Wedge* has been hitherto, by all Writers, reckoned among the *ſimple mechanical Powers*; but when it is conſidered that any *Wedge* A C B is only a *double inclined Plane*, or compoſed of the *two ſingle Ones* A C H and B C H, it will eaſily appear *that the Wedge is the ſame Thing with the inclined Plane*; for to *double any Thing* makes no Alteration in its Nature or Properties, and conſequently cannot make it a *different Species*; this Sort of Logic does not become Mathematicians whoſe Characteriſtic it is to uſe the *juſteſt Method of Reaſoning*. Beſides, it is idle to pretend that the Force which overcomes the Coheſion of Wood is derived from the *Wedge*, when it is ſo evidently

dently no other than the *Momentum* of the *Bittle*, or *Sledge* which drives it into the Wood; the *Wedge* being uſed as a *convenient Inſtrument* only, to keep the diſſevered Parts of the Wood, *&c.* aſunder. Therefore we take the Liberty of diſcharging the *Wedge* from the Office and Rank of a *mechanical Power*.

1059. We muſt alſo treat the *Screw* with the ſame Freedom, as it is ſo evidently nothing more or leſs than an *inclined Plane of a ſpiral Form* applied to a Cylinder A B, and 'tis as plain that whatever moves up and down upon a Screw moves all that while upon an *inclined Plane* in a *circular* inſtead of a *rectilineal Direction*. In this *Screw-plane* the Length is the Circumference of the Cylinder, and the Height the Diſtance between two neareſt Threads or *Helices*, which is ſeldom in greater Proportion than 10 to 1, a Trifle to mention for a *mechanical Power*. The Lever F G added to the *Screw-plane*, make a *compound mechanical Power* of very great Force and Uſe, for Motion, Power, Compreſſion, *&c.* as is too well known to be farther inſiſted upon. Here the *Power is to the Force as the Diſtance of the Helices to the Circumference deſcribed by the End* F *of the Lever* F G, which may be encreaſed at Pleaſure. We therefore reduce the *Number of ſimple mechanical* Powers to *four* only, making the moſt of them at the ſame Time; for it is certain the *Lever*, *Pulley*, and Axis *in Peritrochio*, differ only in Form, and not in their Properties whence their Power is derived, which is the ſame in them all, as we have ſhewn. We therefore refer it to the *Metaphyſician* to determine if there be really any more than *two abſolutely different mechanical Powers*, *viz.* the *Lever* and the *inclined Plane*.



1060. It is alſo, on the other Hand, a Wonder that we meet with nothing in our *mechanic Treatiſes*, on the Subject of an *Arch* conſidered as a *mechanical Power*, as its Nature and the common Uſe we make of it ſeems to entitle it to that Denomination, and eſpecially that the *Catenarian Curve* ſhould be paſſed

by

by in ſuch general Silence, when it is a Subject that merits in the higheſt Degree the Conſideration of every *Mechanic*, *Architect*, or *Engineer*, as we have heretofore obſerved *, and may more largely ſhew in a future Part of this Work.

1061. In the *ſimple mechanical Powers*, I have ſuppoſed no *Friction*, or Impediment thence ariſing to the moving Power, to interrupt the *Theory*; yet becauſe there are no Bodies in Nature which move one upon another without ſome Degree of Reſiſtance or Friction ariſing from the Roughneſs of the Parts, this muſt be accounted for in Calculation of Forces, and it is not to be found but by Experiments, by which it appears to be (at a Mean) about a third Part of the Weight in Machines in general; in ſome it is much more, and in others leſs.

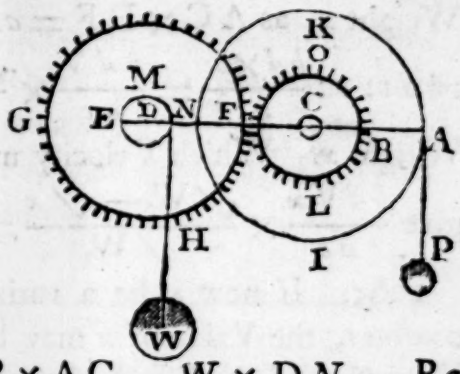
1062. To one or other of theſe Machines, moſt of the Inſtruments uſed in the common Affairs in Life are reducible. Thus a LADDER to be raiſed up upon one End, is one Sort of *Lever*; a WHEEL-BARROW is another; as alſo the CROW, for forcing up Shrubs and ſmall Trees by the Roots. The HAMMER applied in drawing a Nail is a third Sort; the SCISSARS, and SHEARS act on the ſame Principle. The CAPSTAN and WINDLASS, is the ſame with the *Wheel and Axle*; the JACK for raiſing up Bodies is the ſame in Effect. The KNIFE and the Ax are both in the Nature of the *inclined Plane*; and every Body that acts with any Kind of Force upon another, will be found when rightly conſidered, to do it on the Principles above explained.

1063. As to compound Machines, their Force or Power may eaſily be computed, and the Reaſon of their Effects may be clearly underſtood, from the Nature of the ſimple Machines of which they are compoſed. Or thus, let the Machine be ever ſo complicated, conſider the *Velocities of the Motion of the Power and Weight*, and their Ratio will expound the Force of the Engine, as well as in any of the foregoing ſimple Machines. For the Principle of (1045, *&c.*) holds good of all Kinds of Motion whatſoever. But leſt we ſhould be thought deficient in not giving ſome further Account of compound Engines, the following general Theory is ſubjoined.

1064.

* See *Miſcellaneous Correſpondence* for *July*, 1756, and the Plate of a *Bridge* there conſtructed on ſuch Arches.

1064. Suppoſe the Power P by Means of the Wheel A K I and its Pinion B L O, ſuſtain *in Equilibrio* the Weight W, hanging from the Axis E M *N* of the Wheel G F H; then putting A C = *a*, C B = *b*; D F = *c*, D *N* = *d*; we have (by 1045, 1046, 1047,) $\frac{P \times AC}{CB} = \frac{W \times DN}{DF}$, or $\frac{Pa}{b} = \frac{W\,d}{c}$, and therefore P *a c* = W *b d*. Or thus, let (*w*) be the Weight equivalent to P on the Wheel A K I, and its Axle B L O, then P : *w* : : *b* : *a*; therefore $\frac{P\,a}{b} = w$. Again, let *w* be now conſidered as a Power with Regard to W on the Wheel G F H, and its Axle D *N*, then *w* : W : : *d* : *c*; and therefore $w = \frac{W\,d}{c} = \frac{P\,a}{b}$, as above. Now ſince the Machine is at Reſt, the Point A is urged with the whole Power of Gravity or *Momentum* of P; but if the Weight W be diminiſhed to a leſſer Weight *x*, the Machine will move, and the Point A will have a Motion conſpiring with that of P. Let V = Velocity with which the Point A endeavoured to deſcend with the Gravity of P, and *v* = Velocity of the Point A ariſing from the Diminution of the Weight W; then will V — *v* be the relative Velocity with which it would endeavour to deſcend by the ſame Power P, counterpoiſing the Weight *x* at Reſt. But the Effects of the Power P with the Velocities V, and V — *v*, in a given Time, will be as the Spaces through which it would deſcend in that Time, which are as the Squares of thoſe Velocities, *viz.* as V^2 to $\overline{V-v}^2$. And theſe Effects are as W and *x*, who counter-act them; therefore ſince $V^2 : \overline{V-v}^2 :: W : x$, we have $V^2 x = W \times \overline{V-v}^2$, and $V\sqrt{x} = \sqrt{W} \times \overline{V-v} = V\sqrt{W} - \sqrt{W} \times v$; whence $v = \frac{V\sqrt{W} - V\sqrt{x}}{\sqrt{W}}$. But when the Machine is in Motion, the Velocity of the Power P, or *v*, is to the Velocity of the Weight



Weight x, as $AC \times DF = ac$, to $CB \times DN = bd$; or $ac : bd :: v : \frac{bdv}{ac} = \frac{bdV}{ac} \times \frac{\sqrt{W} - \sqrt{x}}{\sqrt{W}}$ = Velocity of the Weight x. Which Velocity multiplied by the Weight x, will give $\frac{bdVx}{ac} \times \frac{\sqrt{W} - \sqrt{x}}{\sqrt{W}}$ = Effect of the Machine.

1065. If now x be a variable Quantitity, and all the reſt conſtant, the Value of x may be determined, when the above Expreſſion of the Effect of the Machine is a *Maximum*, or the greateſt poſſible, by putting its Fluxion equal to nothing. Now the Fluxion of $\frac{bdV}{ac\sqrt{W}} \times x \times \overline{\sqrt{W} - \sqrt{x}}$, is $\dot{x} \times \overline{\sqrt{W} - \sqrt{x}} - \frac{x\dot{x}}{2\sqrt{x}} = 0$, whence we have $\sqrt{W} - \sqrt{x} = \frac{x}{2\sqrt{x}}$, and ſo $2\sqrt{W}x = 3x$, and $x = \frac{4}{9}W$. *Hence if any Engine be charged with $\frac{4}{9}$ of ſuch a Charge or Weight as will juſt keep it in Equilibrio, it will produce the greateſt Effect poſſible.*

1066. Hence, if inſtead of x, we ſubſtitute its Equivalent $\frac{4}{9}W$, we have $\frac{bdVx}{ac} \times \frac{\sqrt{W} - \sqrt{x}}{\sqrt{W}} = \frac{4bdVW}{27ac} = \frac{4VP}{27}$ (becauſe $P = \frac{Wbd}{ac}$) when the Engine is in its greateſt Perfection. Alſo putting $\frac{4}{9}W$ for x in $v = \frac{V\sqrt{W} - V\sqrt{x}}{\sqrt{W}}$, we have $v = \frac{1}{3}V$. Laſtly, if we divide $\frac{4bdVW}{27ac}$ by $\frac{4}{9}W$, we have $\frac{bdV}{3ac}$ = Velocity of the Weight x, when the Machine is in its utmoſt Perfection.

1067. If the Power P inſtead of a Weight, as here repreſented, were a Current of Water, Air, *&c.* then is V the Velocity thereof $= V - v$, the Difference or relative Velocity with which the Fluid ſtrikes the Floats of the Wheel. Whence 'tis eaſy to apply the foregoing Calculus to any mechanical Engine whatſoever, as we ſhall more particularly ſhew under the Subject of *Hydraulics*, in the Theory of *Mill-work*.

1068. The Velocity here mentioned is ſuppoſed to be that of an *uniform Motion*; for though all Machines actuated by a conſtant

conſtant Power will have their Motion at firſt accelerated for ſome Time; yet will the Increments of Velocity continually decreaſe by the Friction, or Reſiſtance of the ſeveral Parts, 'till at laſt they become totally deſtroyed, and the Motion, by that Means, rendered equable and uniform.

CHAP. VI.

The Method of inveſtigating the CENTER OF GRAVITY *in Bodies.*

1069. THAT Point F, which in the Lever was called the *Fulcrum*, (ſee Fig. to Art. 1051,) is otherwiſe called the *common Center of Gravity*; for as in every ſingle Body there is one common Point which tends to the Center of the Earth, with the united Forces of all the gravitating Particles which compoſe that Body, and which therefore is called its *Center of Gravity*; ſo in any Syſtem of two or more Bodies, which are any how connected or depend on each other, there is one certain Point in which their whole Forces of Gravity are united, and which being ſuſpended, keeps the Bodies *in Equilibrio*, ſuch as is the Point F with reſpect to the Bodies A and B, and it is therefore their *common Center of Gravity*.

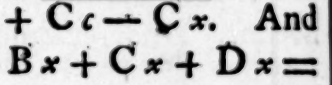


1070. If to the Bodies A and B, we ſuppoſe two others added, as C and D, at ſuch Diſtances from F, that C : D :: DF : CF; then C × CF = D × DF, (by 1047,) and becauſe it is alſo A × AF = B × BF; therefore theſe Equations added together, make A × AF + D × DF = B × BF + C × CF. Thus it appears the Point F is the common Center of Gravity of all the Bodies.

1071. Hence a general Rule for finding the Diſtance of the common Center of Gravity from any given Point E in a right

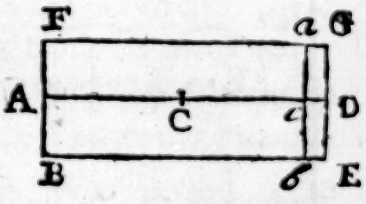
Line E B paſſing through all the Centers of the Bodies A, B, C, D. Thus put E A $= a$, E D $= b$, E F $= x$, E C $= c$, E B $= d$, then A F $= x - a$, D F $= x - b$, C F $= c - x$, B F $= d - x$. Whence the general Equation above (1070) will become $\overline{A\dot{x} - A\dot{a}} + \overline{D\dot{x} - D\dot{b}} = \overline{B\dot{d} - B\dot{x}} + \overline{C\dot{c} - C\dot{x}}$. And by tranſpoſing the Terms we have $A\dot{x} + B\dot{x} + C\dot{x} + D\dot{x} = A\dot{a} + B\dot{d} + C\dot{c} + D\dot{b}$. Conſequently $x = \frac{Aa + Bd + Cc + Db}{A + B + C + D}$ $=$ E F the Diſtance required.

1072. The general Rule therefore for finding the Diſtance of the Center of Gravity from the extreme Part of any Body, is this, *divide the Sum of all the Momenta by the Sum of all the Weights, and the Quotient will be the Diſtance required.* Hence in a right Line A B we may conſider all the Particles which compoſe it, as ſo many very ſmall Weights, each $= \dot{x}$, which is therefore the Fluxion of the Weights, or Line A B $= x$. Therefore the Weight $\dot{x}$ multiplied by its Diſtance from A, *viz.* x, is $x\dot{x}$, its *Momentum*; that is, $x\dot{x}$ is the Fluxion of all the *Momenta* in the Line A B; whoſe Fluent $\frac{1}{2}xx$ is the Sum of all the *Momenta*, which divided by the Sum of all the Weights x, gives $\frac{1}{2}x = \frac{1}{2}$ A B, the Diſtance of the Center of Gravity C from the Point A.



In a PARALLELOGRAM.

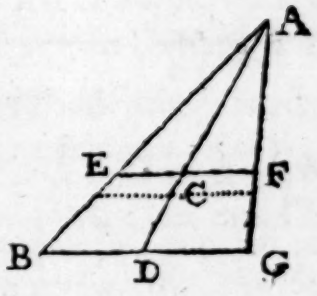
1073. Thus in the PARALLELOGRAM B E G F, whoſe Length A D $= x$, and Breadth G E $= y$, if you draw $a\,b$ infinitely near E G, the Areola $a\,b$ E G $= y\dot{x}$ will be the Fluxion of all the Weights y, which make the whole Weight of the Parallelogram, which multiplied by the Diſtance A D $= x$, gives $yx\dot{x}$, the Fluxion of the *Momenta*; whoſe Fluent $\frac{1}{2}xxy$ is the Sum of all the *Moments*, which divided by the Sum of all the Weights (xy), quotes $\frac{1}{2}x =$ $\frac{1}{2}$ A D $=$ A C, the Diſtance of the common Center of Gravity from A, as required.



In

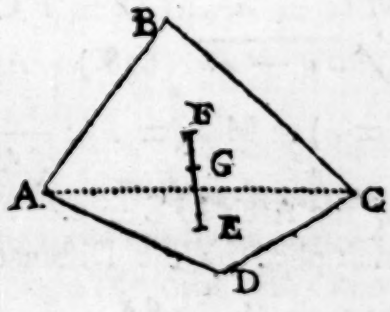
In a Triangle.

1074. In the Triangle A B G, the common Center of Gravity is found thus; draw A D (= *a*) to biſect the Baſe B G (= *b*) in the Point D, and the Parallel E F in the Point C. Put A C = *x*, and we have $a : b :: x : \frac{bx}{a}$ = E F. Which (as a *Weight*) multiplied by $\dot{x}$, gives $\frac{bx\dot{x}}{a}$ = Fluxion of the Weights; this again multiplied by *x* = A C (the Diſtance from A) gives $\frac{bx\dot{x}}{a}$ = Fluxion of the *Momenta*; whoſe Fluent or Sum of the Moments $\frac{bx^3}{3a}$ divided by the Fluent of the Weights $\frac{bx^2}{2a}$, quotes $\frac{2}{3}x = \frac{2}{3}$ A C, for the Diſtance of the Center of Gravity from A in the Triangle A E F; and when *x* = A D, then $\frac{2}{3}$ A D gives the ſame for the Triangle A B G.



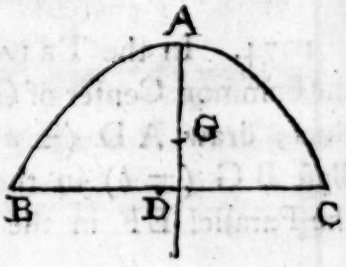
In a Trapezium.

1075. To find the Center of Gravity G of the *Trapezium* B D. Let the ſame be divided into two Triangles A B C and A C D, and find their Centers of Gravity F, E (448.) Join E F, which divide in ſuch a Manner in the Point G, that it may be F G : G E :: A D C : A B C, which is done by this Analogy, A B C D : A C D :: E F : F G, and the Point G is determined as required. This is evident from (1050, and 648.)



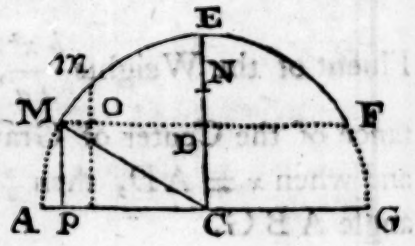
In the PARABOLA.

1076. The Center of Gravity in a PARABOLA BAC. Let AD $= x$, and BC $= 2y = z$; then will $px : zz$ (740;) and putting $p =$ 1, it is $x : zz$, and $z : \sqrt{x} (= x^{\frac{1}{2}})$. Whence $z\dot{x} : \dot{x}x^{\frac{1}{2}}$, the Fluxion of the Weights, which multiplied by x gives $\dot{x}\,x\,x^{\frac{1}{2}}$, the Fluxion of the *Momenta*; whose Fluent $\frac{2}{3}x^{\frac{5}{2}}$ divided by the Fluent of $\dot{x}\,x^{\frac{1}{2}}$, *viz.* $\frac{2}{3}x^{\frac{3}{2}}$, gives $\frac{3}{5}x = \frac{3}{5}$ AD $=$ AG, for the Distance of the Center of Gravity from the Vertex A.



In the ARCH *of a* CIRCLE.

1077. *To find the Center of Gravity* N *of the Arch of a Circle* MEF *fixed to the Radius* CE. 'Tis evident the Particles M, F, equidistant from E, have their common Center of Gravity at D, in the Radius CE; and since the same is true of all the other Particles, it is manifest the common Center of Gravity of the whole Arch MEF is somewhere in the said Radius CE. Put MC $= a$, MD $=$ PC $= x$; then PM (= DC) $= \sqrt{aa - xx}$ (828.) And PM $(= \sqrt{aa - xx})$: MC $(= a)$ :: M o $(= \dot{x})$: $\frac{a\dot{x}}{\sqrt{aa-xx}} =$ M$m = \dot{z}$, the Fluxion of the Arch ME $= z$. Now the Fluxion of the Weights $\dot{z}$ multiplied by the Distance of the Center of Gravity CD $=$ PM, gives $\frac{a\dot{x}}{\sqrt{aa-xx}} \times \sqrt{aa - xx} = a\dot{x} =$ Fluxion of the *Momenta*; the Fluent of which, *viz.* $ax =$ MC $\times$ MD, divided by the Weights or Arch ME, gives $\frac{ax}{z} = \frac{\text{MC} \times \text{MD}}{\text{ME}}$ $=$ CN, the Distance from C required.



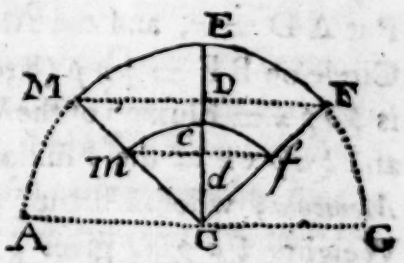
In

In a QUADRANT.

1078. Hence when the Arch E M becomes the QUADRANT A E then $\frac{a\,x}{z} = \frac{M\,C^2}{A\,E} = C\,N$, the Diſtance of the Center of Gravity N of the Semicircle A E G. If C M = 1, then will the Quadrant A M E = 1,57079, (ſee 826,) and $\frac{M\,C^2}{A\,E} = 0{,}6366 = C\,N$, the Diſtance of the Center of Gravity from the Center of the Circle C.

In the SECTOR *of a* CIRCLE.

1079, To find the *Center of Gravity N*, of the SECTOR of a Circle M C F. Deſcribe the Arch *m e f*, and draw the Chord *m f*. Put C E = *a*, M E = *z*, C *e* = *x*; then $a : z :: x : \frac{z\,x}{a} = m\,e$; and



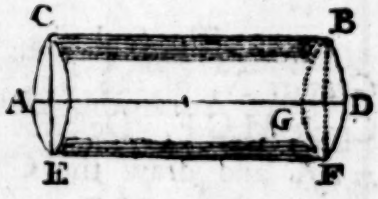
putting M D = *b*, we have $a : b :: x : \frac{b\,x}{a} = m\,d$. Now $\frac{m\,d \times C\,e}{m\,e} = \frac{b\,x}{z}$, is the Diſtance of the Center of Gravity of the Arch *m e f* (1077.) And the Arch $m\,e = \frac{z\,x}{a}$ multiplied by $\dot{x}$, gives $\frac{z\,x\,\dot{x}}{a}$ = the Fluxion of the Weights, which multiplied by the Diſtance of common Center of Gravity $\frac{b\,e}{z}$, gives $\frac{b\,x\,x\,\dot{x}}{a}$ = Fluxion of the *Momenta*, whoſe Fluent $\frac{b\,x^3}{3\,a}$ divided by the Fluent of the Weights $\frac{z\,x\,x}{2\,a}$, gives $\frac{2\,b\,x}{3\,z}$, (or when $x = a$) $\frac{2\,b\,a}{3\,z}$ = Diſtance required.

In a SEMICIRCLE.

1080. Hence when the Arch M E F becomes a *Semicircle* A E G, we ſhall have $\frac{2\,A\,C \times C\,E}{3\,A\,E}$ = Diſtance of the common Center of Gravity of the Semicircle A E G, from the Center C. If Radius A C = 1, then that Diſtance will become $\frac{2}{4.71237}$ = 0.424 +.

In a CYLINDER.

1081. To find the *Center of Gravity* in the CYLINDER E C B F. Put A D = x, and the Area of the Circle on B F = $\frac{1}{4}\,a\,p$ (830;) then is $\frac{1}{2}\,a\,p\,\dot{x}$ = Fluxion of the Weights; and $\frac{1}{2}\,a\,p\,x\,\dot{x}$ = the Fluxion of the *Momenta*; whoſe Fluent $\frac{1}{4}\,a\,p\,x\,x$, divided by the Fluent of the Weights $\frac{1}{2}\,a\,p\,x$, gives $\frac{1}{2}\,x = \frac{1}{2}$ A D = the Diſtance of the ſaid Center from A.



In a CONE.

1082. To find the *Center of Gravity* in a CONE A H B. The Fluxion of the Cone (or Weights) is $\frac{p\,a\,x^2\,\dot{x}}{2\,b\,b}$ (834,) and the Fluxion of the *Momenta* $\frac{p\,a\,x^3\,\dot{x}}{2\,b\,b}$; whoſe Fluent $\frac{a\,p\,x^4}{8\,b\,b}$ divided by $\frac{a\,p\,x^3}{6\,b\,b}$, will quote $\frac{3}{4}\,x = \frac{3}{4}$ H C, the Diſtance of the Center from H.



In the SEGMENT *of a* SPHERE.

1083. To find the *Center of Gravity* in any SEGMENT of a Sphere *a* D *b*. The Fluxion of the Segment is $p\,x\,\dot{x} - \frac{p\,x^2\,\dot{x}}{2\,a}$,

(ſee

(ſee Fig. to Art. 836,) and the Fluxion of the *Momenta* will therefore be $p x^2 \dot{x} - \frac{p x^3 \dot{x}}{2a}$, whoſe Fluent $\frac{1}{3} p x^3 - \frac{1}{8} \frac{p x^4}{a}$ divided by the Fluent (of the Weights) $\frac{1}{2} p x^2 - \frac{1}{6} \frac{p x^3}{a}$, quotes $\frac{5}{8} x =$ Diſtance of the Center of Gravity from D.

In an HEMISPHERE.

1084. Hence when $x = a$, or D c = D C, that is, when the Segment becomes the *Hemiſphere* A D B, then the Diſtance of the Center of Gravity will be $\frac{5}{8}$ of C D from the Point D, or $\frac{3}{8}$ C D from the Center C. In like Manner you will find the Diſtance of the Center of Gravity in the Semi-Spheroid A B D to be $\frac{3}{8}$ of C B from the Center C.

1085. From this Method of *finding the Center of Gravity* in Bodies, there reſults a *general Rule or Canon* for finding the ſuperficial and ſolid Content of Bodies, *viz. The Periphery deſcribed by the Center of Gravity, multiplied into the generating Line or Plane, is ever equal to the Superficies or Solid generated by the Rotation of the ſaid Line or Plane about an Axis.* For Example; the Diſtance of the Center of Gravity in a Semicircle is $\frac{8}{3} \frac{a a}{5p}$ (1080.) And it is $a : p :: \frac{8 a a}{3 p} : \frac{8}{3} a =$ Periphery deſcribed by the Center of Gravity. Now the generating Plane or Semicircle is $= \frac{a p}{4}$ (830,) then $\frac{8}{3} a \times \frac{p a}{4} = \frac{2}{3} p a a =$ Solidity of the Sphere, (by 836, 837.)

CHAP.

CHAP. VII.

The Method of computing the CENTER *of* FORCE, *or* PERCUSSION, *in* moving *Bodies.*

1086. THERE is nothing of greater Conſequence to be rightly underſtood in all mechanic Arts, than the Doctrine of *Momenta and Forces* of moving Bodies. The Former of which we have treated of; and the Latter, *viz.* the *Forces of Bodies* in Motion, and what is called the *Center of Force, or Percuſſion* in a *ſtriking Body*, comes now to be conſidered.

1087. In order to render the Idea, or Notion of this percuſſive Force as natural and eaſy as poſſible, we are to conſider, that what has been hitherto delivered concerning the *Momenta* of Bodies, reſpects them in a State of Reſt, under the Influence of *Gravity*; but when we conſider the Body in *actual Motion*, there will another Force ariſe from the Velocity of that Motion; by which Means the Body will be rendered capable of acting upon another, in the Mode we call *Striking, or Percuſſion.*

1088. This *percuſſive Force*, therefore, ariſes from three Sources, *viz.* (1.) From the *Maſs of Matter* in the percutient Body. (2.) From its *gravitating Force*, in regard to its *Diſtance from the Center of Motion*; and (3.) From the *actual Velocity of the Motion* itſelf. The two firſt of theſe make the *Momentum*; and this compounded with the Latter, conſtitutes the *percuſſive Force.*



1089. Thus, ſuppoſe any uniform Rod A B, were to move or vibrate about the Point or Axis A; let it firſt be ſuſpended in the horizontal Poſition A B at the Extremity B; then will every Particle D, C, B have a Tendency to deſcend in Proportion to the *Quantity of Matter in* each (968;) this Tendency will be farther augmented in proportion to the Diſtance from the *Center of Motion* (1047, *&c.*) and this makes the *Momentum* of the Particle at D, C, and B. Laſtly, if the

Rod

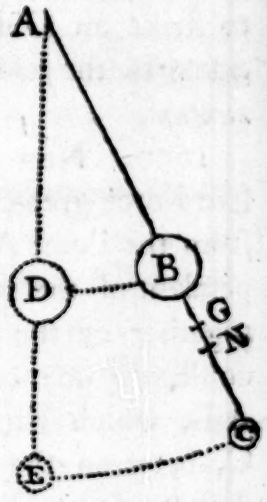
Rod A B be left at Liberty, it will commence Motion, and the actual Velocity of the Particle D, C, or B will enable it to ſtrike an Obſtacle at G, F, or E, with a Force proportional to the ſaid Velocity, conjointly with the reſpective *Momentum*.

1090. Now though each Particle in the Rod has a percuſſive Force greater on all theſe Accounts, as it is more remote from the Point A, if we conſider it ſingly in itſelf, and independent of the reſt; yet when we conſider them all connected together by the Force of Coheſion, their ſeveral Forces will conſtitute one compound Force of the whole Line or Rod A B, which will not be greateſt at B, but in ſome other Point C, between the Extremes A and B; and this Point C will ſtrike a fixed Obſtacle at F in ſuch a Manner, that the whole Force of the ſaid Rod will be exerted upon it, and by the equal Re-action of the Obſtacle, it will be deſtroyed (965,) and ſo the Rod in the Poſition A E, will be motionleſs at the Moment of the Stroke, though diſengaged from the Point A. And this Point C is therefore called the *Center of Percuſſion* in the Rod A B.

1091. In order to this, it is neceſſary this Point C ſhould have the whole percuſſive Force on each Side, in the Parts A C and C B, equal; for if it were greater in the Part A C, than in C B, the Part A C would, after the Stroke, move forwards, not being counteracted by an equivalent Force in the Part C B; alſo if the Force in the Part C B were ſuppoſed ſuperior to that in A C, then it would continue to move forwards alſo after the Stroke, and ſo the Motion of the whole Rod would not be ſpent upon the Obſtacle, as it is when the percuſſive Force is greateſt of all.

1092. Now ſince when this percuſſive Force is not equal, the Rod (ſuppoſed diſengaged from the Point A at the Time of the Stroke) muſt turn upon the Obſtacle as a Center of Motion, it follows, that the Force of each Particle on each Side of the ſaid Center, or Obſtacle, will be as its *Momentum* multiplied into its Velocity, or Diſtance from that Point (1089,) about which it then vibrates; and that therefore *the Momenta of the Particles muſt be reciprocally as their Diſtances from the Point*

(C) *which ſhall be the Center of Percuſſion, or Point of concentrated Forces.*



1093. But to exemplify this Matter, and to inveſtigate a *general Rule* for finding the Center of Percuſſion in all Bodies, let us firſt ſuppoſe the leaſt compound Caſe, *viz.* of two Bodies B, C, fixed to an inflexible right Line A C, and call their Maſſes of Matter Q, q; let their Tendency to Motion, or the Diſtance A B, A C, be G, g; and their actual Velocity in Motion, be V, v. Then if each of theſe Bodies be conſidered by itſelf in Motion, the *whole percuſſive Force* of B, and C, upon Obſtacles at D and E would be denoted by QGV, and qgv. And the Sum of theſe Forces united, *viz.* $QGV + qgv$, is the whole conjoint Force exerted upon an Obſtacle, ſtruck by the Center of Percuſſion.

1094. Let G be the Center of Gravity between the two Bodies, and N the Center of Percuſſion to be found; put $AG = m$, and $AN = n$; then will $\frac{QG + qg}{Q + q} = m$ (1045;) alſo ſince $BN = n - G$, and $NC = g - n$; it will be $QG : qg :: g - n : n - G$ (1092.) Whence $QGn - QG^2 = qg^2 - qgn$, and ſo $QGn + qgn = QG^2 + qg^2$, and ſo $n = \frac{QG^2 + qg^2}{QG + qg}$, or (becauſe G is as V, and g as v) it is $\frac{QGV + qgv}{QG + qg} = AN$; that is, the *Sum of the Forces of the two Bodies divided by the Sum of the* Momenta, *quotes the Diſtance of the Center of Percuſſion from the Point of Suſpenſion* A. Which is the *general Cannon* required.

1095. *Coroll.* Put the Maſſes of Matter in both Bodies $Q + q = M$, and the Sum of the Forces $QGV + qgv = F$; then becauſe $QG + qg = \overline{Q + q} \times m = Mm$; we ſhall have $\frac{F}{Mm} = n$, or $F = Mmn$, whence we have $\frac{M}{F} = \frac{1}{m} \times \frac{1}{n}$. Which Theorem it will be of Uſe hereafter to remember.

1096.

1096. Let the Line A B (in Article 1089,) be denoted by x; then will the gravitating Force of the Point B be as x, (1088;) and the Velocity of the ſaid Point, when in Motion about A, will be as x likewiſe; therefore ſuppoſing the Line in its naſcent State at B, it will be then as $\dot{x}$, and $\dot{x} \times x\,x = \dot{F} =$ Fluxion of the Forces; the Fluent of which $\frac{x^3}{3}$ will be as the whole percuſſive Force in the moving Line, or Rod A B. But the Sum of all the *Momenta* is as $\frac{x^2}{2}$ (by 1072.)

Therefore $\frac{x^2}{2}\Big)\frac{x^3}{3}\Big(= \frac{2}{3}x =$ *Diſtance of the Center of Percuſſion* from the Point of Suſpenſion A, by the general Rule at Art. (1094.)

1097. Thus alſo in the Parallelogram B E G F, (1073) moving about the Axis B F, the Fluxion of the *Momenta* was $y\,x\,\dot{x}$, and ſo the Fluxion of the Forces will be $y\,x\,x\,\dot{x}$, whoſe Fluent $\frac{1}{3}\,y\,x^3$, divided by the *Momenta* $\frac{1}{2}\,y\,x^2$, will give $\frac{2}{3}\,x = \frac{2}{3}$ A D, for the *Center of Percuſſion* from the Axis of Motion.

1098. Again, the Fluxion of the *Momenta* in the Triangle A B G (1074) was found $\frac{b\,x^2\,\dot{x}}{a}$, therefore the Fluxion of the Forces will be $\frac{b\,x^3\,\dot{x}}{a}$, the Fluent whereof $\frac{b\,x^4}{4\,a}$ divided by the *Momenta* $\frac{b\,x^3}{3\,a}$, will quote $\frac{3}{4}\,x = \frac{3}{4}$ A D, the Diſtance of the Center of Percuſſion, from the Axis of Motion at A.

1099. Thus alſo the Center of Percuſſion is found in a Cylinder E B C G (1081,) vibrating about an Axis in the Point A. For the Fluxion of the *Momenta* was $\frac{1}{2}\,a\,p\,x\,\dot{x}$, and therefore the Fluxion of the Forces will be $\frac{1}{2}\,a\,p\,x^2\,\dot{x}$, whoſe Fluent $\frac{a\,p\,x^3}{6}$, divided by the Moments $\frac{a\,p\,x^2}{4}$ will quote $\frac{4}{6}\,x = \frac{2}{3}$ A D for the Center of Force from the Axis A, the ſame as in the right Line (in 1072.)



1100. *To find the Center of Percuſſion in a* CYLINDER E B C G, *vibrating about any diſtant Point* L.

Here it muſt be conſidered, that if two or more Bodies connected together, ſo as to move with equal Velocities, then will the Sum of the Forces of theſe Bodies be equal to the Force of their common Center of Gravity. And therefore the Sum of the Forces of all the Particles in the fluxionary Circle $(\frac{p a \dot{x}}{2}$ =) C F G H, will be equal to the Force of their common Center of Gravity D, or the *Momentum* of that Point multiplied by its Diſtance from the Center of Motion L D. Wherefore putting A L $= b$, and A D $= x$, as before; we have $\frac{a p \dot{x}}{2} \times \overline{b + x} = \frac{a b p \dot{x}}{2} + \frac{a p x \dot{x}}{2}$ = Fluxion of the *Momenta*, which again multiplied by $b + x$, gives the Fluxion of the Forces $= \frac{b b a p \dot{x} + 2 b p a x \dot{x} + p a x x \dot{x}}{2}$, whoſe Fluent $\frac{b b p a x}{2} + \frac{b p a x^2}{2} + \frac{p a x^3}{6}$, divided by the Sum of the *Momenta*, $\frac{b a p x}{2} + \frac{p a x^2}{4}$, will quote $\frac{3 b b + 3 b x + x x}{3 b + \frac{3}{2} x}$, for the Diſtance required from the Point L. Or, if we put L D $= g = b + x$, then will $x = g - b$, and the Expreſſion will become $\frac{g g + g b + b b}{\frac{1}{2} \times g + b}$.

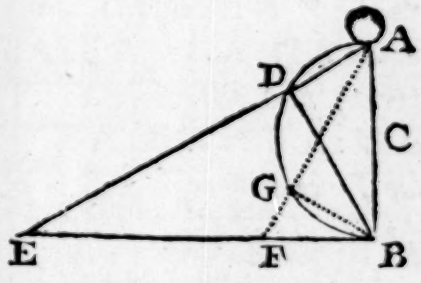
1101. From what has been ſhewn, it is evident, that ſince a Walking-cane is generally but little tapering, and when a Stroke is made therewith, the Center of Motion is in the Hand, the Center of Percuſſion will be near the Part which is $\frac{2}{3}$ of the Cane from the Hand, but ſomewhat nearer to the Hand, on Account of the Cane's not being a perfect Cylinder, but really the Fruſtum of a Cone. So likewiſe a Stroke made with a Sword, (becauſe the Blade is nearly of a triangular, or rather of a pyramidal Form) will be in a Part much nearer to the Hand than in the other Caſe; but to determine preciſely where that Point is in a Sword, (or any Body not of an *uniform Figure*) by an analytical Proceſs, would

would prove too intricate an Affair for this Place; but it may be eaſily found by Means of a *Pendulum*, as ſhewn in the next Chapter.

CHAP. VIII.

Of the DESCENT *of* BODIES *on* INCLINED PLANES, *and the* DOCTRINE *of* PENDULUMS.

1102. THE Doctrine of PENDULUMS depends on that of the Deſcent of Bodies on *inclined Planes*; which therefore comes next to be conſidered. It has been ſhewn (1057) that an heavy Body (as A) upon an inclined Plane A E, has its Gravity diminiſhed in the Ratio of the Length of the Plane A E to the Height A B. About the Height of the Plane A B, as a Diameter, deſcribe the Semicircle A D B cutting the Plane in D, and join D B; then is the Angle A D B a right one (645,) and ſo the Triangles A B D and A E B are ſimilar; and therefore A E : A B : : A B : A D : : *abſolute Gravity* of the Body : *reſidual Gravity*, by which it deſcends on the Plane.



1103. Again, ſince Spaces deſcribed in the ſame Time are proportional to the accelerating Forces of Gravity (999;) the Forces which are as A B and A D, will carry the Body A through the perpendicular Space A B, and the ſlant Space A D in the ſame Time; that is, *in the Time a Body would fall freely from* A *to* B *through the Height of the Plane, another will arrive from* A *to* D *upon the Plane.*

1104. By the ſame Argument it is ſhewn, that a Body will deſcend on any other Plane A F (of the ſame Height) to the Point G in the ſame Time it would deſcend freely through the Height A B; and therefore it follows, *any two Chords* A D, A G *of the Semicircle will be deſcribed in the ſame Time.*

1105.

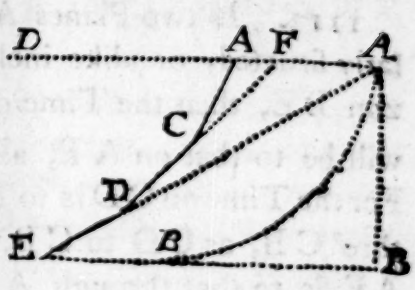
1105. Suppoſe A G = D B, then becauſe the Angle F A B = A B D, we have the Angle A F B = D B F; and ſince both the Planes A G and D B are equal, and both alike inclined to the Horizon, it is evident a Body will deſcribe them both in the ſame Time; and ſo the Time of deſcribing the Chord D B will be the ſame as is ſpent in deſcribing A D. Thus the Time of Deſcent in A G and G B will be the ſame. Conſequently, *the Times of Deſcent thro' any Chords* D B, G B, *will be equal.*

1106. Since the Times of Deſcent through A D and A B are equal (1103,) the Time (t) of Deſcent through A D (or A B) is to the Time (T) of Deſcent through A E, as $\surd \overline{A D}$ to the $\surd \overline{A E}$ (991;) and ſo $t^2 : T^2 :: A D : A E$; but (becauſe A D : A B :: A B : A E) we have A D : A E : $A D^2 : A B^2$, (672) and therefore $t^2 : T^2 :: A D^2 : A B^2$; and ſo t : T :: A D : A B :: A B : A E. That is, *the Time of the perpendicular Deſcent through* A B *is to the Time through the Plane* A E, *as the Height of the Plane to the Length.*

1107. The Velocity acquired in falling from A to D is to the Velocity acquired in deſcending from A to B, as A D to A B, (for ſince they are generated in the ſame Time, they will be as the Powers which produce them, (998.) Alſo the Velocity at D is to that at E, as $\surd \overline{A D}$ to $\surd \overline{A E}$ (991;) that is $v : V :: \surd \overline{A D} : \surd \overline{A E}$, and ſo $v^2 : V^2 :: A D : A E$; but becauſe A D : A B :: A B : A E, it will be A D : A E :: $A D^2 : A B^2$. Therefore $v^2 : V^2 :: A D^2 : A B^2$; and ſo $v : V :: A D : A B$. Hence, ſince the Velocities at B and at E have the ſame Ratio to the Velocity at D, *they muſt be equal to each other* (198.)

1108. In the ſame Manner it is ſhewn, that the Velocity acquired in the Point F, in deſcending through the Plane A F, is the ſame with that at B. Conſequently *the Velocities acquired in Deſcents through any inclined Planes* A E, A F, *of the ſame Height, are equal to each other.*

1109.



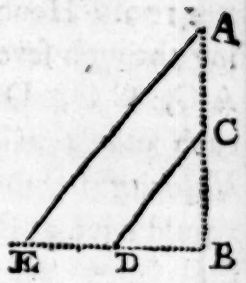
1109. Hence a Body deſcending through ſeveral inclined Planes A C, C D, D E, contiguous to each other, will acquire the ſame Velocity in the Point E, as it would have at B in falling through the ſame perpendicular Height A B. For at C the Velocity is the ſame as it would be in deſcending through F C (1107;) and at D, it is the ſame as it would be in deſcending through A D; conſequently it is the ſame at E as it would be in deſcending through A E, that is, the ſame as it would be in deſcending from A to B, the ſame perpendicular Height.

1110. If now we ſuppoſe the Number of thoſe contiguous Planes infinite, and their Lengths infinitely ſmall, they will then conſtitute a *Curve Line*; whence it follows, *that a Body deſcending through the Arch of any* Curve Line A *B, will acquire the ſame Velocity at the loweſt Point B, as it would have at* B, *by deſcending through the ſame perpendicular Height* A B.

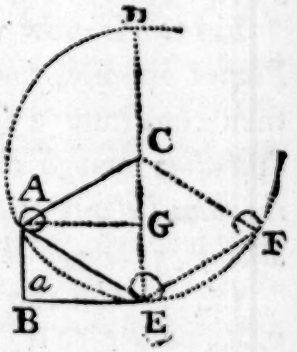
1111. On the Diameter A B deſcribe the Semicircle A G B, and draw any two Chords D B and G B; join A D, and from D and G let fall the Perpendiculars D E, G F to the Diameter A B. *Then will the Velocities acquired in deſcending through the Chords be as their Lengths reſpectively.* For the Velocities acquired through D B and G B will be the ſame as would be acquired in the perpendicular Deſcents through E B and F B (1105,) that is, as $\sqrt{EB}$ to $\sqrt{FB}$ (991.) But ſince A B : D B : : D B : E B (659,) we have $AB = \frac{DB^2}{EB}$; in the ſame Manner it is ſhewn, that $AB = \frac{GB^2}{FB}$; therefore $\frac{DB^2}{EB} = \frac{GB^2}{FB}$, and ſo $BD^2 : GB^2 :: EB : FB$; conſequently $DB : GB :: \sqrt{EB} : \sqrt{FB}$: : Velocity acquired through D B : Velocity acquired through G B. *Q. E. D.*



1112.



1112. If two Planes A E and C D ſimilarly ſituated, or alike inclined to the Horizon B E, then the Time of Deſcent on C D will be to that on A E, as $\surd CD$ to $\surd AE$. For the Time on C D is to that of the Deſcent thro' C B, as C D to C B, and the Time on A E is to that through A B, as A E to A B (1111.) But the Times through C B and A B are as $\surd CB$ to $\surd AB$. But we have (by ſimilar Triangles) $\surd BC : \surd AB :: \surd DC : \surd AE$. Conſequently the Time on D C is to the Time on A E, as $\surd DC$ to the $\surd AE$.



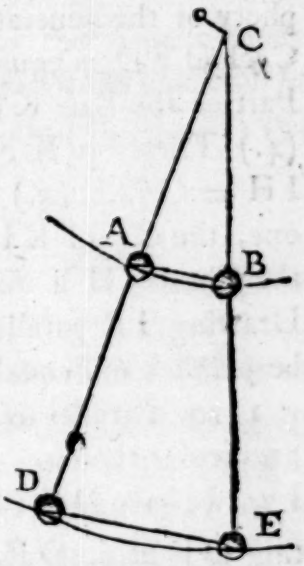
1113. A PENDULUM (or *pendulous Body*) is in any Body A hanging at the End of a String A C, and moveable about the fixed Point C as a Center. Hence 'tis manifeſt, if the ſaid Pendulum be in the Poſition A C, and there left to move freely by the Force of Gravity, it will in its Deſcent deſcribe the Arch of a Circle A *a* E; and when it arrives at the loweſt Point E, it will there acquire a Velocity equal to what it would have acquired, by falling freely through the ſame perpendicular Height G E (1108, 1109.)

1114. The Body having deſcended to E, will, with the Velocity there acquired, continue its Motion forwards (963,) and deſcribe an Arch E F in its Aſcent equal to A *a* E, (ſuppoſing the Motion were in an unreſiſting Medium, and without Friction at C;) for there muſt be the ſame Time ſpent in generating and deſtroying any given Quantity of Motion by the ſame Agent acting uniformly, and conſequently the ſame Space or Arch E F will be deſcribed in the Aſcent, as was before deſcribed in the Deſcent.

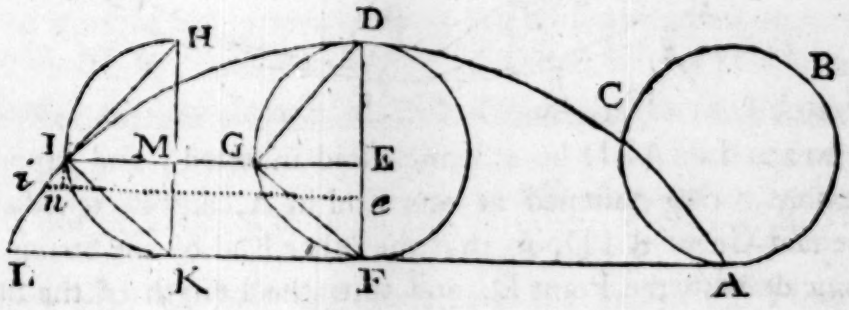
1115. Draw the Chord of the Arch A E, then will that be an inclin'd Plane, whoſe Height is A B = G E. Now 'tis evident, that when the Arch A *a* E is very ſmall, the Chord A E will nearly coincide with it, and the Motion of the Body deſcending on the inclined Plane or Chord, and that of the Pendulum

dulum in the Arch will nearly agree in their Properties, or at leaſt be inſenſibly different.

1116. Hence, if there be two Pendulums of different Lengths CA, CD, the Times in which they will deſcribe the Arches of Deſcent AB and DE, will be as the Square Roots of the Lengths of theſe Arches, or their reſpective coincident Chords AB, DE (by 1112.) But becauſe of ſimilar Triangles ABC and DEC, it will be $\sqrt{AB} : \sqrt{DE} :: \sqrt{CA} : \sqrt{CD} :: t : T$. Therefore $CA : CD :: t^2 : T^2$. That is, *the Lengths of Pendulums are as the Squares of the Times of Vibration.*



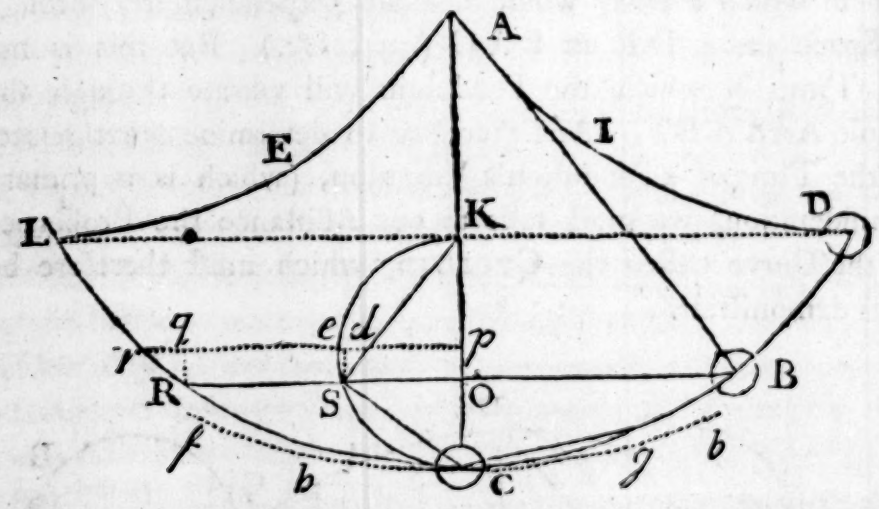
1117. The Time in which the Pendulum would deſcend along the Chord AE, (ſee Fig. of Art. 1113,) is eaſily determined, being that in which it would deſcend through DE (1103, 1105,) = 2AC; and the Time in which it would paſs from A to F on the two Chords AE and EF, will be equal to that in which a Body would deſcend perpendicularly through a Space = 4DE = 8AC (991, *&c.*) But this is not the Time in which the Pendulum will vibrate through the whole Arch AEF. And therefore to determine what relates to the Time of a Pendulum's Vibration, (which is a primary Conſideration) we muſt take to our Aſſiſtance the Properties of the Curve called the CYCLOID, which muſt therefore be next demonſtrated.



1118. If a Circle ABC inſiſting on a Right Line AL begin to revolve from A towards L, the Point A will by its twofold

Motion deſcribe the aforeſaid Curve or Cycloid A C D I L. Whence we obſerve, (1.) The Baſe A L is equal to the Periphery of the generating Circle A B C. (2.) The Axis of the Cycloid F D is equal to the Diameter of the Circle. (3.) The Part of the Baſe K L is equal to the Arch of the Circle I K. (4.) Therefore K F (= M E) is equal to the remaining Arch I H = G D. (5.) The Angle H I K being always a right one, the Chord K I will be *perpendicular* to the Cycloid, and the Chord I H a *Tangent* thereto in the Point I. (6.) Laſtly, Drawing I E parallel to the Baſe A L, the Tangent I H will be parallel and equal to the Chord G D; and I K to G F.

1119. Parallel to E I draw *e i* indefinitely near, and I *n* perpendicular thereto; then by ſimilar Triangles D G E, D G F, I *n i*, we have D E : D G :: D G : D F :: I *n* : I *i*; that is, (putting D F = *a*, D E = *x*, D I = *z*) $x : \sqrt{ax} :: \sqrt{ax} : a :: \dot{x} : \dot{z} = Ii = \frac{a\dot{x}}{\sqrt{ax}}$ = Fluxion of the Arch D I, whoſe Fluent is $2\sqrt{ax}$ (803,) = 2 D G = D I. Whence the Semicycloid D I L = 2 D F, the Diameter of the generating Circle.



1120. Let A I D be a Semicycloid inverted; and ſuppoſe a flexible String faſtened at one End in A, and ſtretched along the ſaid Curve A I D, ſo that the other End of the String may coincide with the Point D, and thus the Length of the String will be equal to that of the Curve. If now a Ball or heavy Body were tied to the End D, and left freely to deſcend, it would deſcribe in one Vibration the Cycloid D C L, provided there

there be placed on the other Side another Semicycloid A E L. Of this I might here give a Demonſtration, but it is needleſs.

1121. The Velocity acquired by the Pendulum deſcending through any Arch R C of the Cycloid, is the ſame a Body would have in deſcending thro' the ſame perpendicular Height O C (1109,) and therefore is as $\sqrt{OC}$; but (becauſe O C : S C :: S C : K C = 1,) $\sqrt{OC}$ is as the Chord S C = $\frac{1}{2}$ R C (1119.) Therefore *the Velocity in the loweſt Point* C, *is ever proportional to the Space paſſed through, or to the Arch of the Cycloid deſcribed in the Deſcent.*

1122. But in all Kinds of Motions, the Space (S) is as the Rectangle of the Time (T) and Velocity (V,) that is, S : T V (by 971, *&c.*) therefore if in any Caſe (as that above 1121) it be S : V, it will be T : 1, that is, the Time of the Motion will be a given Quantity, or always the ſame. *Hence all the Vibrations through any Arches of a Cycloid, great or ſmall, are performed in equal Time.*

1123. If we put C K = a, K O = x, then 2 S C = R C = $2\sqrt{aa-ax}$. If the Deſcent be from L to R, the Velocity at R will be as $\sqrt{KO} = \sqrt{x}$ (1111.) Now (by ſimilar Triangles) it is C O ($a-x$) : C S ($\sqrt{aa-ax}$) (:: C S : C K = a) :: R q = ($\dot{x}$) : R r = $\frac{a\dot{x}}{\sqrt{aa-ax}} = \dot{z}$ = the Fluxion of the Arch L R, which divided by the Velocity $\sqrt{x}$, gives $\frac{a\dot{x}}{\sqrt{aax-axx}} = \frac{a\dot{x}}{\sqrt{ax-xx}} \times \frac{1}{\sqrt{a}} = \dot{T}$ = the Fluxion of the Time. But $\frac{\frac{1}{2}a\dot{x}}{\sqrt{ax-xx}}$ is the Fluxion of the circular Arch K S, (875.) (therefore 2 K S is the Fluent of twice that Fluxion, *viz.* $\frac{a\dot{x}}{\sqrt{ax-xx}}$). Conſequently the Fluent of the Time of Deſcent through L R is $\frac{2KS}{\sqrt{a}}$. Hence when S coincides with C, L R will become L C; and ſo the Time of Deſcent thro' the Semicycloid L C is $\frac{2KSC}{\sqrt{a}}$.

1124. Therefore the Time of a Vibration through the whole Cycloid L C D is $\frac{4\,K\,S\,C}{\sqrt{a}}$. But the Time of Deſcent thro' the Perpendicular $K\,C = a$, is as $2\sqrt{a}$ *; therefore we have $\frac{4\,K\,S\,C}{\sqrt{a}} : 2\sqrt{a} :: 2\,K\,S\,C : a$. That is, *The Time of Vibration in the Cycloid, is to the Time of Deſcent through Half the Length of the Pendulum, as the Circumference of a Circle to the Diameter, or as* 3,14159 *to* 1.

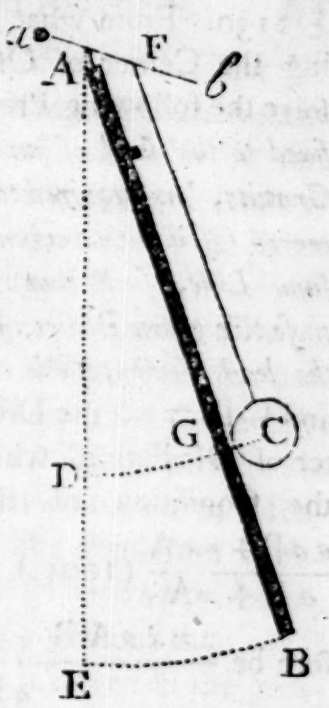
1125. Let the Time of Vibration of the Pendulum A C be = 1 Second, then its Length is thus found. It is known by Experiments, that a Body deſcends freely through the perpendicular Height of $193\frac{1}{2}$ Inches in a Second of Time. And ſince the Spaces deſcended are as the Squares of the Times (991,) therefore $\overline{3.14159}^2 : 1^2 :: 193\frac{1}{2} : 19{,}6 = \frac{1}{4}\,A\,C$; whence A C = 39,2 Inches. And ſince the Lengths alſo are as the Squares of the Times of Vibration (1116,) therefore 4 : 1 :: 39,2 : 9,8 Inches = Length of a Pendulum vibrating in $\frac{1}{2}$ a Second.

1126. Hence ſince the Length of a *Second Pendulum* is ſo conſiderable, the Bob will deſcribe (without the cycloidal Cheeks A I D, A E L,) an Arch of a Circle $f\,C\,b$, which will nearly coincide with the Cycloid for a ſmall Space $b\,g$. Hence all the Properties of the common Pendulum vibrating through very ſmall Arches $b\,g$, will have the ſame Properties as though it moved in the Arch or Curve of the Cycloid.

1127.

---

* To make this plain, we have $S = V\,T$ (971.) Therefore $\dot{S} = V\,\dot{T}$. Therefore $\dot{T} = \frac{\dot{S}}{V} = \frac{\dot{S}}{\sqrt{S}}$ (becauſe $V : \sqrt{S}$) but the Fluent of $\frac{\dot{S}}{\sqrt{S}}$ is $2\sqrt{S} = 2\sqrt{a}$, when $S = a$. (See 803, 804.)



1127. Let A B be any Sort of Body ſuſpended from an Axis *a b*, and moveable freely about it; then will it become a Pendulum, and vibrate in the ſame Manner with the common Pendulum C F. For if it be taken out of the perpendicular Poſition A E into any other A B, and there let go, it will by its Gravity deſcend, and paſs the ſaid Perpendicular A E to a Diſtance on the other Side equal to E B, and ſince every Particle in the Rod endeavours thus to deſcend, and all thoſe Particles are connected together, their Forces will be all united in one particular Part as G, and that Force will ſo act upon the Body A B, as if all the Matter thereof were collected in that Point; and hence the Point G is called the CENTER OF OSCILLATION in ſuch a Sort of Pendulum.

1128. Hence it follows, that any Body A B being hung upon an Axis to vibrate, the Time of a Vibration will always be equal to the Time of Vibration in a ſimple Pendulum F C, whoſe Length is equal to the Diſtance A G of the Center of Oſcillation from the Axis of Motion in the Pendulum A B. And therefore a Method is hence obvious of finding the ſaid Point G, or Diſtance A G in any Body whatſoever, *viz. by taking a ſingle Pendulum* F C *that ſhall vibrate in the ſame Time.*

1129. From the above Definition of the *Center of Oſcillation*, it is eaſy to underſtand that it is the very ſame with the *Center of Percuſſion* in the Body or Rod A B, for ſince the Point G is that in which the Forces of all the Particles are united to generate Motion in the Body, and the *Center of Percuſſion* is that in which alone the Motion of the Body can be all deſtroyed (1090;) it neceſſarily follows, they are both one and the ſame Point; and therefore if A B be of an uniform Figure, it will be iſochronal (or vibrate in the ſame Time) with the common Pendulum F C = A G = $\frac{2}{3}$ A B, (1099.)

1130.

1130. From what has been premiſed concerning the Center of Oſcillation, 'twill be eaſy to ſolve the following Problem, *viz.* *Any Body* B *being fixed to the End of an inflexible Line* C B *void of Gravity, 'tis required to find the Diſtance of the Center of Oſcillation, when another Body* A *is fixed to the ſame Line, ſuch that the Pendulum compounded of thoſe two given Bodies, ſhall perform its Vibrations in the leaſt Time poſſible.* Let C B = a, C A = x, and C D = n, the Diſtance of the common Center of Oſcillation, which muſt be a *Minimum* by the Condition of the Problem; now $n = \frac{a\,a\,B + x\,x\,A}{a\,B + x\,A}$ (1094,) whoſe Fluxion muſt therefore be $\frac{2\,x\,\dot{x}\,a\,A\,B + x^2\,\dot{x}\,A^2 - a\,a\,\dot{x}\,A\,B}{a\,B + x\,A^2} = 0$,



which will give $x\,x + \frac{2\,a\,B}{A} = \frac{a\,a\,B}{A}$; whence by compleating the Square, and extracting the Root, we ſhall find $x = \frac{a}{A}$ $\sqrt{A\,B + B\,B} - \frac{a\,B}{A}$.

1131. Having thus found x = C A, if we ſubſtitute its Value in the Equation $n = \frac{a\,a\,B + x\,x\,A}{a\,B + x\,A}$, the Diſtance C D of the Center of Oſcillation of the compound Pendulum becomes known, and thus any ſingle Pendulum of the Length C D = n, will vibrate in the ſame Time with the compound one.

1132. Since the *Lengths* of Pendulums will alter with *Heat* and *Cold* (as we ſhall hereafter ſhew by the PYROMETER) the *Times* of their Vibrations will vary alſo on that Account (1116,) and therefore when applied to Clock-work and other Uſes, the *Compound Pendulum* will be preferable to the ſingle one, in as much as the Body A may be conſidered as a *Corrector* of the Motion of the Pendulum C B, ſince its Center of Oſcillation D may be always kept on the ſame Point by moving the Ball A up or down by Means of a Screw on the Rod C D. But in this Caſe great Skill and Caution will be required for a proper Adjuſtment.

1133.

1133. Hence it appears, the Nature of the *Pendulum* (of any Sort) conſtitutes it one of the beſt Kind of CHRONOMETERS, or Inſtrument for *meaſuring Time*; and alſo by this Means it is applicable to ſome Caſes of ALTIMETRY, LONGIMETRY, *&c.* and thoſe too, where Trigonometry is either wholly deficient, or cannot be ſo eaſily applied. It moreover ſerves for the *Meaſure of Forces* of *Percuſſion*, *Reſiſtance*, *Velocity*, *&c.* and in ſuch Caſes where the common Methods of Art will fail us, and which yet make the moſt eſſential and fundamental Part of the *genuine Theory of* GUNNERY. The PENDULUM is alſo the only Original and *Philoſophical* STANDARD of Meaſure of LENGTH; and if the Length of the Pendulum vibrating Seconds, had at firſt been made the STANDARD YARD, to all Nations, we ſhould have had no Doubt about their other Meaſures, or Dimenſions, which now lie involved in the greateſt Uncertainty and Obſcurity; all theſe Particulars will fully appear in the Sequel of this Work.

CHAP. IX.

The Phyſico-Mechanical PRINCIPLES *of* BALISTICS, *or the* Doctrine *of* PROJECTILES *applied to the Solution of all Caſes in* GUNNERY.

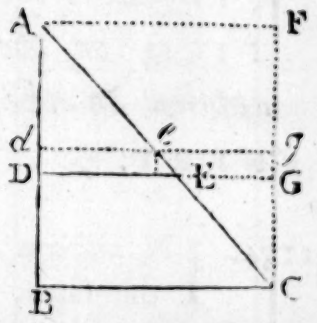
1134. IN *uniform Motion*, or that whoſe Velocity is always the ſame, if the *Time* (T) be given, the *Space* (S) paſſed over will be as the Velocity (V); and if the *Velocity* be given, the Space will be as the Time; *but if neither the Time nor Velocity be given, the Space deſcribed will be as the Product or Rectangles under both, viz.* S : *s* : : T V : t *v*. This we have largely ſhewn (971, *&c.*)

1135. If the Motion be not uniform and equable, but accelerated by a continual Action of the Force which generates the Motion; and this continual Action of the Force be equable and uniform, or in every Moment the ſame, then will the Velocity

Velocity be equably and uniformly accelerated, or increaſe equally with the Time, and will therefore be proportional to the Time; and this is the Caſe of Bodies falling by their Weight or Gravity, which near the Earth's Surface is every where the ſame. Therefore in this Sort of Motion $T : t :: V : v$.

1136. Now though in reſpect of any large Interval of Time, the Motion of falling Bodies is accelerated, yet in the fluxionary Moments of Time the Acceleration is ſo ſmall or inconſiderable, that with reſpect to any proximate Moments the Motion may be eſteemed equable; and conſequently the Space deſcribed in thoſe Moments will be as the Rectangle under the Moments or Fluxion of the Time and Velocity, that is, $\dot{s} = \dot{t}\,v$ (1134.) Alſo ſince $T : t :: V : v$; we have $t = \frac{T}{V} v$; and ſo $\dot{t} = \frac{T}{V} \dot{v}$, which if ſubſtituted for $\dot{t}$ in $\dot{s} = \dot{t}\,v$, we ſhall have $\dot{s} = \frac{T}{V} v\,\dot{v}$; and therefore $S = \frac{T}{2V} v\,v$. *Whence the Space deſcribed by a Body falling freely by its Gravity, is always proportional to the Square of the Velocity*, ſince T and V are given Quantities. This agrees with what was ſhewn in (991, &c.)

1137. Since $t = \frac{T}{V} v$, 'tis plain the Motion is rectilinear, and the Space deſcended ſuch as may be repreſented by a Triangle, one of whoſe Sides repreſents the *Time*, the other the *Velocity*. Thus, if the Triangle A B C repreſent the Space deſcended through in any given Time A B, and B C be the Velocity acquired at the End of that Time; then will the Triangle A D E be the Space deſcribed in the Time A D (992,) and D E (parallel to B C) will be the Velocity acquired in that Time. And ſo the Triangle A B C (S) : A D E (s) : : $\overline{AB}^2$ (T^2) : $\overline{AD}^2$ (t^2) : : $\overline{BC}^2$ (V^2) : $\overline{DE}^2$ (v^2). And therefore A B (T) : A D (t) : : B C (V) : D E (v); (671) which gives the Equation, expreſſing the Nature of a Triangle, as above.

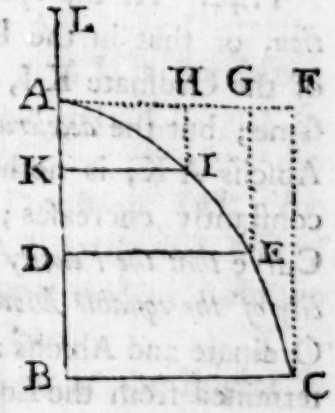


1138.

1138. Hence we may easily compare the Spaces pass'd thro' by a Body moving with an accelerated Velocity in the Time A B, and afterwards with an uniform Velocity, equal to the last acquired Velocity B C, in the same Time. For the Fluxion of the latter Space is $\dot{s} = \dot{t}\ V = D\,d\,g\,G$; and the Fluxion of the Former, or that described by the accelerate Velocity, is $\dot{s} = \frac{T}{V}\,v\dot{v} = D\,d\,e\,E$, whose Fluents t V and $\frac{T}{2V}\,v\,v$ are the Spaces themselves; but since by the Supposition $T = t$, and $V = v$; therefore $t\,V : \frac{T}{2V}\,v\,v :: 2\,T\,V\,V : T\,V\,V :: 2 : 1$. That is, *the Space described by the equable Motion is to that by the accelerate, in the same Time, as* 2 *to* 1; *or as the Rectangle* A B C F *to the Triangle* A B C. (See 993.)

1139. If the Times are not in the simple Ratio of the Velocities, but as any Power t^n to any Power v^m; that is, if $t^n : v^m$, and so $t : \sqrt[m]{v^n}$; then since this is a Ratio expressing the Nature of a *Paraboliform Figure*, 'tis evident the Motion will now be *Curvilinear*; and the Tract which the Body describes will be the Curve of some Kind of *Parabola*. If $n = 1$, and $m = 2$, then $t : v^2$; or $A\,D : \overline{D\,E}^2$, then will the Curve A E C which the Body will describe, be that of the *common Parabola*. If $n = 1$, and $m = 3$, then will $t : v^3$; and the Curve will be that of the *cubical Parabola*. If $n = 2$, and $m = 3$; then $t^2 : v^3$, or $t : v^{\frac{3}{2}}$, and the Curve will be that of the *Semicubical Parabola*; and so of others *in infinitum*.

1140. Here also, the Space described is represented by the *Parabolic* Area A B C, which is to the Space A B C F described by an equable Motion in the same Time A B, and with the last Velocity B C, as 2 : 3, or as $n : n + m$, as we have before shewn (827.)

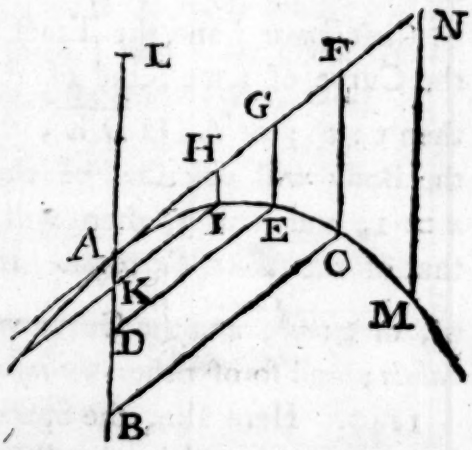


1141. *This accelerated Motion in the Curve of a Parabola is the same with the Motion of a* Projectile. For if the Body be projected (as

from a *Cannon*) from the Point A, in the horizontal Direction A F, it will proceed with an equable Motion; and therefore the Spaces A H, A G, A F will be as the Times of deſcribing them; but while the Body is going from A towards F, it is conſtantly acted upon by Gravity, and drawn downwards towards the Earth's Center, and therefore in a Direction A B, perpendicular to A F; by which Means the Motion towards F is not at all impeded.

1142. Therefore ſuppoſe in the Time it would by the equable Motion arrive at F, it deſcends by its Gravity through the Space A K, then drawing K I parallel to A F, and H I to A B, 'tis evident by this compound Motion the Body will be found at the End of the Time A H in the Point I. But A H, the Time, is as $\sqrt{AK}$, the Space deſcribed by the accelerated Motion, or A K is as $\overline{AH}^2$, or $\overline{KI}^2$, conſequently the Point I is in the Curve of a *common Parabola* (740, *&c.*) and the ſame may be proved of all the other Points E, C. *Therefore the Path of a Projectile is the Apolonian Parabola.*

1143. If the Projection be not in the *Horizontal Direction*, but in oblique Direction *A N*, then alſo will the Curve or Path of the Projectile A E M be that of a Parabola, as is demonſtrable in the very ſame Manner, as before in the other Caſe.



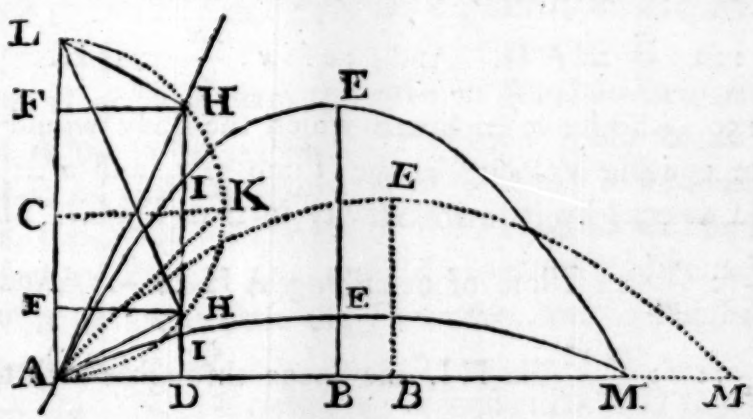
1144. As the *equable Motion*, or that in the Direction of the Ordinate K I, is in the Point A and every where the ſame; but the *accelerate Motion*, or that in the Direction of the Abſciſs A K, is nothing in the Point A, but there begins and conſtantly encreaſes; it muſt happen at ſome Point in the Curve *that the Velocity of the accelerate Motion is equal to the Velocity of the equable Motion, viz.* there where the Fluxion of the Ordinate and Abſciſs are equal. Which Point is thus eaſily determined from the Equation of the Curve $p x = y y$; which, in Fluxions, is $p \dot{x} = 2 y \dot{y}$, and making $\dot{x} = \dot{y}$, we have $p = 2 y$,

or

or $y = \frac{1}{2}p$; therefore $y\,y = \frac{1}{4}p\,p = p\,x$; hence $x = \frac{1}{4}p =$ A K; whence K is the Focus of the *Parabola*.

1145. Hence, if in the Axis produced, we take A L $= x =$ $\frac{1}{4}$ of the Parameter of the Point A, then will the Point L be the Altitude, from which if a Body fall, it will in the Point A acquire ſuch a Velocity, that if it be from thence reflected in the Direction A F, it will deſcribe the given Parabola A E C.

1146. Now ſince it requires the ſame Force to project a Body from the Point A to the Altitude L, as the Body will acquire in the Point A, by falling from the ſaid Altitude L, it follows, that this Altitude is given from the Time of the Aſcent and Deſcent of the Ball projected perpendicularly upwards from the Muzzle of the Cannon; and conſequently the Trajectory or Curve which the Ball will deſcribe, when projected with the ſame Charge of Powder in any Direction A F whatſoever, will be given alſo; and the Solution of the *common Problems of Gunnery* very eaſy.



1147. Thus, for Example, if about A L as a Diameter you deſcribe the Semicircle A K L; then as A L is the Meaſure of the Effect of the *direct Force* of the given Charge; ſo the Chord of the Circle A H will meaſure or repreſent the Effect of the ſame Charge in the oblique Direction A H. *For the direct is to the oblique Force, as Radius* A L *to the Sine* A H *of the Angle of Incidence* A L H $=$ H A D; as is evident from the Reſolution of compound Forces, (1027, *&c.*)

1148. Now ſince A H is as the Force of the Engine in that oblique Direction, ſo this may be reſolved by two o-

thers, *viz.* H D perpendicular to the Horizon, and A D parallel to the ſame. Therefore will H D = B E, be the greateſt Height to which the Ball will riſe in that Projection; and 4 A D = A M, the horizontal Diſtance, Random, or Amplitude thereof. For ſince A L : A H :: A H : H D; the Height to which the Ball will riſe with the Celerity H D is to the Height it will riſe with the Celerity A H, as $HD^2 : AH^2 :: HD : AL$; but A L is the Height with the Celerity A H; therefore H D is the Height with the Celerity H D. Again, the horizontal Velocity A D, being uniform, will carry the Body through twice the horizontal Diſtance A D in the Time it will riſe to the Altitude H or E; and therefore thro' four times A D, while it aſcends and deſcends thro' the Altitude D H or B E.

1149. But theſe Matters are beſt inveſtigated by Fluxions, in the Method *de Maximis & Minimis*, by finding the Relation of the Abſciſs A D to the Ordinate D I. In order to which let *Radius*, *Sine*, *and Coſine*, of the Angle of Elevation H A D be called 1, s, *&c.* and put A D = x. Then will $c : 1 :: x : \frac{x}{c}$ = A H. And $c : s :: x : \frac{sx}{c}$ = D H.

1150. Alſo let m = Space which the Body would deſcribe by an equable Velocity in the Time = t, and n = Space it would deſcend by its own Gravity in that Time. Then $m : t :: \frac{x}{c} : \frac{tx}{mc}$ = Time of deſcribing A H = $\frac{x}{c}$. And as $t^2 : \frac{t^2x^2}{m^2c^2} :: n : \frac{nx^2}{m^2c^2}$ = H I, the Space through which the Body deſcends in the ſame Time. (1136.)

1151. Hence $DH - HI = \frac{sx}{c} - \frac{nx^2}{m^2c^2} = \frac{cm^2sx - x^2n}{m^2c^2}$ = D I, which is to be determined to a *Maximum*, by making its Fluxion $\frac{csm^2\dot{x} - 2nx\dot{x}}{m^2c^2} = 0$, (818.) which gives $csm^2 = 2nx$; and therefore $x = \frac{csm^2}{2n}$ = A B, becauſe now D I is a *Maximum*, and = B E = $\frac{s^2m^2}{4n}$ = the Height of the Projection.

1152.

1152. If we make D I $= \frac{c\,m^2\,s\,x - n\,x^2}{m^2\,c^2} = 0$; then $c\,s\,m^2 = n\,x$; and therefore $x = \frac{c\,s\,m^2}{n}$ = A M, the Amplitude of the Projection. And then $\frac{t\,x}{m\,c}$ will become $\frac{s\,t\,m}{n}$ = the Time of the whole Projection.

1153. If we make the Angle H A D *a Right* one, then will A H coincide with A L, and A L will be the Height, or *Impetus* of the perpendicular Projection, and will be equal to $\frac{m^2}{4\,n}$, becauſe in this Caſe $s = 1$ in the Expreſſion of the Height $\frac{s^2\,m^2}{4\,n}$.

1154. If now we put A L $= \frac{m^2}{4\,n} = a$; then will the Height of the Projection in the Direction A H be B E $= a\,s^2$, and the Amplitude A M $= 4\,a\,c\,s$. And ſince $1 : s :: a : s\,a$ = A H; therefore $1 : c :: s\,a : a\,c\,s$ = A D $= \frac{1}{4}$ A M, as was ſhewn before, (1148.)

1155. Hence 'tis plain the greateſt Random or Amplitude will be A M, made upon the Elevation A K, of an Angle K A D $= 45°$, becauſe then A D becomes a *Maximum*, or equal to C K, and the Height of the Projection will be $BE =$ A C $= \frac{1}{2}$ A L $= \frac{1}{4}$ A M; and therefore the horizontal Random A M is in this Caſe the *Parameter*, and B the Focus of the Parabola A E M.

1156. In this Caſe only the Perpendicular H D touches the Circle in the Point K; in every other Caſe it will interſect it in two Points H, H, which therefore give two Elevations A H, A H, for ſtriking the ſame Object M on the Horizon, either of which may be taken as the Exigence of the Caſe ſhall require.

1157. There is, beſides the above, another Method very conciſe and ſimple, and yet general for all Caſes in the Science of BALISTICS, by Means of the *Tangent* inſtead of the *Sine* and *Coſine* of the Angle of Elevation as before. Thus, put A L $= a$, the Velocity of the Projection will be as $\sqrt{a}$

(1145.)

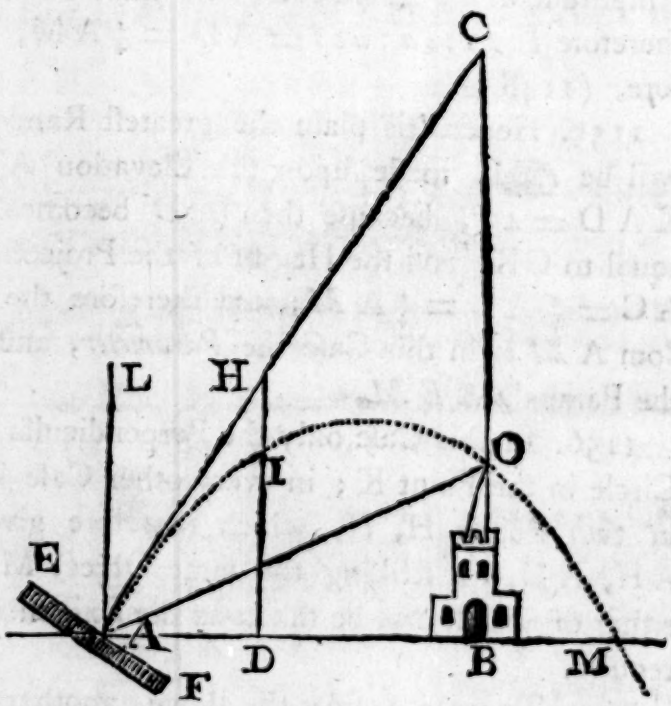
(1145.) Alſo let A H $= t$, and H I $= z$. Then for the Direction A H we have $t : 2z :: \sqrt{a} : \sqrt{z}$ (992.) Whence $t^2 = 4az$.

1158. Again in the Triangle A D H, we have Radius : Tangent : : 1 : n : : A D : D H, and putting A D $= x$, and D I $= y$, we have D H $= nx$; and H I $=$ D H $-$ D I $= nx - y = z$.

1159. Alſo $AH^2 = HD^2 + AD^2$; that is, $t^2 = x^2 + n^2x^2$. Hence $x^2 + n^2x^2 = 4az$, (1156;) and therefore $z = nx - y = \frac{x^2 + n^2x^2}{4a}$ (1157;) from whence we get $4anx - 4ay = x^2 + n^2x^2 = x^2 \times \overline{1+nn}$. From hence the principal Caſes of *Gunnery* are eaſily ſolved.

1160. Case I.

With a given Impetus (*or Charge of Powder*) *a, to ſtrike a given Point* O, *at a given horizontal Diſtance* A B.



Put A B $= b$, and O B $= c$; then will $x = b$, and $y = c$, and the Equation above (1159,) will become $\overline{nn+1} \times bb = 4nab - 4ac$, from whence we get the Tangent of the required Angle of Elevation $n = \frac{2a}{b} \pm \frac{1}{b}\sqrt{4a^2 - 4ac - b^2}$. Whence it appears there are two ſuch Angles or Poſitions of the Cannon that may be taken, agreeable to (1156.)

1161.

1161. If the Point O be in the Horizon, then O B $= c = o$, and $n = \frac{2a}{b} \pm \frac{1}{b}\sqrt{4a^2 - b^2}$; and if O be below the Horizon, it will be negative, or $-c$; and then $n = \frac{2a}{b} \pm \frac{1}{b}\sqrt{4a^2 + 4ac - bb}$. In every Caſe $4aa$ muſt be equal to, or greater than $4ac + bb$, elſe the radical Quantity will be negative and render the Solution impoſſible.

1162. Case II.

To find the Elevation that ſhall produce the greateſt horizontal Random poſſible, with a given Charge of Powder.

Here we have A M $= x = \frac{4n}{nn+1} \times a$, whoſe Fluxion made $= o$, will give $n^2 = 1$, or $n = \sqrt{1} = 1$, and ſhews that the Angle ſought is 45°, (as in 1155.)

1163. Case III.

To find the Charge of Powder requiſite to ſtrike the given Point O *on a given Elevation of the Piece.*

In this Caſe we have $\frac{nn+1}{4nb-4c}bb = a$, which gives the Velocity of the Ball at the Muzzle of the Gun, (1146.)

1164. If we make the Fluxion of this Value of $(a) = o$, it will give $n = \frac{c}{b} \pm \frac{1}{b}\sqrt{bb + cc}$; and if this be ſubſtituted for n, in the foregoing Equation, we ſhall have $a = \frac{1}{2}c + \frac{1}{2}\sqrt{bb + cc}$, *the leaſt Charge of Powder that will throw the Ball on the given Object* O.

1165. Join A O $= \sqrt{bb + cc}$; then ſince the leaſt Impetus $a =$ A L $= \frac{1}{2}$B O $+ \frac{1}{2}$A O; if in O B continued out we take O C $=$ A O; then B C is the *Tangent of the Angle* B A C, and A C the *Elevation* of the Mortar, for ſtriking the given Object O with the leaſt Charge. For Radius : Tangent :: 1: n::

$n :: b\,(=AD) : c + \sqrt{bb+c}\,(=BO+OC)$ whence $n = \frac{c+\sqrt{bb+cc}}{b}$, as before.

1166. In this Caſe, ſince AO = OC, the Angle OAC = OCA, and ſince BC and AL are parallel, the Angle ACB = CAL; and therefore the Angle CAL = CAB, whence it is evident, the Line of Direction AC biſects the Angle LAO.

1167. Therefore if EF be a reflecting plain Surface, placed on the Cannon, perpendicular to its Axis, and the Cannon be moved up and down, 'till an Eye placed over the Glaſs, ſees the Object O in the Pependicular AL (conſidered as a *Plumb-Line*) then that will be the required Elevation of the Piece for ſtriking the Object O with the leaſt Charge. And this *mechanical Method* was firſt invented, or obſerved from the Theory by the late Dr. HALLEY.

1168. We might here largely inſiſt on ſtating the Ratios, and finding the Values of the *Time*, *Direction*, *Height*, *Amplitude*, *Impetus*, *&c.* of Projections; but as they are eaſily deducible from the foregoing Theorems, and we are not now on the *practical Part* of Gunnery, it will be beſides our Purpoſe to dwell any longer on this Head.

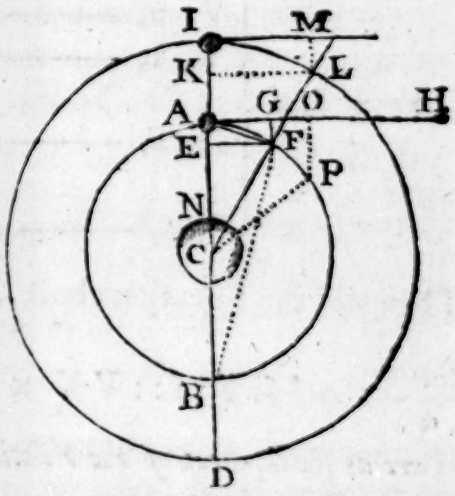
1169. And eſpecially, if it be conſidered, that this Theory of Projectiles *in Vacuo* is only of uſe in two Caſes, *viz.* of *Swift Motions in Vacuo*, ſuch as thoſe of the *Planets*; and *Slow Motion* in a *Reſiſting Medium*, as that of *Spouting Fluids*, *&c.* but for *Swift Motion* in a *Reſiſting Medium*, ſuch as is that of a *Bullet or Bomb* in the *Air*, it can be of little or no Service; as we purpoſe to ſhew further on, when we ſhall treat profeſſedly on the ELEMENTS of MILITARY PHILOSOPHY, and from thence deduce a NEW and GENUINE THEORY of GUNNERY, which though the moſt neceſſary, has hitherto been the leaſt improved Part of the MECHANICAL PHILOSOPHY.

CHAP.

CHAP. X.

Of the PROJECTILE *and* CENTRAL FORCES *by which Bodies deſcribe* CIRCLES, *or any Species of* Circular MOTION; *a Compariſon and Computation of the* FORCES, VELOCITIES *and periodical* TIMES.

1170. LET C be any large attracting Body, A any leſſer Body attracted by it at the Diſtance A C; again, ſuppoſe a *projectile Force impreſſed* on the Body A in the Direction A H, perpendicular to A C, and in ſuch Proportion to the *attractive Force* of C, as that the Body being at Liberty, and urged by both, may deſcribe the Circle A F P B A about the central Body C.



1171. Let A F = $\dot{x}$, be the fluxionary Arch deſcribed in the firſt Moment; and from F let fall the Perpendicular F E, F G, on the Right Lines A C, and A H; then will A G be to A E (= G F,) as the *Projectile Force* to the *Attractive Force*, at the Diſtance A C. Which latter Force is called the *Centripetal Force*, as being thereby carried towards the Center of Attraction C. Draw the Radius C F and the Chords A F and B F, and put A C = x, and A B = d. Then by the Nature of the Circle, A B : A F :: A F : A E = $\frac{\overline{AF}^2}{AB}$: Now becauſe the Sine F E, the Arch A F, the Chord A F, and the Tangent A G are all equal to each other in their *firſt, or naſcent State*; therefore $\frac{\overline{AF}^2}{AB}$ will in that Caſe be = $\frac{\dot{x}\dot{x}}{d} = \dot{x}$; whence $\dot{x} : \dot{\dot{x}}$ :: $\dot{x} : d$:: the Centrifugal Force : the Projectile Force; and becauſe

cauſe $\dot{z}$ is infinitely leſs than d, therefore $\dot{z}$ the Projectile Force does infinitely exceed $\dot{x}$, the Central Force in *circular Motion*.

1172. In any other Circle ILD, the ſame Things are repreſented by Roman Symbols $\frac{\dot{z}\,\dot{z}}{d} = \dot{x}$; then we have $\dot{x} : \dot{x} :: \frac{\dot{z}\,\dot{z}}{d} : \frac{\dot{z}\,\dot{z}}{d}$; that is, *the Central Forces in two different Circles, are as the Squares of the Projectile Forces, or circular Velocities applied to the Diameters of the Circles.*

1173. Let
- F, f, repreſent the *Central Forces* $\dot{x}$, $\dot{x}$.
- V, *v*, —— the circular Velocities $\dot{z}$, $\dot{z}$.
- T, t, —— the periodical Times or Revolutions.
- D, *d*, —— the Diameters of the Circles, or *Radii*.
- P, *p*, —— the Peripheries of the Circles.

Then will the Equation above $\frac{\dot{x}\,\dot{z}\,\dot{z}}{d} = \frac{x\,\dot{z}\,\dot{z}}{d}$ become $\frac{f\,V\,V}{D} = \frac{F\,v\,v}{d}$; and ſo $F : f :: V\,V \times \frac{1}{D} : v\,v \times \frac{1}{d}$; that is, *the Forces are as the Squares of the Velocities directly, and as the Diameters reciprocally*, as before.

1174. If the Bodies I and A ſo move as to deſcribe equal Areas in equal Times, (*viz.* ILC = ACP,) that is, if $d\,\dot{z} = d\,\dot{z}$, then $\dot{z} : \dot{z} :: d : d$; and ſo $\dot{z}^2 : \dot{z}^2 :: d^2 : d^2$; conſequently $\dot{x} : \dot{x} :: \left(\frac{\dot{z}^2}{d} : \frac{\dot{z}^2}{d} ::\right) \frac{d^2}{d} :: \frac{d^2}{d} :: d^3 : d^3$. *That is, the Centrifugal Forces are reciprocally as the Cubes of the Diameters*, (*or* Radii) *of the Circles.*

1175. If $D = d$, then $F : f :: V^2 : v^2$; *or the Forces are directly as the Squares of the Velocities in the ſame Circle.* If $V = v$; then $F : f :: \frac{1}{D} : \frac{1}{d} :: d : D$; *or the Forces are inverſely as the Diſtances or Diameters, when the Velocities are equal.*

1176. Since the Motion in a Circle is equable, the Spaces will be as the Times; and therefore as $V : P :: 1 : T$; hence $TV = P$, and ſo $V = \frac{P}{T} = \sqrt{DF}$ (becauſe $\dot{z} = \sqrt{d\,\dot{x}}$ 1171;)

1171;) therefore $P^2 = T^2 DF = \overline{3,1416}^2 DD$; whence $T^2 F = \overline{3,1416}^2 D$; and ſince $\overline{3,1416}^2$ is a conſtant Quantity, we ſhall have F always as $\frac{D}{T^2}$; or $F : f :: \frac{D}{T^2} : \frac{d}{t^2}$; *that is the central Forces in different Circles will be as the Diameters directly, and reciprocally as the Squares of the periodical Times.*

1177. When T = t, then F : f :: D : *d*; that is, *when the periodical Times are equal, the central Forces are as the Diſtances from the Center directly.*

1178. When D = *d*, $F : f :: t^2 : T^2$; or *the Forces are reciprocally as the Squares of the periodical Times in the ſame Circle.*

1179. If F = f, then $\frac{D}{T^2} = \frac{d}{t^2}$, and ſo $T : t :: \sqrt{D} : \sqrt{d}$; that is, *the periodical Times are in the ſubduplicate Ratio of the Diſtances, when the Forces are equal.* Alſo then we have $\frac{V^2}{D} = \frac{v^2}{d}$; and ſo $V^2 : v^2 :: D : d$; or the *Squares of the Velocities are then as the Diſtances directly from the Center.* Hence alſo $T : t (:: \sqrt{D} : \sqrt{d}) :: V : v$; that is, *when the cent. Forces are equal in two different Circles, the periodical Times will be as the Celerities.*

1180. Again; when V = *v*, we have T : t :: (P : *p* : :) D : *d*; that is, *when the Velocities are equal, the periodical Times will be as the Diameters directly.* And when D = *d*, then T : t :: *v* : V; that is, *in the ſame Circle the Times will be inverſely as the Velocities.*

1181. Since when F = f, we had $T : t :: \sqrt{D} : \sqrt{d}$; and when V = *v*, we had T : t :: D : *d*; therefore when neither F, nor V are given, it will be $T : t :: D\sqrt{D} : d\sqrt{d}$, or $T^2 : t^2 :: D^3 : d^3$; that is, *the Squares of the periodical Times are univerſally as the Cubes of the Diſtances of Bodies circulating about the ſame Center* C.

1182. Or thus more generally; let $F : f :: D^m : d^m :: \frac{D}{T^2} : \frac{d}{t^2}$; whence $d^m D t^2 = D^m d T^2$; or $d^{m-1} t^2 = D^{m-1} T^2$, and ſo

$T : t (:: d^{\frac{m-1}{2}} : D^{\frac{m-1}{2}} ::) D^{\frac{1-m}{2}} : d^{\frac{1-m}{2}}$. Now if $m = 0$; then $T : t :: D^{\frac{1}{2}} : d^{\frac{1}{2}} :: \sqrt{D} : \sqrt{d}$; as before when $F = f$. When $m = 1$, then $T : t :: D^{\circ} : d^{\circ} :: 1 : 1$. *viz. when the Forces are as the Diſtances, the Times of the Periods will be equal, in any Circles whatſoever.* Laſtly, if $m = -2$; then $T : t :: D^{\frac{3}{2}} : d^{\frac{3}{2}}$, or $T^2 : t^2 :: D^3 : d^3$; but in this Caſe $F : f :: D^{-2} : d^{-2} :: d^2 : D^2$; that is, *when the centripetal Forces are inverſely as the Squares of the Diſtances, then will the Squares of the periodical Times be as the Cubes of the Diſtances.* And this is that *univerſal Law of Nature* which is found to obtain *in the Motions of all the heavenly Bodies.*

1183. In the above Theorems and Analogies, we have exhibited the Ratios of the Forces, Velocities, and Times of Revolution in different Orbits, but to expreſs them in their proper Meaſures for any given Orbit whoſe Radius is r, we muſt proceed in the following Manner: The Force is meaſured by the Velocity that may be uniformly generated in a given Time, 1. which let us expound by the Power r^n (the n Power of the Radius (r.) Then the Diſtance through which a Body will freely deſcend in the ſame Time will be expreſs'd by $\frac{1}{2} r^n$ (993.) Therefore if A F be an Arch deſcribed in the Time (t,) the Diſtance A E deſcended in that Time, will be found by this Analogy; as $1^2 : T^2 :: \frac{1}{2} r^n : \frac{T^2 \times \frac{1}{2} r^n}{1^2} = AE$ (992.)

1184. Now from the Nature of the Circle, we have $AF^2 = AB \times AE = 2AC \times AE = 2AE \times AC = \frac{T^2 r^n}{1^2} \times r = \frac{T^2 r^{n+1}}{1^2}$, therefore $AF = \frac{\sqrt{T^2 r^{n+1}}}{1^2} = \frac{T \times r^{\frac{n+1}{2}}}{1} =$ Space deſcribed in the Circle in the Time t.

1185. But in equable Motions, the Spaces deſcribed with a given Velocity (r^n) will be as the Times (971;) therefore $T : T r^{\frac{n+1}{2}} :: 1 : r^{\frac{n+1}{2}} =$ the true Meaſure of the Celerity in the Circle. This might have been deduced from (1173,) where $v = \sqrt{rf} = \sqrt{r \times r^n} = \sqrt{r^{n+1}} = r^{\frac{n+1}{2}}$.

1186.

1186. Then by putting $q = 3{,}14159$, &c. we ſhall have $r^{\frac{n+1}{2}} : 2rq :: 1 : 2qr^{\frac{1-n}{2}}$ = the true Meaſure of the *periodical Time*.

1187. To find the Space S through which a Body muſt deſcend, to acquire the Velocity $r^{\frac{n+1}{2}}$, we have $r^{2n} : r^{n+1} :: \frac{1}{2}r^n : S = \frac{\frac{1}{2}r^{2n+1}}{r^{2n}} = \frac{\frac{1}{2}r \times r^{2n}}{r^{2n}} = \frac{1}{2}r$ (992.) *The Velocity in the Circle therefore is acquired by a Deſcent through half the Radius.*

1188. To accommodate theſe Expreſſions for Uſe; ſuppoſe the Force we ſpeak of be that of GRAVITY; then its Meaſure $r^n = 2s = 32\frac{1}{6}$ Feet, becauſe it is known by Experience, that $\frac{1}{2}r^n = s = 16\frac{1}{12}$ Feet for the Deſcent in the firſt Second of Time. Therefore $r^{n+1} = 2sr$, and $r^{\frac{n+1}{2}} = \sqrt{2sr} =$ the Velocity *per Second*, in any given Circle whoſe Radius is $(r.)$

1189. Again, $\sqrt{2rs} : 2rq :: 1 : \frac{2rq}{\sqrt{2rs}} = q\frac{\sqrt{2r}}{s} =$ T, the *periodical Time*, as is evident from (1186); becauſe $2r = D$, the Diameter, and $2rq = C =$ the Circumference of a Circle, therefore $\sqrt{Ds} =$ Velocity, and $\sqrt{\frac{qC}{s}} =$ periodical Time; which Expreſſions are now in the moſt ſimple Forms.

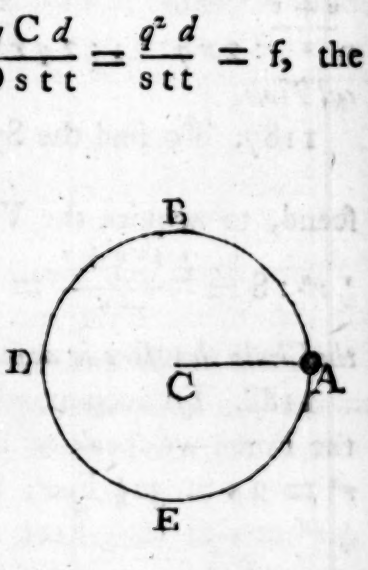
1190. Suppoſe a Circle equal to 8000 Miles, or 42000000 Feet in Diameter, which is nearly equal to that of our *Earth*'s Circumference, and the centripetal Force be equal to that of *Gravity*; then will $D = 42000000$, and $s = 16{,}083$; and ſo $\sqrt{Ds} = 26000$ Feet the Velocity; and $\sqrt{\frac{qC}{s}} = 5075'' = 1^h : 24' : 35''$ nearly the periodical Time, at the Diſtance of the Earth's Surface.

1191. In any Circle, whoſe Diameter is d, and t the Time of Revolution in Seconds, then the central Force in ſuch a Circle may be compared with Gravity. For ſince $\sqrt{\frac{qC}{s}} =$

T,

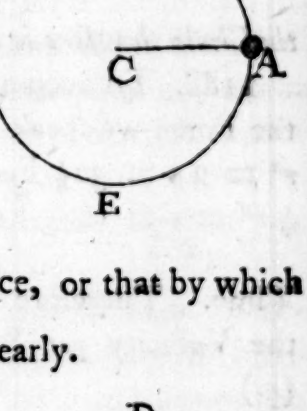
T, we have by (1176,) $\frac{D}{T^2} : \frac{d}{t^2} :: F : f$, that is, (ſuppoſing Gravity, or $F = 1$) $\frac{Ds}{qC} : \frac{d}{tt} :: 1 : \frac{qCd}{Dstt} = \frac{q^2 d}{stt} = f$, the central Force required.

1192. For Example, ſuppoſe A a Ball of *one Ounce*, whirled about the Center C, ſo as to deſcribe the Circle A B D E, and each Revolution be made in *Half* a Second; and let the Length of the Cord A C be two Feet. Then $t = \frac{1}{2}$, and $d = 4$, and $\frac{q^2}{s} = 0{,}6136$;



therefore $\frac{q^2 d}{stt} = 9{,}818 =$ the central Force, or that by which String A C is ſtretched, *viz.* 10 Ounces nearly.

1193. If the String and Ball be ſuſpended from a Point D, and deſcribes in its Motion a conical Surface A D B; then putting $DC = b$, $AC = r$, and $AD = g$; and putting $F = 1$, the Force of Gravity, as before; then will the Body A be affected with three Forces, *viz.* Gravity acting in the Direction D C, a *centrifugal Force*, in the Direction C A, and the *Tenſion of the String*, or Force by which it is ſtretched in the Direction D A; hence theſe three Powers will be as the three Sides of the Triangle C A D reſpectively, (1031) and therefore as C D $(= b) : A D (= g) :: 1 : \frac{g}{b} =$ Tenſion of the String compared with the Weight of the Body.



1194. And $CD (= b) : CA (= r) :: 1 : \frac{q^2 d}{stt}$ (1191,) that is, $b : r :: 1 : \frac{2q^2 r}{stt}$, whence $\frac{2q^2 rb}{stt} = r$, therefore $2b \times q^2 = stt$; and ſo $t = q\sqrt{\frac{2b}{s}}$, or becauſe $\frac{2q^2}{s} = 1{,}2272$, we have $t = 1{,}108\sqrt{b} =$ to the periodical Time. 1195.

1195. We farther obſerve, that $stt = S =$ the Space any Body deſcends in the Time t, ſince $1^2 : t^2 :: s : S$ (992,) therefore ſince $2h \times q^2 = stt \times 1^2$, we have $1^2 : q^2 (= \overline{3,14159}^2) :: 2h : (stt =) S$; that is, *as the Square of the Diameter is to the Square of the Periphery of any Circle, ſo is twice the Height of the Cone to the Space through which a heavy Body will deſcend in the Time of one Revolution.*

1196. Since the mean Diſtance of the Moon is nearly 60 Semidiameters of the Earth, or 1257696000 Feet; if C be the Earth and A the Moon, (Fig. to 1170) then will AB = 251539200; and the Circumference of her Orbit A F B A = 7897834380, which is deſcribed in $27^d : 7^h : 43' = 2360580''$. Hence the Feet paſſed over in one Second will be $3346 = AF = V$. Therefore $AF^2 = 11128976$; whence $\frac{11128976}{2515392000} = 0.00443$ $= AE$, the Diſtance through which the Moon would deſcend in the Time of *one Second*, if the circular Motion were to ceaſe. Now on the Earth's Surface Bodies deſcend in the ſame Time $16\frac{14}{100}$ Feet; but 0,00443 : 16,14 :: 1 : 3643; which is very near as 1 to 3600, the Squares of the Diſtances from the Earth's Center reciprocally; and therefore confirms the Law of central Forces above laid down (1182.)

1197. Again, ſince $T^2 : t^2 :: D^3 : d^3$, we have as $D^3 = 60^3 : d^3 = 1^3 :: T^2 = 2360580''^2 : \frac{2360580}{60^{\frac{3}{2}}} = t = 88' ;$ $53''$, the periodical Time at the Earth's Surface, nearly the ſame as in (1190.)

1198. And ſince when $D = d$, we had $T^2 : t^2 :: f : F$; therefore as $24^h = \overline{1440'}^2$, the Square of the Time of the Earth's diurnal Rotation is to $\overline{84'}^2$, the Square of the periodical Time when the Force is equal to Gravity, ſo is 1 to $\frac{1}{293,8}$, or ſo is 293,8 to 1 = the central Force on the Surface of the Earth ariſing from the Rotation. Hence the centrifugal Force, *viz.* that by which Bodies endeavour to fly off from the Earth's Surface directly from its Center, is to Gravity, nearly as 1 to 293,8 under the *Equator*.

1199.

1199. But at all Diſtances from the Equator to the Pole, this centrifugal Force continually diminiſhes in the Ratio of the Velocities of the Places (1177); but theſe Velocities are as the Diſtances from the Earth's Axis, or as the Co-ſines of the Latitude; *therefore the centrifugal Force is every where to Gravity as the Co-ſine of the Latitude to* 293,8; *the Radius being* = 1.

1200. Let AFB, and ILD be the Orbits the Moon deſcribes about the *Earth* and *Sun* (Fig. to 1170); then will the Semidiameters of thoſe Orbits be A C = 240000, and I C = 80000000; and if we put the Earth's annual Revolution, or periodical Time = 1, then will that of the Moon be $\frac{27,3}{365} = 0.0748$; the Squares of which Times will be 1, and 0,005594. And therefore ſince $F : f :: \frac{D}{T^2} : \frac{d}{t^2}$; we have $\frac{82000000}{1} : \frac{240000}{0,005594} ::$ 80000000 : 43800000 : : 800 : 438 : : F : f. That is, the Gravity of the Moon to the Sun is to its Gravitation to the Earth, as 800 to 438, or as 1,82 to 1 nearly.

1201. Laſtly, we may compare in this Manner the centrifugal Forces ariſing from the diurnal and annual Motions of the Earth, for the Times are, as 1 to 365, whoſe Squares are 1 and 133125; and the Diſtances in Semidameters of the Earth are, as 1 to 20500; therefore $F : f :: \frac{D}{T^2} : \frac{d}{t^2} :: \frac{20500}{133125} : 1 : : 15 : 100$. And ſuch are the Forces ariſing from the annual and diurnal Motions of the Earth, by which Bodies endeavour to fly off from its Surface.

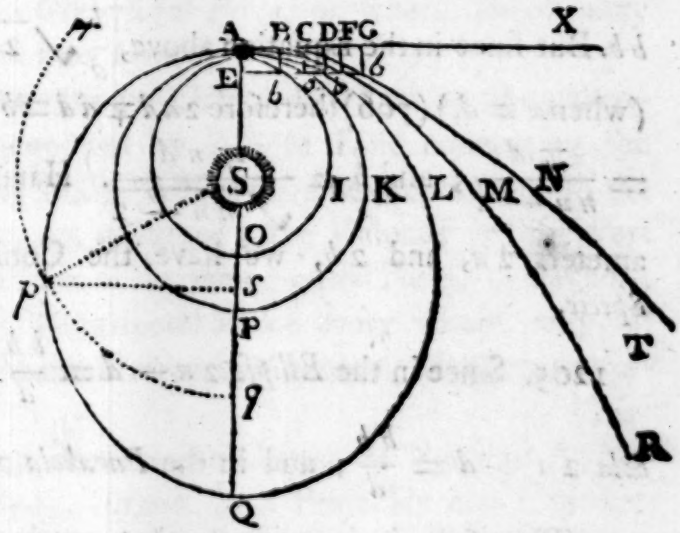
1202. What we have ſaid may ſuffice for the Doctrine of circular Motion and central Forces; in which, I preſume, we have been as full and as plain as the Nature of the Subject will admit; for this is a Species of Knowledge of great Delicacy as well as Uſe; and is to be reckoned among the *firſt Principles* of *Aſtronomy*, and ſome other Sciences. We next proceed to conſider the Curves, or Trajectories, that will be deſcribed by a Body urged with different projectile Forces, compounded with a given centripetal Force.

CHAP. XI.

The general DOCTRINE *of* TRAJECTORIES, *or* ORBITS *deſcribed about a* CENTER, *by a* Given FORCE *tending thereto, and compounded with a variable* PROJECTILE FORCE: *Alſo of the* Proportionality *of* AREAS *and the* TIMES *in which they are deſcribed.*

1203. HAVING diſpatched the Doctrine of *Circular Motion*, where both the *Projectile* and *Centripetal* Forces were given, or conſtant Quantities, in every Part of the Curve; we will now conſider one of them, (*viz.* the Projectile Force) as variable, while the other remains the ſame, and obſerve from thence the different Kinds of *Trajectories*, or *Curves*, that a Body impelled by ſuch a Compoſition of Forces will deſcribe about a given Center. And ſince at any Diſtance from the Center the Projectile Force, by which it may deſcribe a Circle, is given (1187.) therefore that Force may be made the Standard for Compariſon, by which the Quantity of other Forces whereby other Curves are deſcribed may be diſcovered; as we ſhall ſhew in the following eaſy and perſpicuous Method.

1204. Let a Body be projected from the Point A, with ſuch a Velocity AC = 1, as compounded with the centripetal Force A E, may cauſe it to deſcribe a Circle AKP about the Sun, or Center of Force S. Then let it be projected from



the ſame Point, and in the ſame Direction A X, with any other Degree of Velocity A D $= n$, to determine the *Conic Section*, or *Curve* it will deſcribe.

By Suppoſition A X is perpendicular to A S, let E a b be drawn parallel to A X, cutting the Circle and Curve in the Points a and b. Put A S $= d$; the Semi-tranſverſe Diameter $= a$, the Semi-conjugate $= b$; and A E $= x$. Then will $\sqrt{2dx - xx} = y =$ E a in the Circle; and $\frac{b}{a}\sqrt{2ax \mp xx} = y =$ E b, in the Conic Section. Now the Fluxions of the Ordinates E a and E b, viz. $\frac{d - x \times \dot{x}}{\sqrt{2dx - xx}}$, and $\frac{b}{a} \times \frac{a \mp x \times \dot{x}}{\sqrt{2ax \mp xx}}$ will be as the Velocities in every Point of Curves in the Direction A X; but theſe Fluxions are as $\frac{d - x}{\sqrt{2d - x}}$ and $\frac{b}{a} \times \frac{a \mp x}{\sqrt{2a \mp x}}$, (dividing each by $\frac{\dot{x}}{\sqrt{x}}$;) and therefore when A E $= x = 0$, then the Ratio of theſe Fluxions will become $\frac{d}{\sqrt{2d}} : \frac{b}{a} \times \frac{a}{\sqrt{2a}}$, or as $\sqrt{d} : \frac{b}{\sqrt{a}}$ in the Point A. Conſequently $\sqrt{d} : \frac{b}{\sqrt{a}} :: 1 : n$; and ſo $n\sqrt{d} = \frac{b}{\sqrt{a}}$, and therefore $nnd = \frac{bb}{a}$. Whence $nnad = bb$. But ſince in the Equation above, $\frac{b}{a}\sqrt{2ax \mp xx} = y = \frac{bb}{a}$ (when $x = d$,) (766) therefore $2ad \mp dd = bb = nnad$; whence $a = \frac{\mp d}{nn - 2}$; and $b = \frac{\mp nd}{\sqrt{nn - 2}}$. Having therefore the Diameters $2a$, and $2b$, we have the Conic Section given *in Specie*.

1205. Since in the *Ellipſe* $2a - d = \frac{bb}{d}$; and in the *Hyperbola* $2a + d = \frac{bb}{d}$; and in the *Parabola* $2a = \frac{bb}{d} =$ Infinity. Therefore it is evident, that putting $nn = \overline{AF}^2 = 2$, the

the Equation above for (a), will be $a = \frac{\mp d}{0}$; whence a is infinite; and therefore the Curve deſcribed with a Velocity in the Vertical Point, which is as $\sqrt{2} = n$, will be a PARABOLA A M R.

1206. If nn be leſs than 2, then it is plain that a will be affirmative, and b negative, or $+a$, and $-b$; and therefore the Curve deſcribed will be an ELLIPSIS. If n be greater than 1, that is, if E b = A D, be greater than E a = A C, then will the Ellipſis A L Q be without the Circle, and the Sun, or Center of Force in the upper Focus. But if n be leſs than 1, or E b = A B leſs than E a = A C = 1, then will the Ellipſis A I O be deſcribed within the Circle, and have the Sun or Center of Force S in its lower Focus. Laſtly, when $n = 0$, the Body A will deſcend in a right Line A S to the Sun S.

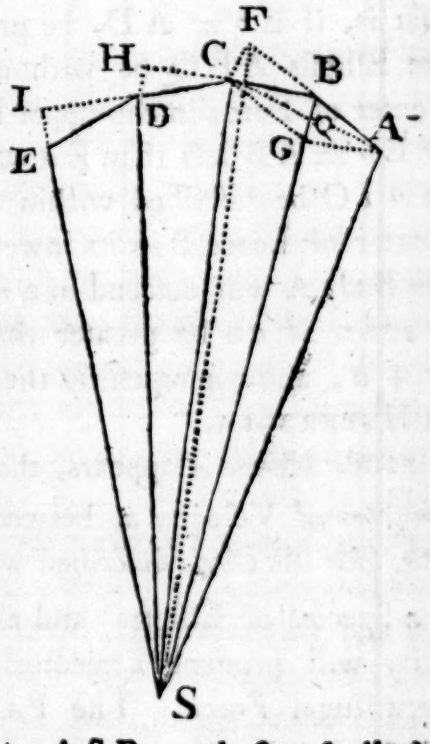
1207. If nn be greater than 2, then will d be affirmative, or $+d$; and conſequently the Curve A N T deſcribed will be an HYPERBOLA.

1208. Hence it appears, that the Curves deſcribed with all Degrees of Velocity n, between 0 and $\sqrt{2}$, will be ELLIPSES, (for the *Circle* deſcribed with the Velocity $n = 1$, is really one Species of Ellipſes) and all the Velocities from $\sqrt{2}$ to Infinity will produce Trajectories of the *hyperbolic Kind*, by a Centrifugal Force. The PARABOLA with the Velocity = $\sqrt{2}$ being the common Limit between them; and where the *Centripetal* becomes a *Centrifugal Force*, or where *Gravity* may be ſaid to be changed into *Levity*.

1209. In any determinate Orbit, deſcribed as abovementioned, the Areas generated by a right Line connecting the Center and revolving Body, will always be proportional to the Times in which they are deſcribed; the Velocity in any Part of the Orbit; the Times of deſcribing equal Parts, or Arches; and the Expreſſion of centripetal Force every where, may all be determined and demonſtrated in a general Manner, as follows.

1210. Let S be an immoveable Center of Force or Attraction, and A any Body, urged by a Projectile and Centripetal Force conjointly. In the Point A let the Projectile Force

be ſuch, as would carry the Body through the Space A B in an indefinite ſmall Particle of Time; then in the ſame Time it would deſcribe B F = A B, were it not drawn from the Tangent A F to ſome other Point C, by the Centripetal Force F C, acting in a Direction parallel to B S. In like Manner, in the Third equal Particle of Time, it would deſcribe C H = C B, but is drawn to the Point D by the Centripetal Force H D in a Direction parallel to C S; and ſo on.



1211. Now all theſe triangular Areas A S B, B S C, C S D, D S E, &c. deſcribed about the Center S in equal Times, are equal to each other. For joining S F, the Triangle A S B is equal to the Triangle B S F, as being upon equal Baſes and of the ſame perpendicular Altitude. (635) Alſo the Triangles B S F = B S C; as being both on the ſame Baſe B S, and between the ſame Parallels B S and C F; therefore the Triangles A S B = B S C. In the ſame Manner the Triangle C S D is proved equal to B S C, and therefore to A S B; and ſo of all the Reſt; whence the Propoſition is evident.

1212. Hence, when the Particles of Time, or the Spaces A B, B C, &c. are taken infinitely ſmall, the Polygon A B C D E, &c. will become a Curve; and *the Areas deſcribed by any Radius* A S *will be always proportional to the Times of their Deſcription.*

1213. Alſo, becauſe in equal Triangles $\frac{1}{2}$ H B = $\frac{1}{2}$ *h b*, whoſe Baſes B, *b*, are unequal, thoſe Baſes are reciprocally as their Heights H, *h*, (*viz.* B : *b* : : *h* : H;) it follows, *that the Velocities in every Part of the Orbit of the revolving Body are reciprocally as the Perpendiculars let fall from* S *to the Tangents of the Orbit in thoſe Points.*

1214.

1214. And becauſe Triangles upon equal Baſes are as their Altitudes, (*viz.* $\frac{1}{2}$ H B : $\frac{1}{2}$ *h* B : : H : *h*;) it follows, *that the Times in which equal Parts, or Arches of the Orbit are deſcribed, are directly as thoſe Perpendiculars to the Tangents.*

1215. Draw A G, G C, parallel to B C, B F; then will B G = F C, or be as the Centripetal Force in the Point B, and becauſe the Diagonal A C biſects B G in Q, and this is every where the Caſe; *therefore in every Part of the Orbit the Centripetal Force will be as the* Sagitta B Q *of the indefinitely ſmall Arch* A B C.

1216. From what has been ſaid, it follows, that every Body which moves in a Curve about an immoveable Point S, ſo that by a Radius drawn to that Point, it deſcribes Areas proportional to the Times, I ſay, ſuch a Body is urged by a Centripetal Force tending to that Point. For ſuch a Body in the Point B is drawn from the Tangent by a Force which acts in the Direction of a Line parallel to C F, that is, in the Direction B S; and in the Point C it acts in ſome Line parallel to H D, *viz.* in the Line C S; and therefore in every Point, it acts in Lines tending to the Center S.

1217. But if the Areas are not proportional to the Times, 'tis evident that C F, H D, *&c.* are not parallel to B S, C S, *&c.* and therefore the Direction of the Centripetal Force is not to the fixed Point S; but to ſome other uncertain and variable Point, on this Side or that, as the Areas deſcribed in equal Times increaſe or decreaſe, as is evident from the Conſtruction of the Figure.

CHAP.

CHAP. XII.

A Determination of the LAW *of the* Centripetal Force, *tending to a given Point any where placed in the* AXIS *of a* CONIC SECTION, *by which the Curve pertaining thereto may be deſcribed.*

1218. TO determine what the Law of the Centripetal Force muſt be, that if the Point to which it tends be placed any where in the Axis of a Conic Section, the Body ſhall be made to deſcribe the Curve proper thereto, is a Problem of the greateſt Importance, and may be ſolved either by *Lineal Geometry*, or by *Fluxions*. In the firſt Way it has been often done, but it is moſt general and expeditious in the Latter, which here follows.

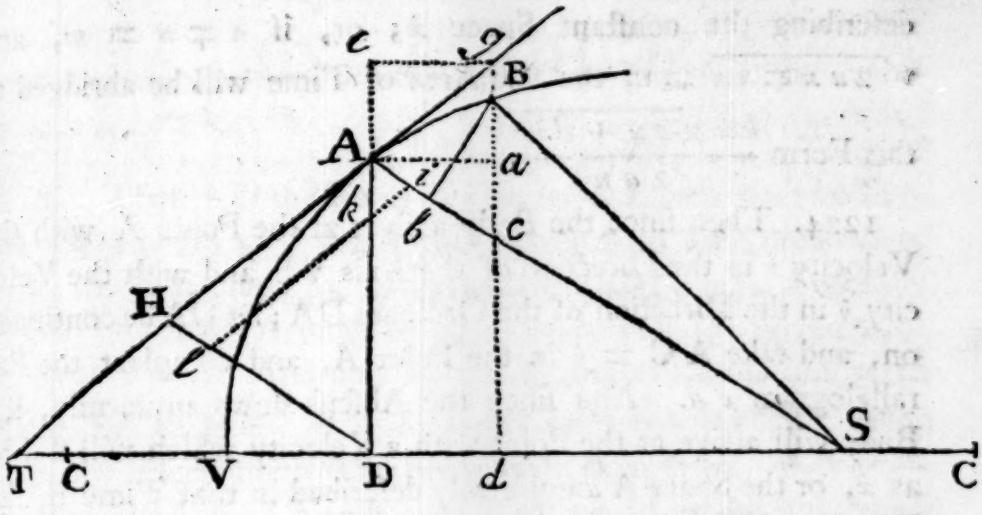
1219. Let V A B be a Part of the Curve, T V C its Axis, C the Center, S a Point in the Axis to which the revolving Body tends, A the Place of the Body, A B the Arch deſcribed in an indefinite ſmall Part of Time; A D $= y$, an Ordinate, V D $= x$, the Abſciſs belonging thereto, A T a Tangent, and A S the Diſtance of the Body; D H a Line drawn parallel to A S. Let V S $= d$, A S $= z$; the *Semi-tranſverſe* Diameter (a,) the *Semi-conjugate* (b;) and let the Abſciſs V D ($= x$) flow uniformly, ſo that its Fluxion $\dot{x} =$ A a, may be conſtant; while the Fluxion of the Ordinate A D, *viz.* B a $= \dot{y}$, is variable.

1220. Then the Equation of the Conic Section being $2ax \mp xx = \frac{aa}{bb}yy$ (767,) we have $2a\dot{x} \mp 2x\dot{x} = \frac{aa}{bb}2y\dot{y}$; and ſo $\dot{x} = \frac{a^2 y\dot{y}}{ab^2 \mp b^2 x}$. Alſo $Ae : eg :: AD : TD$; that is, $\dot{y} : \dot{x} :: y : \frac{\dot{x}y}{\dot{y}} =$ T D, the Sub-tangent. Therefore

$$TD = \left(\dot{x} \times \frac{y}{\dot{y}} = \frac{a^2 y\dot{y}}{ab^2 \mp b^2 x} \times \frac{y}{\dot{y}}\right) \frac{a^2 y^2}{ab^2 \mp b^2 x} = \frac{a^2}{b^2} y^2 \times \frac{1}{a \mp x}$$

$$= \frac{1}{a \mp x} \times 2ax \mp xx = \frac{2ax \mp xx}{a \mp x}.$$

1221.



1221. Again, $DT - VD = VT = \left(\frac{2ax \mp xx}{a \mp x} - x =\right)$ $\frac{ax}{a \mp x}$; therefore $TS = \frac{ax}{a \mp x} + d = \frac{ax + ad \mp dx}{a \mp x}$; and by Reaſon of the ſimilar Triangles TSA, and TDH, we have $TS = \frac{ax + ad \mp dx}{a \mp x} : AS = z :: DT = \frac{2ax \mp xx}{a \mp x}$ $: DH = \frac{z \times 2ax \mp x^2}{ax + ad \mp dx}$.

1222. And laſtly, $DS = d - x : DA = y :: Aa = \dot{x} :$ $ac = \frac{y\dot{x}}{d - x}$; and ſo $Bc = \dot{y} + \frac{y\dot{x}}{d - x} = \frac{\dot{y}d - \dot{y}x + y\dot{x}}{d - x}$; now the Triangle Bbc is ſimilar to Scd, and therefore to SAD, whence $SA : SD :: Bc : Bb$; and ſo $\frac{SD \times Bc}{2} =$ $\frac{SA \times Bb}{2} = \frac{\dot{y}d - \dot{y}x + y\dot{x}}{2}$ = Area of the Triangle ASB, which is the Fluxion of the Area SVA, or the Space that is uniformly deſcribed in the ſame Time with $\dot{x}$.

1223. And becauſe $y = \frac{b}{a}\sqrt{2ax \mp xx}$, we have $\dot{y} =$ $\frac{b\dot{x} \times a \mp x}{a\sqrt{2ax \mp xx}}$; and $\ddot{y} = \frac{ab\dot{x}\dot{x}}{2ax \mp xx^{\frac{3}{2}}}$; therefore theſe Values of y and $\dot{y}$ ſubſtituted in the above found Area, will bring it to this Expreſſion $\frac{b\dot{x} \times \sqrt{ax + ad \mp dx}}{2a\sqrt{2ax \mp xx}}$ for the Time of deſcribing

describing the constant Space $\dot{x}$; or, if $a \mp x = m$, and $\sqrt{2ax \mp xx} = n$, the said Area or Time will be abridged to this Form $\frac{b\dot{x} \times \overline{ax + dm}}{2an}$.

1224. Then since the Body arrives at the Point A with the Velocity $\dot{x}$ in the Direction of the Axis V S, and with the Velocity $\dot{y}$ in the Direction of the Ordinate D A; let D A be continued on, and take A C $= \dot{y}$ in the Point A, and compleat the Parallelogram $e\,a$. And since the Abscis flows uniformly, the Body will arrive at the Point with a Velocity which will still be as $\dot{x}$, or the Space A a uniformly described in that Time in the Direction V S: But the Velocity with which it arrives at B in the Direction d B or A D is different from $\dot{y}$, or A e; and is $\dot{y} \mp q =$ B a, and so $\dot{y} -$ B $a = \mp q =$ B g, which differential Quantity being the Variation of the Fluxion $\dot{y}$, will be the *second Fluxion* of y, or $\ddot{y} = \mp$ B $g = \frac{ab\dot{x}\dot{x}}{2ax \mp xx^{\frac{3}{2}}} = \frac{ab\dot{x}^2}{n\sqrt{n}}$, and therefore will be as the Space passed over in the Direction A D or d B, in the same Time with $\dot{x}$ by a uniform Velocity, generated in that Time in the Fluxion $\dot{y}$.

1225. Since then we have the Space $\frac{ab\dot{x}^2}{n\sqrt{n}}$ passed over in the indefinite Particle of Time $\frac{b\dot{x} \times \overline{ax + dm}}{2an}$, we can find the Space passed through in the constant Particle of Time $= 1$, in the Direction D A, by the following Analogy of the Squares of the Times to the Spaces, *viz.* As $\frac{b^2\dot{x}^2 \times \overline{ax + dm}^2}{4a^2n^2} : 1^2 :: \frac{ab\dot{x}^2}{n\sqrt{n}} : \frac{4a^3}{b\sqrt{n \times \overline{ax + dm}^2}} =$ Space described in the given Particle of Time $= 1$, by the Velocity that is uniformly generated in that Time in the Body A, in the Direction A D.

1226. Lastly, since the Spaces run over in equal Times are as the Velocities, and the Velocities as the Forces by which they are generated; 'tis evident, the Force by which the Body is urged in the Direction A D must be every where, or in any given Particle of Time, proportional to the Space

$4a^3$

$\frac{4a^3}{b\sqrt{n} \times \overline{ax+dm}^2}$, which Diſtance or Force let be repreſented by A k, and through k draw $i\,l$ parallel to the Tangent T A. Then will the Force A k be reſolved into the two Forces $k\,i$, and A i; of which the Latter is in the Direction of A S, and therefore the ſame with the Centripetal Force by which the Body A tends to S, in every equal or given Moment of Time. And becauſe the Triangle A $k\,i$ is ſimilar to D $k\,l$, and therefore to A D H, we have the following Analogy for the Expreſſion of the Force A i, *viz.* As $AD = \frac{b\sqrt{n}}{a} : DH = \frac{zn}{ax+dm} :: \frac{4a^3}{b\sqrt{n} \times \overline{ax+dm}^2} : \frac{4a^4z}{b^2 \times \overline{ax+dm}^3} = A\,i$, the *Centripetal Force* required.

1227. Becauſe $\frac{4a^4}{bb}$ is a conſtant Quantity, the ſaid Centripetal Force will be every where as $\frac{z}{\overline{ax+dm}^3} = \frac{z}{\overline{ax+ad\mp dx}^3} = \frac{AS}{\overline{DC}^3 \times \overline{TS}^3}$; becauſe $DC = a \mp x$.

1228. If now we ſuppoſe $d = a$, or V S = V C, then will the Center of Force be in the Center of the Curve, and in the *Ellipſis* we have $\frac{z}{\overline{ax+da-dx}^3} = \frac{z}{\overline{ax+aa-ax}^3} = \frac{z}{a^6} = z$, (becauſe we may put $a = 1$.) Alſo in the *Hyperbola*, becauſe a is negative, or $-a$, we have $-ax-da+dx = ax-aa^2+ax = -a^2$; and ſo the Force will in this Caſe alſo be as $-z$. In the *Parabola*, a will be infinite, and ax and $\pm dx$ will vaniſh out of the Equation, and leave only $\frac{z}{\overline{ad}^3} : \frac{z}{1} : z$, as before. And therefore in every Section the Force will be directly as the Diſtance S A.

1229. If the Center of Force be in the Vertex of the Curve V, then $d = 0$; and the Force will be every where as $\frac{z}{x^3}$. When the Ellipſis becomes a Circle, then $x = \frac{z^2}{2a}$, (660,)

and the Force is every where as $\frac{8a^3}{z^5}$, or inverſely as the fifth Power of the Diſtance from the Vertex. But if the Center S be removed to an infinite Diſtance, then will the Force be as $\frac{z}{\overline{ad \mp dx}^3} = \frac{1}{\overline{a \mp x}^3} \times \frac{z}{d}$, but $\frac{z}{d} = 1$, as being each infinite and equal; whence, in this Caſe, the Force will be every where as $\frac{1}{\overline{DC}^3}$.

1230. If the Center of Force be placed in the Focus of the Section; then (*per Conics*) we ſhall have $AS = \sqrt{AD^2 + DS^2} = \frac{ax + ad \mp dx}{a} = z$; and therefore $\frac{z}{\overline{ax + ad \mp dx}^3} : \frac{z}{z^3} : \frac{1}{zz}$: Force A *i*; *That is, the Centripetal Force is every where inverſely as the Square of the Diſtance, when the Center of Force is in the Focus of the Section.* And this is the Caſe with reſpect to the *Sun* and *Planets*, and eſpecially the *Comets*, whoſe Orbits are very eccentric Ellipſes.

1231. From the above Demonſtrations, we may obſerve the following Particulars, *viz.* that when the Center of Force S was ſuppoſed in the Center of the Curve, then the Force was every where as the Diſtance *z*, which was demonſtrated alſo of the Circle; when, in different Circles, the periodical Times are equal; *whence the periodical Times alſo in ſimilar Ellipſes are all equal, if made about the ſame Center* C.

1232. Or thus, more univerſally of all Ellipſes. Let A = Area of an Ellipſe, T = periodical Time. V = Velocity at the Vertex; therefore ſince A = (S =) T V, we have $\frac{A}{V}$ = T, and ſo in any other Ellipſe $\frac{A}{V} = T$; whence $T : T :: A \times \frac{1}{V} : A \times \frac{1}{V} :: ab \times \frac{1}{V} : ab \times \frac{1}{V}$; and when the Ellipſes have the ſame common *Tranſverſe*, that is, when $a = a$, it will be $T : T :: b \times \frac{1}{V} : b \times \frac{1}{V} :: bV : bV$; but $V^2 = \frac{bb}{ad}$, and

V^2

$V^2 = \frac{b\,b}{a\,d}$ (1204); and therefore ſince a $d = a\,d$, it is V : V : : b : b; and ſo V $b = V$ b; conſequently T = $\mathcal{T}$. *So that univerſally Bodies revolving in ſimilar elliptic Orbits about the ſame common Center* C, *will all perform their Periods in equal Time.*

1233. Since $T : \mathcal{T} :: a\,b\,\frac{I}{V} : a\,b\,\frac{I}{V}$; and $a\,b = (a\sqrt{n^2\,a\,d}$ $= n\,d^{\frac{1}{2}}\,a^{\frac{3}{2}} =)$ $V\,d^{\frac{1}{2}}\,a^{\frac{3}{2}}$, and $a\,b = n\,d^{\frac{1}{2}}\,a^{\frac{3}{2}}$; therefore $T : \mathcal{T}$ $:: d^{\frac{1}{2}}\,a^{\frac{3}{2}} : d^{\frac{1}{2}}\,a^{\frac{3}{2}}$; and ſo $T^2 : \mathcal{T}^2 :: d\,a^3 : d\,a^3$. Now let r and r be the Radii of two Circles, and t, t, the periodical Times of thoſe Circles; then it will be $T^2 : t^2 :: d\,a^3 : d\,r^3 :: a^3 : r^3$, (when d = r); and in like Manner $\mathcal{T}^2 : t^2 :: (d\,a^3 : d\,r^3 ::)\ a^3 : r^3$, (when $d = r$.) But ſince $t^2 : t^2 :: r^3 : r^3$; therefore $T^2 : \mathcal{T}^2 :: a^3 : a^3$; *That is, The Squares of the periodical Times are as the Cubes of the tranſverſe Axes, in Bodies revolving in elliptic* (*as well as circular*) *Orbits about a Center of Force, poſited in the Focus.*

1234. Hence the *periodical Times in Ellipſes are the ſame as in Circles, whoſe Diameters are equal to the greater Axes of the Ellipſes*; and therefore when the conjugate Diameter (or projectile Force) is = o, the Curve will become a ſtrait Line, which the Body will deſcribe in $\frac{1}{2}$ the Time it would deſcribe a Circle, whoſe Diameter is equal to that Line.

1235. Since the Velocity of a Body is every where as the Perpendicular (= p) let fall from S to the Tangent to the Curve in the Point A (1213); and when the Point A is the Extremity of the conjugate Axis, then will the Velocity be V : $\frac{I}{p} : \frac{I}{b} : \frac{I}{\sqrt{a\,l}} : \frac{I}{\sqrt{a}}$ (by putting the *Latus Rectum* $l = 1$). Let a = Semidiameter of a Circle; then ſince the periodical Times in ſuch a Circle, and the Ellipſis are equal; and in the Circle when T^2 is as a^3, we have $V : \frac{I}{\sqrt{a}}$, every where; it follows, *that the Velocity of a Body revolving in an Ellipſis, is at its mean Diſtance from the Focus, or Center of Force, equal to the Velocity of a Body revolving in a Circle, whoſe Semidiameter is equal to that mean Diſtance.*

1236. The *greateſt*, *leaſt*, and *mean Velocities* of a Body revolving in an Ellipſis A *p* Q, and *s p*, will be as S Q, S A, and (ſee Fig. to Article 1204.) becauſe thoſe Lines are the Perpendiculars to the Tangents in the Points Q, A, and *p*. In the *Parabola*, the *Velocity will always be as the Square Root of the Diſtance from the Center of Force*; becauſe the Perpendicular to the Tangent is always as the Square Root of the Diſtance in that Curve (1205.)

1237. As in the *Ellipſis*, the Force is *Centripetal*, ſo in the *Hyperbola* it will be *Centrifugal*; but in the *Parabola* it will be neither one or the other; for ſince the Center of Force is there at an infinite Diſtance, the Body cannot be properly ſaid to move to or from ſuch a Center. And in this Caſe the Directions in which the Power acts are all parallel; and therefore, *è converſo*, when the Directions in which a Power acts upon a Body are parallel, that Body will deſcribe in its Motion the Curve of a *Parabola*. Whence it follows, that ſince the Center of the Earth is not at an infinite Diſtance, the Directions in which Bodies near its Surface are attracted towards it are not quite parallel, and therefore the Curves which Projectiles deſcribe are not truly (tho' very nearly) *Parabolas*, but really the Arches of very eccentric *Ellipſes*.

CHAP. XIII.

The ELEMENTS *of a* PLANET'S MOTION, *deduced from the foregoing* PRINCIPLES.

1238. WE have hitherto been conſidering ſuch Phyſico-Mathematical Principles of Motion, particularly, with regard to revolving Bodies, as will enable us to account very naturally and eaſily for the Motion of a Planet, or Comet, in its Orbit about the Sun; and therefore the preceding Inſtitutions are to be regarded as the firſt, or elementary Principles in Aſtronomy, without which, the *Rationale* of that celeſtial Science can by no Means appear.

1239.

1239. The Motion of a Planet is known to reſult from a Projectile Force, in the Direction of a Tangent to its Orbit, and a Centripetal Force directed to the Center of the Sun, both which are ſo compounded together, that the Curve which the Planet deſcribes, in Conſequence thereof, approaches very nearly to the Form of a Circle; becauſe in the Caſe of the Planet, the Projectile Force is almoſt infinitely greater than the Force of Gravity, by which the Planet tends to the Sun; in which Caſe, an Orbit nearly circular muſt be deſcribed, as was ſhewn (1171.)

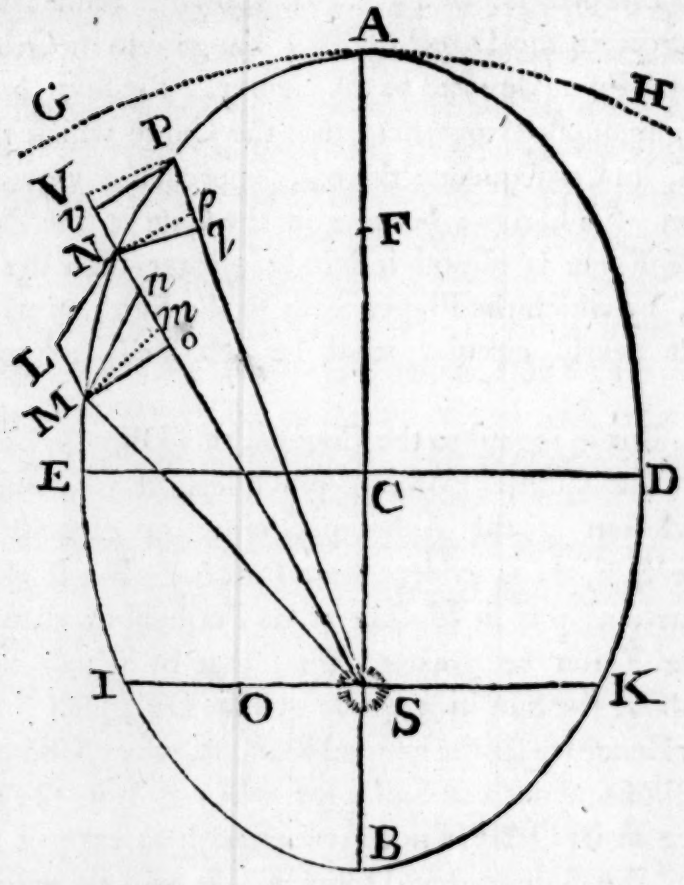
1240. But in regard to the Comets, this Diſproportion of the Projectile and Central Forces is much leſs, if we conſider it, as put in Motion at the Aphelion Point, or greateſt Diſtance from the Sun, as is evident from (1206.) But if we conſider the Comet, as put in Motion at its Perihelion, then will the Projectile Force be greater, than that by which it would be carried about the Sun in a Circle at that Diſtance.

1241. Hence we ſee the general Reaſons, why a Comet revolves in an Ellipſis about the Sun; for when it is in the Aphelion, the Force in its Orbit is not great enough to carry it in a Circle about the Sun at that Diſtance. It will therefore deſcend from that Point towards the Sun with a variable Velocity in its Orbit always increaſing (by 1213,) till at Length it arrives to the Perihelion Point, where its Velocity is greateſt of all.

1242. But, as in this Situation its Projectile Force, or Velocity in its Orbit, is greater than that by which it can deſcribe a Circle about the Sun, it will neceſſarily fly off, and recede from the Sun to greater and greater Diſtances, but with a Velocity always decreaſing, all which is evident, from the variable Ratio of theſe two Forces above and below what is neceſſary to produce a circular Motion.

1243. But farther, it has been ſhewn, that a Body, actuated by a Centripetal Force, which is every where *inverſely, as the Square of the Diſtance*, muſt deſcribe a conic Section about that Body, or Point, ſuppoſed to be placed in the Focus of the Section (1230,) and in all Caſes where the projectile Force is to that which would carry it in a Circle, at the ſame Diſtance in a Ratio leſs than that of $\sqrt{2}$ to 1, the Section will be an *Ellipſis* by (1206.)

1244.



1244. When the projectile Force is nearly equal to that of the Circle at the ſame Diſtance, the Ellipſis will differ but little from a Circle; and this we find is the Caſe of all the *planetary Orbits.* And, on the other Hand, when the Projectile Force compared with that of a Circle ($=1$), is nearly equal to $\sqrt{2}$, then will the Orbit be extremely elliptical, and its extreme Parts, or Apſides will nearly coincide with a *Parabola*, and ſuch are the Orbits we find pertaining to all the *Comets*, and this is the Reaſon why Aſtronomers generally make uſe of the Curve of a Parabola for calculating the Motion, Places, and other Phænomena of Comets, in the perihelion Parts of their Orbits, which admits of greater Eaſe, Expedition, and hardly any ſenſible Error.

1245. But to be more particular, with regard to the Motion of the Comet in every Part of its Orbit, it muſt be remembered, that the *Forces*, *circular Velocities* and *Diſtances*, were all defined by the following Equation $F D v v = f d V V$ (1172.) And ſince the Comet is actuated by the Force of Gravity, which

which every where increaſes in Proportion, as the Square of the Diſtance decreaſes; that is, $F : f :: d^2 : D^2$; this will give $V : v :: \sqrt{d} : \sqrt{D}$, or the circular Velocity will be every where inverſely as the Square Root of the Diſtances; but the Velocity of the Comet in its Orbit is every where inverſely as the Perpendicular to the Tangent.

1246. From hence it will follow, that the Comet deſcends from the Aphelion A, towards the Sun S, becauſe its Velocity is there leſs than the circular Velocity; as it deſcends to leſſer Diſtances P S, its Velocity in its Orbit increaſes in a higher Proportion than the circular Velocity at the Diſtance P S; and this will be the Caſe every where, till the Comet arrives at the loweſt Point, or Perihelion B, where the Proportion of its Velocity to that which it had at A, will be as A S to B S, by (1213,) but the Proportion of the circular Velocities at B and A will be, as $\sqrt{A S}$ to $\sqrt{B S}$. Suppoſe S B = 1, and S A = 4, then will the Velocity of the Comet at B be four Times greater than at A, but the Velocity in the Circle at B will only be twice as great as that in a Circle at A.

1247. Whence it eaſily appears, that the Velocity in the Orbit, getting the better of that in the Circle, will carry the Comet off again from the Sun, when it has attained the loweſt Point B, ſince there it is greater than the Velocity of the Circle at the ſame Diſtance; and, as it recedes from the Sun, the Velocity in the Orbit will decreaſe much faſter than the circular Velocity; the Latter will prevail by Degrees, and cauſe the Comet to deſcribe in its Aſcent a Semi-ellipſe B D A, equal to that in its Deſcent A E B, till at laſt, having attained the higheſt Point A, its Motion is then in the Direction of the Circle A G, but for want of a ſufficient Projectile Force, to continue in that Circle, it will deſcend again from thence towards the Sun as before. Therefore the Velocity in the Circle prevailing in the higher Apſis A, and the Velocity in the Orbit in the lower Apſis B alternately, will occaſion the Comet to deſcribe the ſame Ellipſis perpetually about the Sun. We here ſuppoſe the Planet, or Comet, is not affected by any other Force than that of Gravitation to the Sun.

1248.

1248. To make this Matter ſtill more evident, we may conſider the Proportion of the *Centripetal* and *Centrifugal* Forces. We have ſhewn (1230,) that the Centripetal Force every where encreaſes, as the *Squares of the Diſtances decreaſe*; alſo it has been ſhewn (1174,) that the Centrifugal Force ariſing from the circular Motion about the Sun S, does increaſe in Proportion, as the *Cubes of the Diſtances* decreaſe; ſo that, when the Comet arrives to the Perihelion B, its Gravity is but 16 Times greater than at A; but the Centrifugal Force, or that by which it endeavours to fly off from the Sun, is 64 Times greater than in the Aphelion A, ſuppoſing, as before, that A S be 4 Times greater than B S.

1249. Hence it will follow, that tho' Gravity prevails in the higher Part of the Orbit, the Centrifugal Force (as it increaſes much faſter) will prevail over it in the lower Part, and ſo prevent the Comet from approaching any nearer to the Sun. But at B, as the Comet recedes from the Sun in its Aſcent, this Force will be conſtantly checked by Gravity, which, as it decreaſes in a much lower Proportion than the Centrifugal Force, will, at length, prevail over it at the higheſt Apſis A, and there put a Stop to any farther Receſs from the Sun. Here the Comet again begins to deſcend, by Virtue of a ſuperior Gravity, and ſo alternately deſcends and aſcends, according as the Action of theſe two Powers prevails.

1250. Such are the mechanical Laws and Principles of a *planetary* or *cometary Motion*; and it may be worth while to obſerve, that were the Centripetal Force to be in any other Ratio than that of the Squares of the Diſtances reciprocally, ſuch a regular and beautiful Order could not have obtained in the Syſtem. Thus, for Inſtance, ſuppoſing that the Centripetal Force as the Cubes of the Diſtance inverſely, (or that $F : f :: d^3 : D^3$) then from the foregoing Equation (1245,) we ſhall have $D : d :: v : V$, or the circular Velocities will be in the inverſe Proportion of the Diſtances. But that is the very ſame Proportion that the Velocities in the Orbit have at A and B. And therefore, ſince the Velocities in the Circles and in the Orbit at A and B, vary in the ſame Proportion, it is evident, that the ſame which prevails at one Diſtance, muſt prevail at the other;

others; and therefore, if the Velocity in the Orbit at A be leſs than the circular Velocity, there the Comet will begin to deſcend, and it muſt always continue to deſcend, for the ſame Reaſon that it firſt of all began to do ſo, and conſequently will, after an infinite Number of Revolutions, fall into the Sun. But if, on the other Hand, the Comet be ſuppoſed at B, there the circular Velocity, being greater than that in the Orbit, will carry it off from the Sun; and becauſe it continues always in the ſame Proportion greater, the Comet muſt ever keep riſing in ſpiral Revolutions from the Sun. Therefore, in this Law of Gravity, the preſent Frame of Nature could not in the leaſt exiſt.

1251. The ſame Thing would alſo appear from what we have ſaid of the Centrifugal Force; for as that Force every where increaſes in the reciprocal Ratio of the Cube of the Diſtance, which is the very ſame Ratio as that in which Gravity is ſuppoſed to increaſe, it muſt follow, that, if Gravity once prevail, as in the higher Apſis A, it muſt ever prevail over the Centrifugal Force, and cauſe the revolving Body conſtantly to deſcend in a ſpiral Orbit toward the Sun. But if, on the contrary, the Centrifugal Force prevail in any Point, as at B, then that Force will ever prevail over Gravity, and not only make the Body begin, but cauſe it continually to recede from the Sun.

1252. If the Gravity increaſes in a higher Proportion than as the Cube of the Diſtance decreaſes; then will the circular Velocity increaſe in a higher Proportion than the Diſtances decreaſe, and conſequently, in a higher Proportion than the Velocity in the Orbit increaſes from A to B; ſo that, as the circular Velocity exceeds the Velocity in the Orbit at A, it will much more exceed it at B, and conſequently, the Body will every where continue to deſcend to the Sun with an accelerated Velocity; and the higher the Power of the Diſtance is, to which the Gravity is reciprocally proportional, ſo much the quicker, or in a leſs Number of Revolutions, will the Body deſcend to the Center of Force. On the other Hand, if once the Body recede from the Center, it muſt continue to do ſo for ever.

1253. Again, if Gravity increaſe in the reciprocal Proportion of ſome Power of the Diſtance between the Square and

Cube, the Body will take more than half a Revolution to deſcend from the higher to the lower Apſis; for it takes half a Revolution, when the Gravity is reciprocally as the Square of the Diſtance, and it has no lower Apſis, when it is reciprocally as the Cube of the Diſtance, whence the above Propoſition is evident.

1254. If the Gravity increaſe in Proportion as ſome Power of the Diſtance leſs than the Square decreaſes, the circular Velocities will increaſe in a lower Ratio than that, in which the Velocity in the Orbit increaſes, and conſequently, the latter will more eaſily prevail; alſo the Centrifugal Force will ſooner exceed the Gravity, and therefore the Body will deſcend to the lower Apſis in leſs than half a Revolution, and return to the higher Apſis, in leſs than a complete Revolution.

1255. From all that has been ſaid, it appears, that were the Planets, or Comets, affected in their Motions by one attracting Body only, whoſe Power is reciprocally as the Squares of the Diſtances, then they would deſcribe what one might call a fixed Ellipſis, whoſe Apſides have no Motion at all. But if it happens, that any foreign, attractive Force be added to that of the Sun, ſo as to make the Sum, or Difference of thoſe Gravities, vary in a higher or lower Proportion than that of the Squares of the Diſtances inverſely, it will occaſion the Apſides to move forward or backward, and the elliptic Orbit to become, as it were, moveable. The Excentricity of ſuch an Orbit will alſo be changed, and the periodical Times conſiderably varied; and this is really the Caſe with regard to the primary and ſecondary Planets, and Comets; but particularly in the Caſe of our own Moon, the gravitating Force of the Earth, added to, or ſubſtracted from that of the Sun, makes her Phænomena very variable in all the abovementioned Circumſtances. Alſo the Forces of *Jupiter* and *Saturn* will ſenſibly diſturb the Motions of the Comets, in regard to their Velocity, Aphelion Diſtance, periodical Times, *&c.*

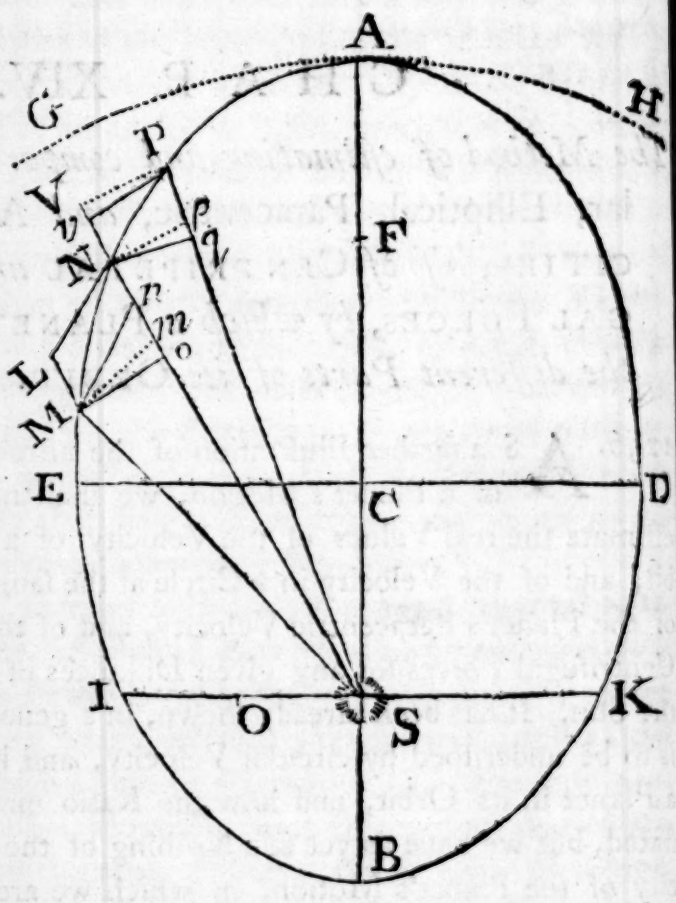
CHAP.

CHAP. XIV.

The Method of eſtimating and comparing the Circular, Elliptical, Paracentric, *and* Angular VELOCITIES; *alſo of* CENTRIPETAL *and* CETRIFUGAL FORCES, *by which a* PLANET *is affected in the different Parts of its* ORBIT.

1256. AS a farther Illuſtration of the aſtronomical Elements of a Planet's Motion, we ſhall next ſhew how to eſtimate the real Values of the Velocity of a Planet in its Orbit, and of the Velocity in a Circle at the ſame Diſtance; alſo, of the Planet's Paracentric Velocity, and of the Centripetal and Centrifugal Forces for any given Diſtances of the Planets from the Sun. It has been already ſhewn, in a general Manner, what is to be underſtood by circular Velocity, and by the Velocity of a Planet in its Orbit, and how the Ratio may at all Times be ſtated, but we have as yet ſaid Nothing of the *Paracentric* Velocity of the Planet's Motion, by which we are to underſtand its Acceſs to, or Receſs from the Sun, eſtimated in a right Line, joining the Planet and the Sun. Thus, let a Planet be in the three Points of its Orbit, P, N, M, in its Deſcent towards the Sun S, and join S P, S N, and S M; on the Center S, with the Diſtance S N, deſcribe the Arch N *p*, and with the Diſtance S M, deſcribe the Arch M *m*, then it is evident, when the Planet comes to N, it will be nearer the Sun than it was at P, by the Diſtance P *p*; and at M it is nearer the Sun than at N, by the Diſtance N *m*; now theſe two Diſtances, or Lines, P *p* and N *m*, are called the Paracentric Velocities of the Planet's Motion in thoſe Parts of its Orbit.

1257. On the Center S, deſcribe the Arch P V, and from P let fall the Perpendicular P *v*; alſo, from N and M draw the Perpendiculars N *q*, M *o*. Laſtly, draw M *n* parallel to the Tangent L P, and L M parallel to S N, then V *v*, or *p q* denotes the Planet's Centrifugal Force at the Points P or N; alſo, *m o* denotes the ſame Thing at M: And L M is the Expreſſion of the Centripetal Force, as is evident from what we have ſaid in the XIIth Chapter.



1258. From the Points P and M, draw the Lines P v, M o, perpendicular to S N; then becauſe the Triangles P S N, N S M are equal, (the Times being ſuppoſed equal) therefore (becauſe the Baſe S N is common to both) the Altitudes P v, M o are equal; take N n = L M, then the Triangles P N v, M n o will be equal and ſimilar, and P N = M n and N v = o n; again, in the right Line S N (produced) ſince S V = S P, and S m = S M, we have N V = S P — S N, and N m = S N — S M; and conſequently N V = (N v) n o + V v, and N m = N n + n o — o m; therefore N V — N m = V v + m o — N n.

1259. If now we put S P = y, then P p = $\dot{y}$, and N V (or P p) — N m = $\ddot{y}$; then becauſe V v = p q = m o, therefore $\ddot{y}$ = 2 m o — N n, or Fluxion of the Paracentric Velocity. Now it is evident, that while the Paracentric Velocity $\dot{y}$ increaſes, its Fluxion $\ddot{y}$ will be Negative or — $\ddot{y}$ = N n — 2 m o (932) till at Length it becomes Nothing, or $\ddot{y}$ = 2 m o — N n = o, in which Caſe 2 m o = N n. After this, the Paracentric Velocity $\dot{y}$ decreaſes, and its Fluxion is affirmative, or + $\ddot{y}$ = 2 m o — N n; till the Planet arrives at B, where it entirely vaniſhes. From hence we learn

1260. Firſt, that in the Deſcent of the Planet from its Aphelion Diſtance A, toward the Sun, its Paracentric Velocity, begins and increaſes till it arrives at a certain Point; after which it de-

decreaſes continually, till it vaniſhes in the Aphelion Point B, or its Velocity of Acceſs to the Sun is accelerated in the firſt Part of its Orbit, and retarded in the laſt; and *vice verſâ* in regard to its Receſs in the other Part of its Orbit.

1261. Secondly, the Fluxion of the Paracentric Velocity $-\dot{y} = N n - 2 m o$ ſhews, that ſo long as the ſaid Velocity continues to increaſe, or be accelerated, *Twice the Centrifugal Force* ($2 m o$) *will be leſs than the Centripetal Force* $N n$.

1262. Thirdly, when the Paracentric Velocity is a *Maximum*, or greateſt of all, *the Centripetal Force* ($N n$) *is equal to twice the Centrifugal Force* ($m o$.)

1263. Fourthly, from the Time the Paracentric Velocity continues to decreaſe, till it vaniſheth at B, *the Centripetal Force, or Gravity* ($N n$,) *will be leſs than twice the Centrifugal Force* ($m o$.) But that Gravity $N n$ can never decreaſe ſo far, as to become equal to the Centripetal Force $m o$, even in the Perihelion Point B, is what we ſhall ſhew by and by.

1264. Let the Triangles P S N and M S N be equal, or deſcribed in equal Times; then will $S N \times N p = S M \times M m$, then it will be $N p : M m :: S M : S N$. Let I K be the *Latus Rectum*, or Parameter of the elliptic Orbit, and $L \times a$ be a conſtant Rectangle, equal to $S M \times M m$. Put $S M = D$, and $I K = L$. Then the Arch $M m = \frac{L a}{D}$, and $M m^2 = \frac{L^2 a^2}{D^2}$, and $m o = \frac{M m^2}{2 S M}$ (1171) $= \frac{a^2 L^2}{2 D^3}$ = Centrif. Force.

1265. Again, the Centripetal Force, or Gravity L M, is inverſely as D^2, or directly, as $M m^2$, (1264,) or as $\frac{a^2 L^2}{D^2}$; that is, (dividing by the conſtant Quantity $\frac{1}{2} L$,) $\frac{2 L a^2}{D^2}$, is as the Force of Gravity. Wherefore Gravity is to the Centrifugal Force, as $\frac{2 L a^2}{D^2}$, to $\frac{L^2 a^2}{2 D^3}$, or as D to $\frac{1}{4} L$. That is, *The Force of Gravity is to the Centrifugal Force every where, as the Diſtance of a Planet to a fourth Part of the* Latus Rectum *of the Ellipſis.*

1266. Let $I O = \frac{1}{4} I S = \frac{1}{4} L$, then in the Point I, Gravity is to the Centrifugal Force, as I S is to I O, or as 2 to 1, and conſequently, the Paracentric Velocity in the Point I, will be the greateſt of all, or a Maximum (1262.) 1267.

1267. That the Centrifugal Force is always leſs than Gravity, will be evident, when we conſider, that the Perihelion Diſtance S B, does, in an Ellipſis, always exceed S O, or $\frac{1}{4}$ of the Parameter, becauſe in a *Parabola*, which lies without the Ellipſis, the ſaid Line S B is then but juſt equal to $\frac{1}{4}$ of that larger Parameter (742.)

1268. Hence it will again appear, that a Planet in the Aphelion Point A, will deſcribe a Circle A G, when the Force of Gravity is there equal to twice the Centrifugal Force. But if it be greater, the Planet will deſcend in an Ellipſis towards the Sun, and in the loweſt Point B, Gravity, being leſs than double the Centrifugal Force, can carry the Planet no nearer to the Sun from that Point; therefore, it muſt of Courſe begin to aſcend with an increaſing Paracentic Velocity of Receſs, till it arrives to the Point K, where Gravity becomes equal again to twice the Centrifugal Force. After this, the Centrifugal Forces leſſening much faſter than Gravity, the Latter will prevail, and the Paracentric Velocity of receding from the Sun will conſequently decreaſe, till the Planet arrives to its Aphelion A, where it will entirely vaniſh.

1269. And thus we ſee, more particularly now, by the Difference of theſe two Forces, how the Planet is made conſtantly to revolve to and from the Sun, and in ſo conſtant and regular an Order, as to give us the cleareſt Ideas of a Uniformly Variable, and Perpetual Motion.

1270. Beſides the Velocity of a Planet's Motion hitherto mentioned, there is one other, which is called the *Angular Velocity* of a Planet in its Orbit. In order to eſtimate this, it muſt be conſidered, that any Angle is greater in Proportion, as the Arch deſcribed with a given Radius is ſo. And alſo, when the Arch is given, the Angle will be leſs in Proportion, as the Radius is greater; and therefore every Angle will be in a Ratio compounded of the direct Ratio of the Arch, and reciprocal Ratio of the Radius, and farther we have juſt now ſhewn (1264,) that in the Caſe of deſcribing equal Areas, the Arches are inverſely as the Radii; therefore, in this Caſe of a Planet's Motion, *the Angles deſcribed in equal Times will be inverſely as the Squares of the* Radii.

1271. But it has been ſhewn, that the Force of Gravity is

every

every where inverſely as the Radius, and conſequently, directly as the angular Motion of a Planet in its Orbit; and therefore the Latter will be an adequate Meaſure of the Former.

1272. Therefore the *Impetus*, or Sum Total of all the Impreſſions of Gravity, which the Planet acquires in moving from A to P, is to the Impetus acquired at M, as the Angle A S P is to the Angle A S M. Hence likewiſe it appears, that the Impetus acquired in deſcending from A to I, is juſt Half that which is acquired in deſcending from A to B; and therefore the Impreſſions of Gravity upon the Planet, as it paſſes from I to B, are equal to all it receives before, in its Paſſage from A to I.

1273. Having thus ſtated the Ratios of the Velocities and Forces concerned in a Planet's Motion, we ſhall next proceed to illuſtrate the ſame by a familiar Inſtance, where all thoſe Quantities will be expreſſed in proper Numbers, and therefore be more eaſily comprehended and underſtood. In order to this, let A S repreſent the Diſtance from the Center of the Earth to the Circumference H A G, a Circle on the Surface of the Earth; it was ſhewn, that a Body revolving in this Circle (1190) with a Centrifugal Force, equal to that of Gravity, muſt be projected from A with a Velocity of 26000 Feet *per* Second.

1274. Now, ſuppoſe it was required to find the Velocity, with which a Body ſhould be projected from the ſaid Point A, to deſcribe the Ellipſes A E B D; ſo that S B may be equal to 1000 Miles. Then as A S is equal to 4000 Miles, the whole tranſverſe Diameter A B will be equal to 5000, and E D will be equal to 4000; and I K will be 3200; ſuch are the Dimenſions of the Ellipſis. Then by the Theory in (1204,) $n = \sqrt{\frac{b\,b}{a\,d}} = \sqrt{\frac{2000 \times 2000}{4000 \times 2500}} = 0{,}63246.$

Therefore 1 : V : : n : v, or 1 : 26000 : : 0,63246 : 16444 Feet *per* Second, the Velocity required.

1275. Thus, the Velocity in the Circle and the Ellipſis, at the Point A, is known, and the Velocity at B in the Circle is to the Velocity at A in the Circle as $\sqrt{A\,S}$ to $\sqrt{B\,S}$, or as 2 to 1, that is, the Circular Velocity at B will be at the Rate of 52000 Miles *per* Hour (1245.)

1276. But the Velocity in the Orbit at B will be to that by

which it was projected at A, in Proportion as A S to S B, or as 4 to 1. Therefore the Velocity of the Planet in B will be $16444 \times 4 = 65776$ Feet per Second, which, as it greatly exceeds the Velocity in the Circle, will prevent the Planet from revolving in a Circle about the Sun, and carry it off in its own proper Orbit, in the Manner before-mentioned (1247.)

1277. Then, as to the Forces, that in the Circle at A is equal to Gravity, which ſuppoſe to be 100 Pound Weight, and we have ſhewn, that Gravity is every where to the Centripetal Force of revolving Bodies, as the Diſtance to $\frac{1}{4}$ of the *Parameter*, that is, in the Point A, it will be as A S to S O, or as 40 to 8, or 5 to 1. Therefore the Centrifugal Force of the Body in A is but 20 Pound.

1278. In the Point B, Gravity is 16 Times greater than at A, or equal to 1600 Pound Weight, and ſince there the Gravity is to the Centrifugal Force as S B to S O, or as 10 to 8, the Centrifugal Force at B will be equal to 1280 Pounds, which tho' it be conſiderably leſs than Gravity, will yet prevent any nearer Approach of the Planet to the Sun.

1279. Laſtly, it appears, that the angular Velocity of the Planet at A, is 16 Times leſs than that at B, or to an Eye placed at S, the Space deſcribed in one Second at A will appear 16 Times leſs than that which is deſcribed in the ſame Time at B, agreeable to (1270.) Since A S = 4 S B, (1274.)

1280. Thus we have applied the Phyſico-Mathematical Principles of Motion to the Theory of Aſtronomy, as far as it can be done without the Aſſiſtance of OPTICS; but, as the greateſt Part of Aſtronomy, both theoretical and practical, depends entirely upon optical Principles, nor can by any Means be underſtood without them, it will be neceſſary here to deſiſt, and proceed to the Elements of the Science of Viſion; nor will the Principles of common Optics be ſufficient to anſwer our Purpoſe, with regard to finiſhing a complete Treatiſe of Aſtronomy. The Doctrine of PERSPECTIVE muſt be well underſtood, as alſo the general Principles of the *Projection* of the SPHERE in *Plano*, and that too in a different Manner from that in which they have uſually been treated, theſe will all be found neceſſary in the various Branches of that Science. We ſhall therefore, in the next Place, proceed to lay down a Series of Inſtitutions, containing the Principles of *univerſal Optics*.

INSTI-

INSTITUTIONS OF Univerſal OPTICS:

CONTAINING

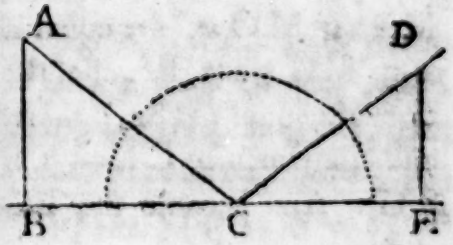
The ELEMENTS of CATOPTRICS, DIOPTRICS, and PERSPECTIVE.

CHAP. I.

The Phyſico-Mathematical ELEMENTS *of* CATOPTRICS, *or* VISION, *by* reflected LIGHT, *from all Sorts of poliſhed* SPECULUMS, *or* MIRRORS.

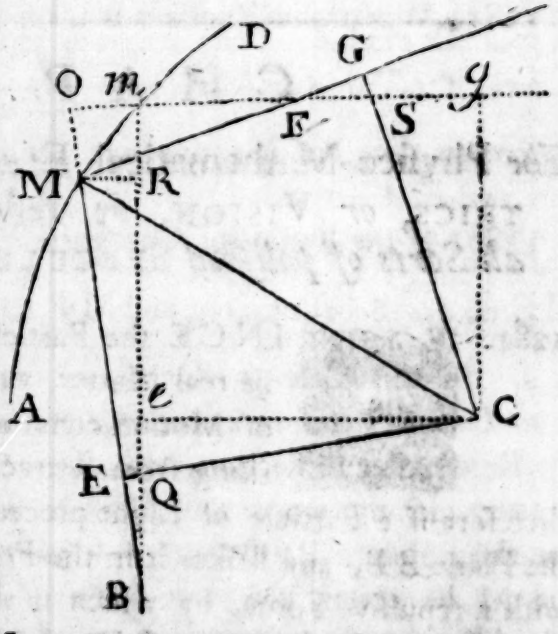
1281. SINCE the Particles of Light are found to be real Matter, they will obſerve the Laws of Motion common to all other Bodies ariſing from Attraction and Repulſion; and therefore if a Particle of Light proceeds from the Point A to the Plane B E, and ſtrikes it in the Point C, it will there meet with a repulſive Force, by which it will be reflected from the ſaid Plane in the Direction C D, making therewith the Angle E C D. Now it is required, to find the Point C in the Plane B E, ſuch, that the Ray of Light impinging thereon ſhall be reflected to the Point D, ſo that its Paſſage from the given Point A to another given Point D ſhall be the leaſt poſſible.

1282. Let the Perpendicular A B $= a$, and the Perpendicular D E $= b$; B E $= c$, and B C $= x$. Then C E $= c - x$, and A C $= \sqrt{aa + xx}$; alſo C D $= \sqrt{bb + cc - 2cx + xx}$.



Now, since AC + CD, that is, $\sqrt{aa+xx}+\sqrt{bb\times cc-2cx+xx}$ must be a Minimum, and therefore its Fluxion $\frac{x\dot{x}}{\sqrt{aa+xx}}+\frac{x\dot{x}-c\dot{x}}{\sqrt{bb+cc-2cx+xx}}=0$, consequently $x+\sqrt{bb+cc-2cx+xx}\times\overline{x-c}+\sqrt{aa+xx}=0$. Therefore $x\times\sqrt{bb+cc-2cx+x^2}=\overline{c-x}\sqrt{aa+xx}$, or BC + CD = CE × AC, and so BC : AC :: CE : CD. Hence the Triangles ACB and DCE are equiangular (657;) *and so the Angle of Incidence* ACB = DCE, *the Angle of Reflection*; and that this is really the Case in regard to the Reflection of Light, we are well assured from Experiments.

1283. The Nature of the Curve AMD, the Distance of the luminous Point B, and the Position of the incident Ray BM being given, it is required to find in the reflected Ray MF, the Point, or Focus F, where all the Rays issuing from the Point B will be united. In order to this, let CM be the Radius of Curvature to the Point of Incidence M, and take the Arch M*m* infinitely small, and draw the right Lines B *m*, *m*F; on the Centers B and F describe the little Arches MR, MO; and draw the Perpendiculars CE, C*e*, GG, C*g*, to the Rays of Incidence and Reflection; and suppose the Distance BM = *d* and ME or MG = *a*; then 'tis evident, that the Triangles MR *m*, MO *m*, are equal and similar, and consequently MR is = MO; and because the Angles of Incidence and Reflection are equal (1282,) therefore CE = CG, and C*e* = C*g*, and consequently CE — C*e*, or EQ is = to CG — C*g*, or SG: And because the Triangles BMR, BEQ, FMO,



FGS

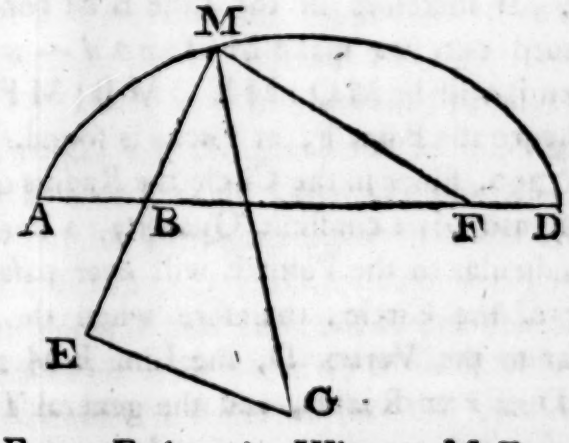
FGS are similar, it is, BM + BE $(2d-a)$: BM (d) :: MR + EQ, or MO + GS : MR, or MO :: MG (a) : MF $= \frac{ad}{2d-a}$.

1284. If the radiant Point B fall on the other Side of the Point E in respect of M, or (which is the same Thing) if the Curve AMD be *convex* towards the Radiant B, then d will be negative, or $-d$; and therefore MF $= \frac{-ad}{-2d-a} = \frac{ad}{2d+a}$. Hence in this Case, the Focus F will always be positive, or on the same Side the Curve with the Point C; and the Rays after Reflection will diverge.

1285. When the Radiant B is on the concave Side of the Curve, the Value of MF $\left(= \frac{ad}{2d-a}\right)$ will be positive when d exceeds $\frac{1}{2}a$; but negative when d is less than $\frac{1}{2}a$; and infinite, when $d = \frac{1}{2}a$.

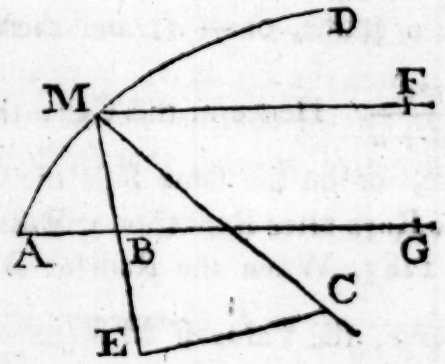
1286. If the Radius of Curvature MC be infinite, then also ME $= a$ will be infinite, and MF $= \frac{ad}{2d \mp a}$ will become $\mp d$. In this Case the Curve, or small Arch M m becomes a *strait Line*. Therefore in both Cases, when it is $-d$, or $+d$, the Rays, after Reflection, will diverge.

1287. If any two of the three Points B, C and M be given, the Third may be found. Thus, if the Curve AMD be an ELLIPSIS, and the Radiant Point B be in one Focus, 'tis evident all the reflected Rays MF will be united in the other Focus F (772.) Whence MF $= \frac{ad}{2d-a} = f$, and $a = \frac{2df}{d+f}$; hence $a \times \frac{d+f}{2} = df$; and therefore $d : a :: \frac{d+f}{2} : f$; that is, BM : ME :: $\frac{1}{2}$ AD : MF.

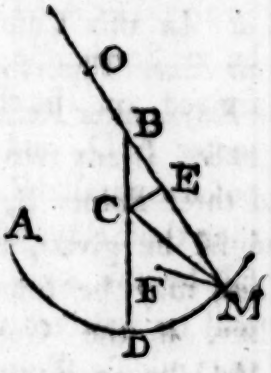


M F. Hence likewiſe, when the Points B and M are given, the Point E or C may be found, for then $a = \frac{2df}{d+f}$, or $\frac{1}{2}$ A D : M F :: B M : M E; and a Perpendicular on the Point E gives the Point C in a right Line M C, biſecting the Angle B M F. *Note*, The ſame Demonſtration ſerves for the *Hyperbola*, where $a = \frac{2df}{d-f}$.

1288. In the PARABOLA A M D, the Focus F is removed to an infinite Diſtance; and the Radiant being in the Focus B as before, 'tis evident the reflected Rays M F will be parallel to the Axis A G; and therefore ſince M F $= f = \frac{ad}{2d-a}$ is infinite, we have $2d = a$, or 2 B M = M E. Therefore a Perpendicular E C erected on the Point E will aſſign the Point C for the Center of Curvature to the Point M, in the Perpendicular M C.

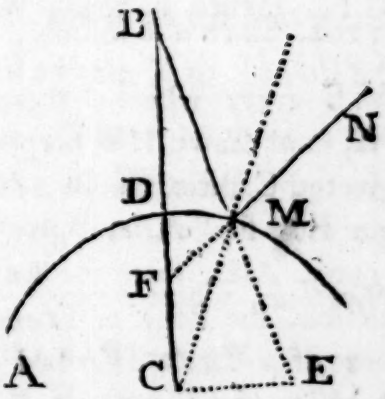


1289. If A M D be a CIRCLE, ſince $f = \frac{ad}{2d-a}$, we have $2d - a : a :: d : f$. If therefore in the Line B M continued out, we make M O $= 2d - a$; then it will be M O : M E :: M B : M F, whence the Point F, or Focus is found.



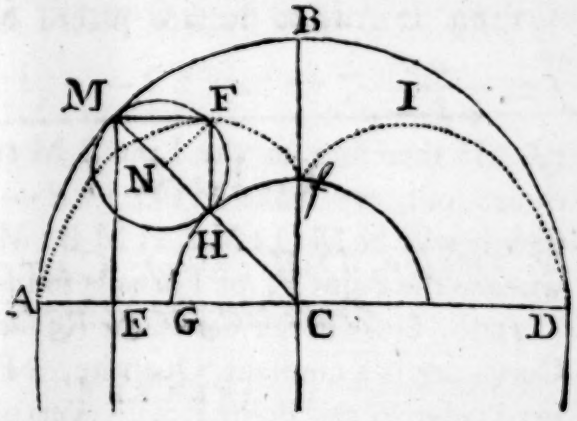
1290. Since in the Circle the Radius of Curvature is a conſtant Quantity, a Perpendicular to the Point E will ever paſs through C the Center of the Circle; therefore when the Point M is infinitely near to the Vertex D, the Line E M $= a$, will be equal to C D $= r$ = Radius, and the general Theorem for finding the Focus F, *viz.* M F $= \frac{ad}{2d-a}$, will become $\frac{dr}{2d-r} = f$, in this particular Caſe; and the Focus F will be in the Axis of the Curve.

1291. If the Radiant B be at an infinite Diſtance, then d being infinite, gives $\frac{dr}{2d-r} = \frac{1}{2}$ $r = f$, or the Focus is then equal to Half the Radius. If $2d$ be greater than r, the Focus f is *poſitive*, or on the ſame Side with the Radiant. If $2d$ be leſs than r, then the Focus will be *negative*, or on the contrary Side, in regard to the Radiant. If d be negative or $-d$, that is, if the Radiant be placed on the convex Side of the Curve, the Theorem becomes $\frac{-dr}{-2d-r} = \frac{dr}{2d+r} = f$, always poſitive, as before (1284.)



1292. Since $dr = 2df + rf$, and conſequently $df + rf = dr - df$, we have $d + r : d :: r - f : f$; that is, B C : B D : : C F : D F. And therefore the Axis of the Curve, or Line B C is *harmonically* divided in the Points C, F, D, B, as will appear, when we treat of *Harmonical Proportions*.

1293. Let A M B D be a Semicircle deſcribed on the Diameter A D, and expoſed to parallel Rays; then thoſe Rays which fall by the Axis C B will be reflected to f, the middle Point of C B (1291,) and thoſe which fall at A, as they touch the Curve only, will not be reflected at all; and any intermediate Ray E M will be reflected to a Point F, ſomewhere between A and f; and alſo, ſince every different incident Ray will have a different focal Point, therefore thoſe various focal Points will conſtitute a Curve-line A F f in one Quadrant, and f I D in the other, which Curve is called the *Cauſtic by Reflection*.



1294.

1294. Since d is infinite, we have $\frac{d\,a}{2\,d - a} = MF = \frac{1}{2}a = \frac{1}{2}ME$ every where; therefore if we biſect the Radius CM in H, and drawn HF perpendicular to MF, the Point F will be in the Cauſtic Curve; for the Triangles MFH and MEC are always ſimilar, and give MH : MC :: MF : ME.

1295. Alſo, ſince the Angle MFH is a right one by Conſtruction, the Point or Focus F will ever be in the Circumference of a Circle deſcribed on the Diameter $MH = \frac{1}{2}MC$. Therefore the Cauſtic AFf is a *Semi-epicycloid*, deſcribed by the Revolution of a Circle MFH, on the Periphery of a Circle GHf about the Center C, whoſe Radius $CH = MH$, the Diameter of the generating Circle.

1296. If the Angle ACM be Half a right one, than becauſe the Angle $EMC = CMF = MCE$, the reflected Ray MF will be parallel to AC, and will therefore touch the Cauſtic AFf in the higheſt Point F.

1297. This Theory, with reſpect to the *Cauſtic* by Reflection, is moſt evidently confirmed by Experiment; for if a cylinder Bowl, or Glaſs, be expoſed to the Sun-beams, or Candle-light, this Curve AFfID will appear very ſtrongly delineated on any white Surface placed horizontally within the ſame.

CHAP. II.

The popular Doctrine of CATOPTRICS, *deduced from the foregoing Theory.*

(PLATE I.)

1298. ACCORDING to the different Modification of the Rays of Light, they receive a threefold Diſtinction. (1.) They are ſaid to be *parallel* when they proceed in Directions equidiſtant, or parallel to each other; as at A in Fig. 1. (2.) *Converging Rays* are ſuch as tend to *one Point* F, as thoſe at B. (3.) *Diverging Rays* are ſuch as proceed from a Point F, in different Directions, as repreſented at C.

1299.

1299. With regard to reflecting Surfaces, uſually called SPECULUMS, or MIRRORS, there are likewiſe three different Forms, *viz.* (1.) A *Plane Speculum*, or Looking-glaſs, as A B in Fig. 2. (2.) *Concave Speculum*, which is the Segment of a hollow Sphere, foliated on the Outſide A V B, or poliſhed on the Inſide, as Fig. 3. (3.) *Convex Speculum*, the ſame Segment of a Sphere, but foliated on the Inſide A C B, or poliſhed on the Outſide, as in Fig. 4.

1300. If we conſider a ſingle Ray of Light D C, falling on theſe three Surfaces, as it reſpects but one ſingle Point C in each of them, ſo the Law of Reflection will be the ſame in all, *viz.* *That the Angle of Incidence* D C E *is equal to the Angle of Reflection* E C F (1282,) E C being ſuppoſed perpendicular to the ſeveral Mirrors in the Point C.

1301. Let A V B (Fig. 5.) be a *Concave Mirror*, E the Center, V its Vertex, and V G the Axis. Then let G be a radiant Point taken any where in the Axis, from whence a Ray of Light G C proceeds to any Point C very near to the Vertex V, and draw E C, the Perpendicular to the Point C, and make the Angle E C f = G C E, and C f will be the reflected Ray, meeting the Axis in f, which is called the *proper Focus*, or that which reſpects the Diſtance G V only. (See 1283.)

1302. All parallel Rays, D C falling on the Part C V, extremely near the Vertex V, will be reflected to a Point F, ſo that D C E the Angle of Incidence be equal to E C F the Angle of Reflection; which Point, or Focus F is the middle Point between E and V, or V F = $\frac{1}{2}$ Radius V E (1291.)

1303. And ſince this is the Caſe, with regard to the Sun-beams, which, by ſuch a Mirror are all reflected or converged to the Point F, that Point is called the *Solar Focus*, or *Focus of parallel Rays*; and is relative to ſuch Objects only as are at a very great or infinite Diſtance.

1304. Again, as the Point F is that where all the Rays of the Sun, falling on the Speculum, are collected into a very ſmall Space, they will be there greatly condenſed, and their Action on Bodies, with reſpect to Light and Heat, ſo very much encreaſed, as to produce *accenſion*, or burning of any combuſtible Body placed in that Point F, whence it is called

called the FOCUS, or *Burning Place*, and all ſuch *Speculums* are called BURNING GLASSES.

1305. What we have ſaid of a ſingle Ray holds good for any Number, or Quantity of Rays iſſuing from a given Point; and hence it will follow, *that all the Rays which flow from any particular Point of an Object on a reflecting Speculum*, will all be *converged to one Point nearly*, or made to *diverge from one Point*; and that Point, therefore, will be a Repreſentation of the ſaid Point in the Object, and conſequently, ſince every Point in the Object may eaſily be conceived to be thus form'd in the *Focus* of the Speculum, the whole Object will be there repreſented, formed, or depicted in *Imagery*; or there will be an IMAGE formed in the Focus of every diſtant OBJECT to which it is expoſed.

1306. To explain this Matter more particularly, let O B be any Object placed before any Speculum C V D, (Fig. 13.) at the Diſtance A V in the Axis; let E be the Center of the Mirror, through which, from each extreme Part of the Object O and B, draw the Lines, or Rays O E D and B E C to the Mirror, and as they paſs through the Center E, they will be perpendicular to the Surface in the Points D and C. Alſo from each Point O and B draw the Rays O V, B V, to the Vertex of the Mirror V. Laſtly, join O C and B D.

1307. Now it is evident, that ſince the incident Ray O V on the Vertex on one Side the Axis A V, makes the ſame Angle O V A, as the reflected Ray V M does on the other, therefore the reſpective Focus M will be on the contrary Side of the Axis from the radiant Point O. And the ſame is to be obſerved with regard to the other extreme Point B, and its Focus I. *Therefore the Poſition of the Image* I M *before the Concave Mirror is inverted with reſpect to that of the Object* O B.

1308. The Points O, A, B, in the Object being repreſented by M, *a*, I, in the Image, it will be found by Computation, *that the Form of the Image is curvelineal*, when the Speculum C D is large, tho' very little ſo, when it is ſmall in Diameter. The Rule for finding the focal Diſtances M D, *a* V, I C, for the reſpective Diſtances of the Radiant O D, A V, B C is this. *Multiply the Diſtance of the Radiant by the Radius of the Speculum, and divide that Product by the Difference between twice*

the

the ſaid Diſtance and the Radius; the Quotient will will be the Focal Diſtance required.

1309. *The Object and Image ſubtend the ſame Angle at the Vertex* V *and Centre* E *of the Speculum.* For at the Vertex they are both ſeen under the ſame Angle O V B; and at the Centre of the Angle which the Object ſubtends O E B = I E M the Angle ſubtended by the Image.

1310. *The Lineal Dimenſions, or Magnitude of the Object and Image, are as their Diſtances from the Speculum.* For O B: I M:: A V: *a* V.

1311. Therefore when the Diſtance of the Object is equal to the Radius, (*viz.* when it is placed in the Centre E) then the Image there meets it, *and is equal to it.*

1312. When the Diſtance of the Object exceeds the Radius E V, then will that of the Image be leſs; *and the Image in all ſuch Caſes will be leſs than the Object.*

1313. On the contrary, when the Diſtance of the Object is leſs than the Radius, that of the Image will be greater; *and the Image will be in Proportion larger than the Object.*

1314. Thus ſuppoſe I M a ſmall Object placed at (*a*) between the *Center and Solar Focus* (1291) then will O B be *its enlarged or magnified Image*; and this is the Caſe and Structure of what is properly called a REFLECTING MICROSCOPE, by a ſmall Speculum C V D.

1315. If C V D (Fig. 14.) repreſent the ſame Concave Speculum and E its Center, then if I M be any Object placed nearer to it than the Solar Focus (or half V E,) then by the ſame Reaſoning we ſhall have O B to repreſent *its enlarged and magnified Image on the other Side of the Speculum* (1291); which Image is in this Caſe *erect*, or in the ſame Poſition with the Object. And thus it is, *that all large Concaves* become MAGNIFYING MIRRORS.

1316. Thus it appears that a *Concave Speculum* has a *poſitive and a negative Focus*, and will *magnify or enlarge the Appearance of an Object* in either. And alſo, that it will *diminiſh Objects* in the poſitive Focus only.

1317. If C V D be a *Convex Speculum* (Fig. 14) then any Object O B placed before it will have a *Virtual Focus* only, or

the Rays will be ſo reflected from it as if they came diverging from a Point behind it (1291) thus the Ray OV will be ſo reflected from V to B as if it came from the Point M, and the Ray BV will be reflected diverging from the Point I, and the ſame may be ſaid of all other Rays from the Points O and B; therefore MI will be the Image of the Object O B.

1318. With reſpect to this Image we obſerve (1.) That it is always on the contrary Side of the Glaſs from the Object. (2.) That it is always erect. (3.) That it is ever leſs than the Object, the Proportion being that of their Diſtances IV to V O from the Vertex, as before. Hence a *Convex Mirror*, when large, will exhibit a *delightful Landſcape of diſtant Objects*, which is its principal Uſe.

1319. As to a *Plain Mirror or Common* LOOKING-GLASS, it appears from the Theory (1286). (1.) That the *Focus is always Negative*, or behind the Glaſs. (2.) That the Diſtance of the Image behind is equal to the Diſtance of the Object before the Glaſs. (3.) That it is erect, and ſimilarly ſituated with the Object. (4.) That it is of *equal Magnitude* with the Object. (5.) Therefore at the Diſtance of the Object, its Image on the Surface of the Glaſs will appear but of half that Length, and conſequently a Perſon of ſix Feet Height will require a Glaſs 3 Feet long to view himſelf compleatly.

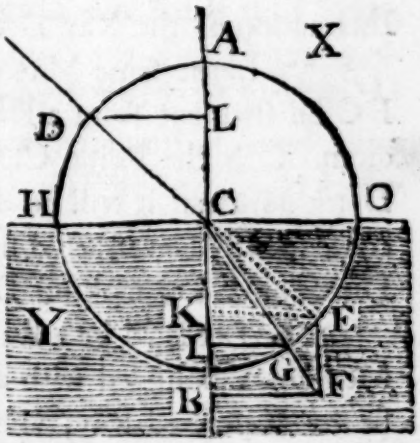
CHAP. III.

The Phyſico-Mathematical ELEMENTS *of* DIOPTRICS, *or* VISION *by* LIGHT *refracted thro' different Mediums, particularly adapted to* LENSES *for* OPTICAL USES.

1320. WE are taught to underſtand, by Sir J. NEWTON, that there is a *reflecting and refracting Power* which acts near the Surface of every Medium, or Body, in ſuch Manner as to reflect the Rays of Light at one Inſtant, and at another to tranſmit and refract them thro' the Subſtance of the Medium.

And

And tho' the *Modus Agendi*, or particular Action of the Power be not ſo accurately aſcertained, yet that Author has made it moſt certain by Experiments, that ſuch a Mode of Action there is, and that its Effects on the Particles of Light are the *Inflection*, *Reflection*, and *Tranſmiſſion* thereof in and thro' different *Media*. And the particular Modifications of Light from thence ariſing, he calls *Fits of eaſy Reflection*, and *Fits of eaſy Tranſmiſſion*.



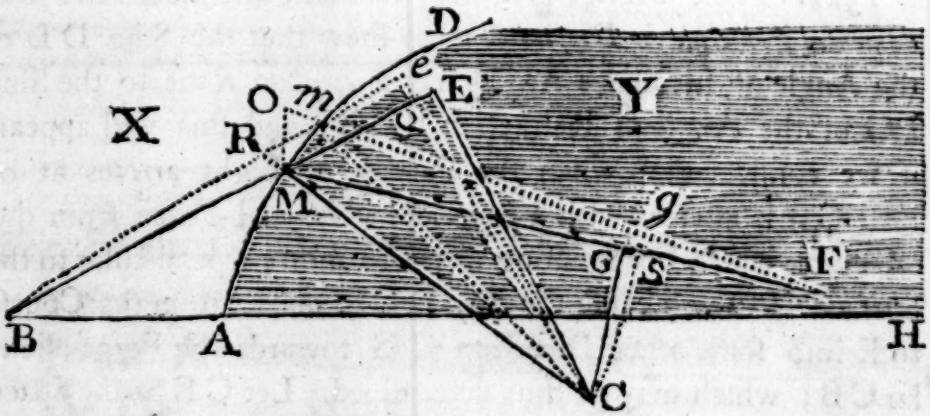
1321. Then admit D C were a Ray of Light incident in one Medium X, upon another Y of a different Denſity and refractive Power, in the Point C; and ſuppoſe it there in a *Fit of eaſy Tranſmiſſion*, then if Y be the Denſer Medium, or has the greateſt Refractive Power, the Ray, by the Action of this Power, will be bent or refracted from its firſt Direction D C E into another C F, ſo as to make the refracted Angle F C B of a leſs Quantity than the Angle of Incidence D C A. The Line AB being ſuppoſed perpendicular to the Surface H O of the Medium Y in the Point C.

1322. Theſe Things premiſed, the next, and indeed the *fundamental Principle in Dioptrics*, is to ſhew that the Sine D L of the Angle of Incidence A C D has a *conſtant Ratio* to the Sine I G of the Angle of Refraction B C G; and this will appear, if we conſider that when the Particle of Light arrives at or near C it is affected with a new and additional Force from the Medium, which Force acting in a Direction perpendicular to the Surface H O will cauſe the Ray D C to deflect from its Courſe to E into ſome other Direction C G towards the Perpendicular C B; which may be thus determined. Let C E be the Space deſcribed in a given Time by the *Uniform Velocity* of Light in the Medium X; then will the ſaid Line C E be as the Force which produces that Velocity, (998). From the Point E draw E F parallel to C B, and let E F repreſent the new acquired Force of the Medium upon the Particle in the Point C, and join C F, then will that be the *new Direction of the Ray*

and the Space deſcribed in the ſame Time thro' the Medium Y, reſulting from the two Forces C E and E F (1028). But C F is to C G (or C E) as B F (or K E) is to I G. That is, *the Sine of Incidence* K E *or* D L *is to the Sine of Refraction* I G *as the Velocity* C F, *in the Medium* Y, *to the Velocity of the Ray* C E *or* D C, *in the Medium* X. But theſe Velocities of Light in different Media are as conſtant as the Powers of Nature which produce them; therefore ſo is the Ratio of the Sines D L to I G in every Inclination of the Ray D C.

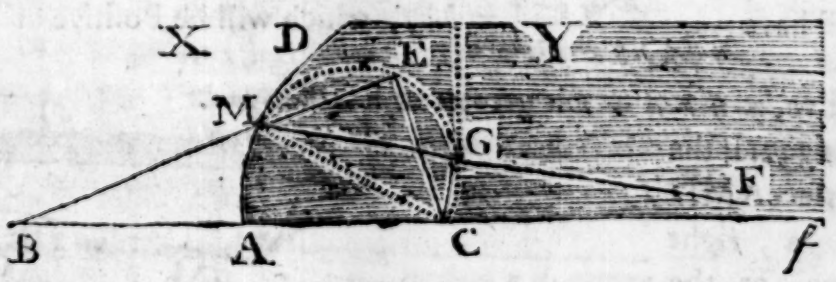
1323. In the ſame Manner it is ſhewn, that if a Ray of Light F C be incident from a Denſe Medium Y upon a Rarer Medium X in the Point C, then by the ſuperior Force of the Denſe Medium it will be deflected towards its Surface H C, and conſequently be refracted from the Perpendicular A C into the Direction C D, making D L to I G as F C to C D, as before.

1324. Let B M be a Ray of Light incident on a Refracting Medium Y bounded with a curved Surface A M D, and let M F be the Refracted Ray, and F the Focus to which all the Rays falling on or near the Point M will be refracted. It is required to find the Point F by having given the Nature of the Curve A M D, the Radiant Point B, the Ray B M, and the Refraction of the Mediums X and Y.



1325. In order to this, let M C be the Radius of Curvature to the Point M, and draw B *m* infinitely near B M, and join *m* F and *m* C; from the point C let fall the Perperdiculars C E, C *e*, on the Incident Rays continued; and C G, C *g*, on the Refracted Rays; and on the Centers B and F deſcribe the ſmall Arches M R, M O. And put B M = *d*, M E = *a*, M G = *b*; the

the little Arch MR $= \dot{x}$, and MO $= \dot{y}$, and the Sine of Incidence CE to the Sine of Refraction CG as m to n (the Radius being MC $= r$) then the right angled Triangles MEC, MRm; MGC, MOm; BMR, BQE, are Similar; becauſe, if from the right Angles RME, CMm, we ſubtract the Angle EMm, there will remain the Angle RMm = EMC; and if from the right Angles GMO, CMm we ſubduct the Angle GMm, there will remain OMm = GMC. Therefore we have the following Analogies, ME : MC :: MC : Mm; That is, $a : r :: \dot{x} : \frac{r\dot{x}}{a} =$ Mm. Again, MG : MC :: MO : Mm; that is, $b : r :: \dot{y} : \frac{r\dot{y}}{b} = \frac{r\dot{x}}{a}$; therefore $a\dot{y} = b\dot{x}$; and $a : b :: \dot{x} : \dot{y} =$ MO $= \frac{b\dot{x}}{a}$. Then BM : BQ(= BE) :: MR : Qe; that is, $d : d + a :: \dot{x} : \frac{d\dot{x} + a\dot{x}}{d} =$ Qe. But (1322) Ce : Cg :: CE : CG :: m : n :: Ce − CE : Cg − CG :: Qe : Sg :: $\frac{a\dot{x} + d\dot{x}}{d} : \frac{na\dot{x} + nd\dot{x}}{md} =$ Sg. Laſtly, the Similar Triangles FMO and FSg, give MO : Sg :: MF : SF or GF; therefore MO − Sg : MO :: MF − SG, or MG : MF; that is $\frac{bmd\dot{x} - aan\dot{x} - and\dot{x}}{amd} : \frac{b\dot{x}}{a} :: b : \frac{bbmd}{bmd - aan - and} =$ MF, the Focal Diſtance required.



1326. If the Curve AMD be a Circle, then CM $= r$ is conſtant, and the Points E and G will be in a Semicircle deſcribed on the Diameter MC; and when the Point M is very near A the Vertex, then ME and MG are both equal to each other, and to MC; that is, $a = b = r$; and the Theorem will become

become $\frac{rmd}{md-nd-rn}$ Mf, in which Caſe the Focus f is in the Axis of the Sphere.

1327. If the Diſtance of the Radiant B (d) be infinite, or the Rays parallel, then the Theorem will be $\frac{mr}{m-n}$ = Mf = Focus of the *parallel Rays*.

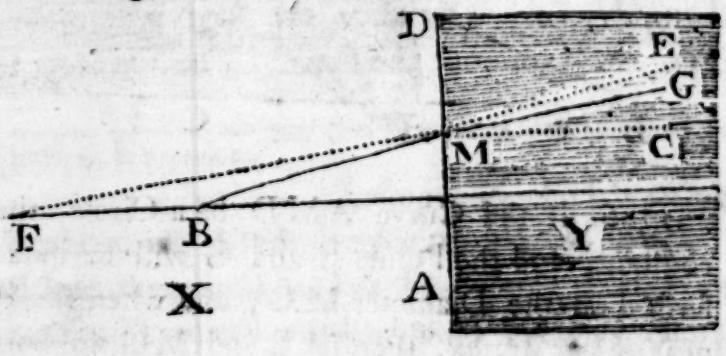
1328. If the Rays fall diverging on the Convex Surface AMD; then d being Negative, or $-d$, the Theorem will become $\frac{-bbmd}{-bmd-aan+and} = \frac{bbmd}{bmd-and+aan}$ = MF (Fig. to 1324) and when AMD is a Circle, and M infinitely near to A, then will the Theorem be $\frac{mrd}{md+dn+rn}$ = Mf.

1329. If the Curve AMD were ſpherically Concave towards the Radiant B, then will the Radius be Negative or $-r$, and the Theorem is $\frac{-rmd}{md-nd+rn}$ = Mf the focal Diſtance; which, becauſe m is greater then n, will be Negative, or on the ſame Side with the Raidant B, or the Rays after Refraction will *diverge*.

1330. In the Caſe of *Parallel Rays* falling on the *Spherical Concave*, we have $\frac{-rm}{m-n}$ = Mf, Negative alſo, becauſe d is infinite.

1331. But *Converging Rays* incident on the ſaid Concave will have $\frac{rmd}{nd+nr-md}$ = Mf, which will be Poſitive or Negative, as $n \times d + r$ is greater or leſs then md.

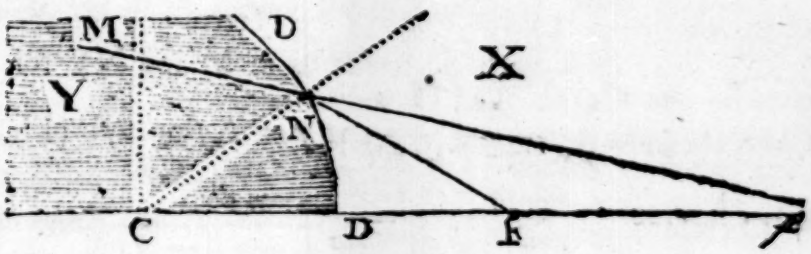
1332. If the Surface AMD be a right Line, or the Radius MC = r infinite, then the Theorem (1325) will be $\frac{mbbd}{-naa}$ = MF; for in this Caſe a and b are alſo infinite;



and

and conſequently ſince $a\,a = b\,b$, we have $\frac{m\,d}{-n} = f$, the focal Diſtance; and therefore $m\,d = -n\,f$, and ſo $n : m :: d : -f$; whence it appears the Ray will be refracted towards the Perpendicular M C, having a negative Focus f on the ſame Side with the Radiant B (1321).

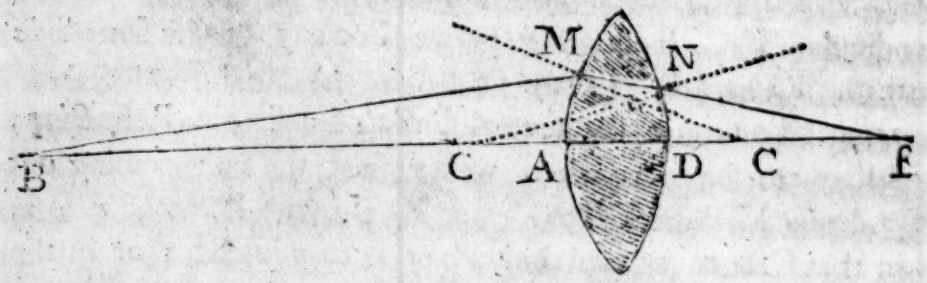
1333. Theſe are the ſeveral Caſes of a ſimple Refraction, and they are the ſame when we conſider the Ray coming out of a denſer Medium Y into a rarer X, only the Ratio of n to m is in that Caſe to be uſed inſtead of m to n in this; or in the foregoing Theorems, putting n for m, and m for n, interchangeably, and other Things altered as required.



1334. Thus for Example; *Suppoſe it required to find the Focus of Rays paſſing out of a denſer Medium* Y *into a rarer* X *in a converging State, and refracted at a concave Surface of the Medium* X. Then it is plain the Theorem in (1326) will equally ſerve here with the following Alterations, *viz.* (1.) Becauſe the Refraction is into a *rarer Medium*, we muſt write n for m, and m for n. (2.) Becauſe the refracting Surface is *Concave*, the Radius will be Negative, and the Sign where (r) is found muſt be changed. (3.) Becauſe the Rays are *converging*, the Diſtance is *negative*, and the Sign where (d) is found muſt be changed. (4.) Therefore the Sign where d and r are both found, will continue the ſame. The Theorem therefore, with all theſe Alterations, will become $\frac{r\,n\,d}{m\,d - n\,d + r\,m} = \mathrm{f} = \mathrm{D}\,\mathrm{f}$, the focal Diſtance required.

1335. Now it is evident, that if the incident Ray M N in this Caſe be conſidered as Part of the refracted Ray M f in the former (1326), then will the Focus f in that Caſe be the radiant Point in this; and therefore if we ſubſtitute f for d, and put

put r for the Radius of Concavity CD, the laſt Theorem will be $\frac{nrf}{mf-nf+mr} = Df = f$.



1336. If now we ſuppoſe the two refracting Surfaces A M and N D to be very near together, ſo that the Diſtance A D be inconſiderable, then may the Focus f be determined for the incident Ray B M after both Refractions at M and N; for the laſt Theorem gives $\frac{mfr}{nr+nf-mf} = f = Af$ or Df. (Fig. to 1326) Therefore $\frac{mfr}{nr+nf-mf} = \frac{mdr}{md-nd-nr}$; and conſequently we have $\frac{ndrr}{mrd-nrd+mdr-ndr-nrr} = f = Df$, as required.

1337. This Theorem may be abbreviated by putting $\frac{m-n}{n} = a$, (or if $n = 1$, and $m-1 = a$) it will become $\frac{drr}{ard+ard-rr} = f$; and is thus accommodated to all Optical Purpoſes, and for a Lens A M N D of any Sort, on which Rays diverging, parallel, or converging can fall.

1338. Thus if *parallel* Rays fall on the Lens, then $AB = d$, being infinite, we have $\frac{rr}{ar+ar} = f$.

1339. If the Rays B M are *converging*, then d being negative, we have $\frac{-drr}{-ard-ard-rr} = \frac{drr}{ard+ard+rr} = f$.

1340. If the Lens be *equally Convex* on both Sides, or $r = r$; then the Theorem for *diverging Rays* is $\frac{ar}{2ad-r} = f$; for parallel

parallel Rays where d is infinite, $\frac{r}{2a} = f$. And for *converging Rays*, where d is negative, $\frac{-dr}{-2ad-r} = \frac{dr}{2ad+r} = f$, always positive.

1341. If one Side N D be plane, or the Radius r infinite, then the Theorem for such a *Plano-Convex* Lens, for *diverging Rays*, is $\frac{dr}{ad-r} = f$; for *parallel Rays* $\frac{r}{a} = f$; and for *converging Rays*, $\frac{-dr}{-ad-r} = \frac{dr}{ad+r} = f$.

1342. If one Radius r be Negative, or the Side N D be Convex towards A M, then the Lens is a *Convexo-Concave*, or *Meniſcus*; and the Theorem for *diverging Rays*, is $\frac{-drr}{adr-adr+rr} = f$; for *parallel Rays*, $\frac{-rr}{ar-ar} = f$; and for converging Rays it is $\frac{drr}{adr-adr+rr} = f$.

1343. If $r = r$, then for diverging Rays, $-d = f$; for parallel Rays, $\frac{-rr}{o} = f$; and for converging Rays, $d = f$.

1344. If both the Radii r and r be Negative the Lens becomes a *double Concave*, and the Theorem for diverging Rays, is $\frac{drr}{-adr-adr-rr} = f$, always negative. For parallel Rays $\frac{rr}{-ar-ar} = f$, ever negative. And for converging Rays $\frac{-drr}{adr+adr-rr} = f$.

1345. If one Radius $-r$ be infinite, then the Lens is a *Plano-Concave*; and the Theorems become $\frac{-dr}{ad+r} = f$, for diverging Rays; $\frac{-r}{a} = f$, for parallel; and $\frac{-dr}{ad-r} = f$ for converging Rays.

1346. If the negative *Radii* are both equal, it makes an *equally Concave* Lens; where the Focus of diverging Rays is

found by this Theorem $\frac{-dr}{2ad+r} = f$, and of parallel Rays $\frac{-r}{2a} = f$, in each Caſe always negative. Converging Rays have $\frac{dr}{-2ad+r} = f$.

1347. Theſe are all the Caſes that can happen in the Theory of common Dioptrics, and may all of them be very eaſily applied in Practice by ſubſtituting ſuch Numbers for the Ratio of m to n as we find by Experiments agreeing to Mediums we uſe.

Thus in { Water, $m:n::4:3$, whence $a = 0{,}3333$.
Glaſs, $m:n::31:20$ —— $a = 0{,}573$
Diamond, $m:n::5:2$ —— $a = 1{,}5$.

But for a larger View of this Subject ſee my *New* PRINCIPLES of OPTICS, lately publiſhed.

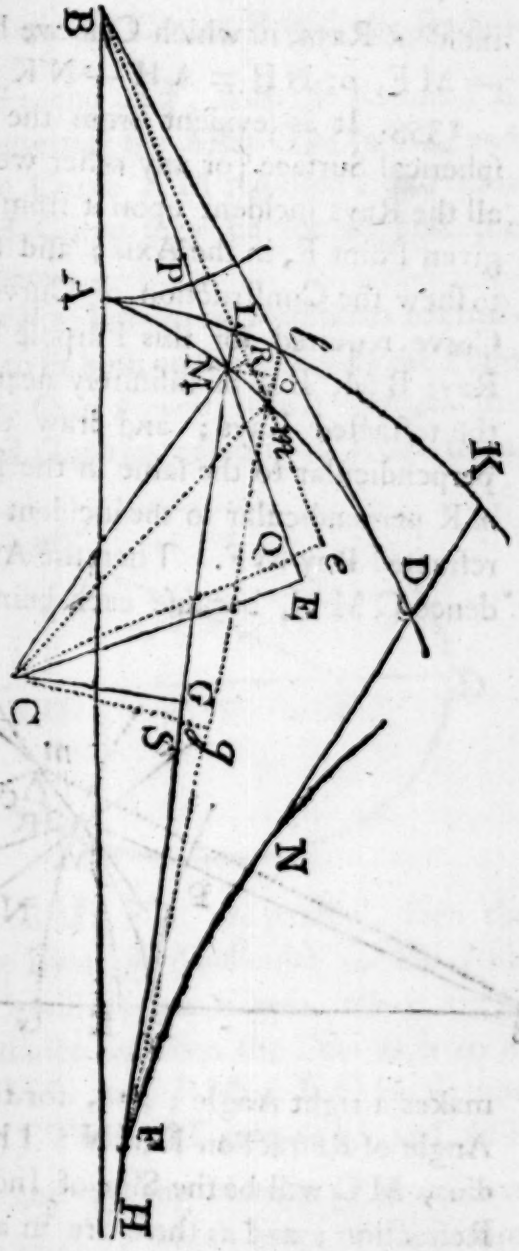
CHAP. IV.

The THEORY *of* DIOPTRICS *continued; the Nature of the* Diacauſtic CURVE *explained; and the Method of finding a* GEOMETRICAL FOCUS *for Rays iſſuing from a given Point in the Axis of a* LENS *of a* Mechanical Figure.

1348. IT has been ſhewn (1325) how the Focus H, F, N, *&c.* of any Rays B A, B M, B N, *&c.* falling on the Curve A M D may be found, after Refraction, from proper *Data*; and the Curve N F H, which is the *Locus* of all thoſe Points, is called the DIACAUSTIC, or *Cauſtic by Refraction.*

1349. If

1349. If the Cauſtic HFN be involved (ſee 914.) beginning at the Point A, the ſaid Point A will deſcribe the Curve ALK, the *Involute* of the ſaid cauſtic Curve; whence the Tangent LF + the Part of the Cauſtic HF, will always be equal to the right Line AH; that is, LF + HF (or KN + HN) = AH.



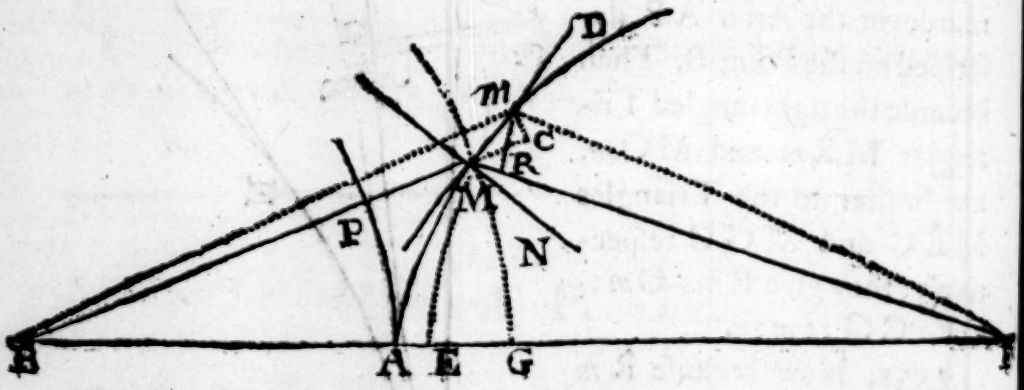
1350. Suppoſe the other Lines of the Figure drawn as directed (1325,) and moreover the Arch AP deſcribed on the Point B. Then becauſe the right-angled Triangles MR*m* and MO*m*, are ſimilar to the Triangles MEC and MGC reſpectively, they give R*m* : O*m* :: CE : CG :: *m* : *n*.

1351. Now becauſe R*m* = *d*, the Fluxion of BM = *d*, and O*m* = Fluxion of LM, and becauſe BP = BA is a conſtant Part of BM, the contemporary or proper Fluents of the foreging Fluxions are PM and LM; and ſince in this Caſe, the Fluxions and Fluents in the ſame Ratio (788) therefore PM : LM :: *m* : *n* :: BM — BA : AH — MF — FH; and conſequently $n \times \overline{BM - BA} = m \times \overline{AH - MF - FH}$; from which Equation, we get $FH = AH - MF + \frac{n}{m} BA - \frac{n}{m} BM$, which is the *Rectification of the diacauſtic Curve.*

1352. When the Radiant B is at an infinite Diſtance, then BA = BM; and AP is a right Line perpendicular to thoſe

incident Rays, in which Caſe we have the Cauſtic $FH = AH - MF$, or $NH = AH - NK$; *&c.*

1353. It is evident from the foregoing Theory, that no ſpherical Surface (or any other we have treated of) can refract all the Rays incident upon it from a given Point B, to another given Point F, in the Axis; and therefore it becomes neceſſary to ſhew the Conſtruction of Curves that will do this. Let the Curve required for this Purpoſe be A M D, and the incident Rays B M, B *m* be infinitely near each other, and M F, *m* F, the refracted Rays; and draw the Tangents D *m* and M N perpendicular to the ſame in the Point M. Laſtly, draw *m* C, *m* R perpendicular to the incident Ray B M, continued out, and refracted Ray M F. Then the Angle M *m* C = Angle of Incidence C M N, becauſe each being added to the Angle *m* M C



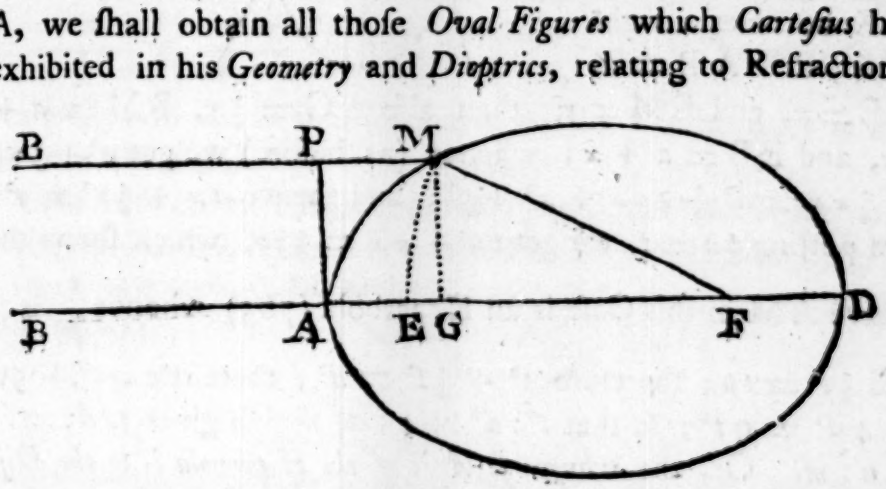
makes a right Angle; and, for the ſame Reaſon, M *m* R = the Angle of Refraction F M N: Therefore if we make M *m* Radius, M C will be the Sine of Incidence, and M R the Sine of Refraction; and as theſe are in a given Ratio *m* to *n* (1322) we have M C : M R :: *m* : *n*. But M C is the Fluxion of the incident Ray, and M R the negative Fluxion of the refracted Ray; the contemporary Fluents therefore of theſe Fluxions will be in the ſame Ratio of *m* to *n*. On the Point B deſcribe the Arches P A and M G, and on the Point F deſcribe the Arch M E, then P M and A E are the Fluents mentioned; and conſequently we have P M or A G : A E :: *m* : *n*. Which is the Property of the Curve A M D.

1354. Therefore when the reſpective *Foci* B and F, and A, the Vertex of the Curve required, are given, the Curve may be thus conſtructed. Take A G at Pleaſure, and ſay *m* : *n* ::

A G

$AG : \frac{n}{m} AG = AE$; then on the Center B with the Radius A G deſcribe the Arch M G; and on F, with the Radius F E deſcribe the Arch M E to interſect the Arch G M in M, and the Point M will be in the Curve A M required; and thus all other Points of the ſaid Curve may be found, and the Curve A M D drawn thro' them.

1355. By cauſing the Point B or F to go off ſometimes to an infinite Diſtance, or ſometimes to lie both on the ſame Side the Point A, we ſhall obtain all thoſe *Oval Figures* which *Carteſius* has exhibited in his *Geometry* and *Dioptrics*, relating to Refractions.

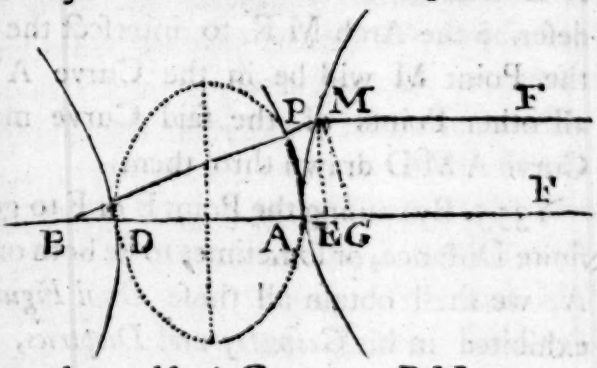


1356. Thus if the Rays B M, B A, are parallel, then the Arch G M becomes a right Line, perpendicular to the Axis B F, and the Curve A M D will be an *Ellipſis*, whoſe tranſverſe Axis A D is to the Diſtance between the Foci as m to n. For put $AF = a$, $AP = MG = y$, $PM = AG = x$; and ſuppoſe $m : n :: 3 : 2$ (1347) Then $GF = a - x$, and MF $\sqrt{aa - 2ax + xx + yy}$; and by the Nature of the Curve (1354) it is $\frac{2}{3} PM + MF = AF$, which gives this Equation $\frac{2}{3}x + \sqrt{a^2 - 2ax + x^2 + y^2} = a$. And by Tranſpoſition and Involution (212) we have $a^2 - 2ax + x^2 + y^2 = a^2 - \frac{4}{3}ax + \frac{4}{9}x^2$; and conſequently $y^2 = \frac{2}{3}ax - \frac{5}{9}xx$; and ſo $\frac{9}{5}yy = \frac{6}{5}ax - x^2$. Put $\frac{6}{5}a = t = AD$, then $tx - xx = \frac{9}{5}yy$; therefore the Curve is an Ellipſis (764.) And $\frac{t}{p} = \frac{9}{5}$, conſequently $t : p :: 9 : 5$. Let $d =$ Diſtance between the Foci, then $t^2 : tp :: 9 : 5$, and $t^2 - tp = d^2$ (768) therefore $t^2 - \frac{5t^2}{9} = d^2$, or $9t^2 - 5t^2 = 9d^2$, therefore $t^2 : 9 :: d^2 : 9 - 5 = 4$, that is

$t : d$

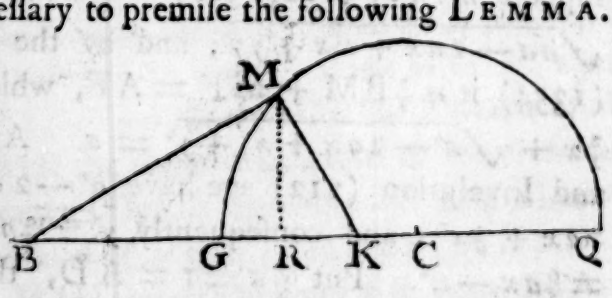
$t : d :: 3 : 2 :: m : n$, or, the *tranſverſe Axis is to the Diſtance between the Foci as the Sine of the Angle of Incidence to the Sine of the Angle of Refraction*, as *Carteſius* has ſhewn in his *Dioptrics*.

1357. On the other hand, if the Point F be ſuppoſed at an infinite Diſtance, or the incident Rays F M, F A, parallel, then M E becomes a right Line; then A B $= a$, A E $= x$, and E M $= y$; then alſo A G $= \frac{3}{2}x$, B M $= a + \frac{3}{2}x$, and E B $= a + x$; whence (as before) we get $a^2 + 3ax + \frac{9}{4}xx = a^2 + 2ax + x^2 + y^2$; and thence $ax + \frac{5}{4}x^2 = y^2$; and putting $\frac{4}{5}a = t$, we get $tx + xx = \frac{4}{5}yy$, which ſhews the Curve A M in this Caſe is an Hyperbola (765) whence $\frac{t}{p} = \frac{4}{5}$, and $\frac{5}{4}t^2 = tp$; therefore $t^2 + \frac{5}{4}t^2 = d^2$; therefore $4t^2 + 5t^2 = 4d^2 = 9t^2$; ſo that $t^2 : d^2 :: 4 : 9$, which gives $t : d :: 2 : 3 :: n : m$. Or, the *tranſverſe Axis of the Hyperbola is to the Diſtance between the Foci of the oppoſite Sections, as the Sine of Incidence to the Sine of Refraction.*



1358. Thus we have ſhewn what Curves will refract all Rays iſſuing from any one Point to any other given in a different Medium, or by a *ſingle Refraction*. It remains now to ſhew how the ſame Thing may be effected by *two Refractions*; in order to which it will be neceſſary to premiſe the following LEMMA.

1359. *To deſcribe the Curve* G M, ſuch, that from any Point M, two right Lines B M, K M *may be drawn to two given Points* B *and* K, *which ſhall be to each other in the given Ratio of m to n, or that* B M : K M :: $m : n$.

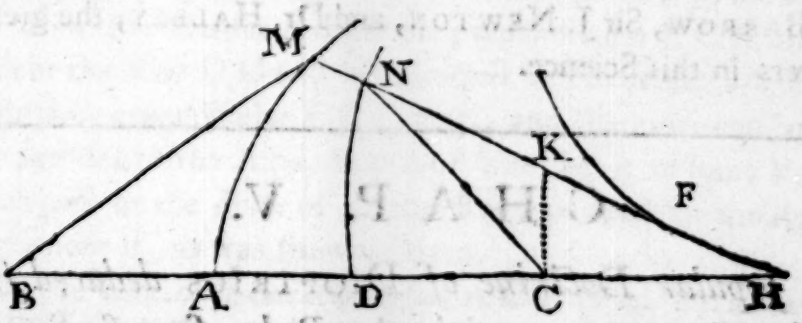


Draw M R perpendicular to B K, and put B K $= a$, B R $= x$, and M R $= y$; then becauſe of the right-angled Triangles B R M, K R M, we have B M $= \sqrt{xx + yy}$, and K M $= \sqrt{a^2 -}$

$\sqrt{a^2 - 2ax + x^2 + y^2}$; and therefore it must be $\sqrt{x^2 + y^2}$: $\sqrt{a^2 - 2ax + x^2 + y^2} :: m : n$; which gives $yy = \frac{2ammx}{mm - nn} - xx - \frac{aamm}{mm - nn}$; and since the Co-efficient of the Square of the Ordinate (yy) is Unity, (764) it is evident the Curve G M is a Circle.

1360. When $yy = 0$, then $\frac{2amnx}{mm - nn} - xx = \frac{aamm}{mm - nn} = \frac{am}{m + n} \times \frac{am}{m - n} = BG \times BQ$ (330.) Therefore GQ (= BQ — BG) is the Diameter of the Circle which is the *Locus* of the Point M.

1361. *The Curve* A M, *the radiant Point* B, *with the Ratio of m to n being given, it is required to deſcribe the Curve* N D, *that ſhall refract the firſt refracted Rays* M N, *and make them converge to any given Point* C.



Suppoſe F H to be the Cauſtic by Refraction to A M, the radiant Point being in B; it is evident that the ſame Curve F H will alſo be the Cauſtic by Refraction to the Curve N D (1352) and therefore $FH = DH - NF - \frac{n}{m} DC + \frac{n}{m} NC = AH - MF + \frac{n}{m} BA - \frac{n}{m} BM$; which Equation will give $\frac{n}{m} AB - \frac{n}{m} BM + \frac{n}{m} DC + AD = MN + \frac{n}{m} NC$.

1362. Therefore to conſtruct the Curve N D, in any refracted Ray A H, take at Pleaſure the Point D, for one of the Points in the Curve required. And in any other Ray as MF take $MK = \frac{n}{m} BA - \frac{n}{m} BM + \frac{n}{m} DC + AD$; and

find

find the Point N (by 1360) ſo that N K : N C : : n : m, or N K $= \frac{n}{m}$ N C, and the Point N will be in the Curve required; and thus a ſufficient Number of Points may be found thro' which to draw the Curve D N.

1363. From the foreging Theory it appears, that it is poſſible to determine the Figure for any *Mechanical* Lenſes, *Convex* or *Concave*, by which either ſingly or conjointly, Rays of Light proceeding from any one given Point B in the Axis of the Lens may be accurately refracted to any other Point or Focus f, which in this Caſe may be called the *Geometrical Focus*. But certain it is, that no Figure for Glaſſes can be found for ſuch a Focus of Rays procceding from a Point out of the Axis; and therefore it is naturally impoſſible that the *Defects of Dioptric Viſion ariſing from the Figure of Glaſſes*, ſhould ever be rectified by Art. We have here given the Subſtance of all that has been publiſhed by KEPLER, DESCARTES, HUGENIUS, Dr. BARROW, Sir J. NEWTON, and Dr. HALLEY, the greateſt Maſters in this Science.

CHAP. V.

The popular Doctrine of DIOPTRICS *deduced from the* THEORY; *with the Rules for finding the* FOCAL DISTANCES *of all Sorts of* LENSES, *alſo the* Proportion, Magnitude, Poſition *&c. of* IMAGES *formed thereby.*

(*Plate* I. *of* OPTICS *and* PERSPECTIVE.)

1364. HAVING thus premiſed the Theory, we now proceed to make ſuch practical Deductions from it as will be ſufficient to acquaint our Readers (not verſed in Mathematics) with all the uſeful Part of *Dioptrics*, and its Application to *Optical Inſtruments* in every Branch of the *Viſual Science*.

1365. Therefore let A B (Fig. 6.) be the Surface of any refracting *Medium* B K denſer than Air, as *Water*, *Glaſs*, *&c.* and let D C be a Ray of Light incident thereon in the Point C;

thro'

thro' the Point draw E H perpendicular to the Surface A B, and the Angle E C D is the *Angle of Incidence*. When the Ray arrives at C it is refracted or bent out of its first right-lined Direction D C K into another C F towards the perpendicular C H, and the Angle F C H is the *Angle of Refraction*, less than the Angle of Incidence K C H by the small Angle F C K, which is called the *refracted Angle*.

1366. On the other hand, if a Ray of Light C F in a dense Medium be incident at C upon any rarer Medium, it will be refracted out of its first Direction F C into another C D which will be farther from the perpendicular C E. And the greater the *different refractive* Powers of the Mediums, the greater will be the Difference of the Angles E C D and F C H. All which is evident from the Theory (1322.)

1367. If the refracting Surface A C B be not plain, but spherical; then let E be the Center of the Sphere, and V E F, the Axis thereof. Let C D be a Ray of Light falling on the *convex Surface* (Fig. 7) in the Point C and parallel to the Axis V F; from the Center E draw the perpendicular E C H and it is evident the Ray D C will be refracted in the denser Medium towards the perpendicular E C (1365), and therefore can be no longer parallel to the Axis, but must intersect it at some Point F which will be the *Focus* of all the Rays parallel to the Axis, and very near it, as was shewn (1327.)

1368. In Case of a *concave Surface* A C B (Fig. 8) a Ray of Light D C parallel to the Axis E V, will, at its Entrance into the denser Medium, be refracted also towards the perpendicular C H, in such a Manner to F, that were F C to be produced, it would cut the Axis produced in the rarer Medium (suppose *Air*) beyond F. (1329) The Distance of the Piont F from the Vertex V in Water, is *four Times the Radius of the Sphere*, *viz.* V F = 4 V E; and in *Glass* it is V F = 3 V E (1347) or the Focus is distant *three Semi-diameter of the Sphere*.

1369. If a Ray of Light be twice refracted, first into a dense Medium and then into a rare one, its Course after the second Refraction will be variable, according to the Figure of the Surfaces which bound on each Side the denser Medium, and may easily be determined by the Theory (1336.) This Case in *practical Dioptrics* brings us to the Consideration of *Lenses*,

which are the eſſential Parts of *Teleſcopes*, *Microſcopes*, and all other Inſtruments of the Dioptric Kind.

1370. Let A V B (Fig. 9) be a *Plano-convex Lens*, then Rays of Light D C, D C, which fall upon it parallel to the Axis V E will, after Refraction thro' it, be converged to a Focus in the Axis at the Point F, which is *nearly equal to the Diſtance of the Diameter of the Sphere* V F (from the Vertex V) *of which the Lens is a Segment* (1341.)

1371. On the contrary, it follows that if Rays of Light diverge from a Point F, at the Diſtance of twice the Radius V E of Convexity in any Plano-convex Lens A V B, *they will, after Refraction thro' it, proceed parallel to the Axis* V E.

1372. In like Manner, if A V B (Fig. 10) be a double and equally convex Lens. Then parallel Rays D C falling upon it *will be refracted to a Focus* F *very near the Point* E *or Center of Convexity, as is demonſtrated* (1340.)

1373. Therefore on the contrary, when Rays of Light F A, F B, diverge from a Point in any Object in the Focus F of an equally convex Lens A V B, *they will, after Refraction thro' it, proceed parallel to the Axis of the Lens* V E.

1374. Let parallel Rays D C, D C, fall upon a *Plano-concave Lens* A C B (Fig. 11) then will they be ſo refracted thro' it to f, f, *as if they came diverging from a Point or Focus* F *at the Diſtance of the Diameter of the Sphere of Concavity of the Lens* (1345) and the Point F is in this Caſe called the *virtual Focus.*

1375. If parallel Rays D C fall upon a *double and equally concave Lens* (Fig. 12.) *then they will be refracted to* f *and* f *diverging from a Point* E *which is the Center of Concavity* (1346) and the *virtual Focus* of the Lens.

1376. In each of theſe two laſt Caſes, if Rays f C, f C, converging to a Focus F or E, are intercepted by a ſingle or double concave Lens A B placed at the Diſtance of the Diameter or Radius of Concavity from the ſaid focal Point, *then the Rays, after Refraction, will proceed parallel among themſelves, and to the Axis of the Lens.*

1377. If the Lens be not *equally convex* on both Sides, then the Focus of *parallel Rays* will be found by the following

RULE.

RULE.

Divide the Product of the Radii, by half their Sum, and the Quotient will be the focal Distance required. (1338) *

Example. Suppose one Radius 15 Inches, the other 9; their Product is 135, which divided by half the Sum 12, gives 11 $\frac{2}{10}$ Inches for the focal Distance of such a Lens.

1378. If the incident Rays are not *parallel*, but come *diverging* from a distant radiant Point in the Axis of a Lens of equal Convexity, then the focal Distance is found by this

RULE.

Multiply the Distance of the Radiant by the Radius of the Lens; and divide that Product by the Difference between the said Distance and Radius; and the Quotient will be the focal Distance required. (1340.)

Example. Let the Radius of the Convexity be 15 Inches, and the Distance of the Radiant 60; then their Product is 900, and their Difference 45; therefore 45)900(20 = the focal Distance in Inches, for that Distance of the Radiant.

1379. When the Radii of Convexity are unequal, the Focus of diverging Rays is found by the following

RULE.

Multiply twice the Product of the Radii by the Distance of the Radiant; and then divide by the Difference between the Sum of the Radii multiplied by the Distance, and twice the Product of the Radii; and the Quotient will be the focal Distance required. (1337)

Example. Let one Radius be 15 Inches, the other 9; and the Distance of the Radiant 60; then twice the Product of the *Radii* is 270, which multiplied by 60 makes 16200; the Sum of the Radii 24 multiplied by 60, is 1440, from which take 270, there will remain 1170; then 1170)16200(13 $\frac{85}{100}$ Inches, the focal Distance required.

1380. If the Rays fall *converging* on the Lens, then if we divide by the *Sum* instead of the *Difference*, the Rule will in all other Respects be the same, in each of the two last Cases, for finding the focal Distance for any given Distance to which the Rays tend (1339.)

* This Rule is in general exact enough for Use, since in common Glass the Value of (a) in (1338) is very little more than ($\frac{1}{2}$).

1381. In all that has been hitherto ſaid, it is ſuppoſed that the Lenſes are of Glaſs, and that the Sine of Incidence is to that of Refraction in Glaſs as 3 to 2; agreeable to (1347.) But as there is a conſiderable Difference in the refractive Powers of different Kinds of Glaſs, if we have Regard to that (as in ſome Caſes will be neceſſary) then the Rules above will be ſomewhat more complicated, and we muſt operate according to the Theorems referred to, and take in the Value of (*a*) as it is found by Experiment, in each particular Sort of Glaſs. †

1382. Let O B be any Object placed at a Diſtance from the convex Lens C D (Fig. 15.) Then it is evident that a Pencil of Rays C A D which flow from the Point A in the Axis will all be converged to another Point (*a*) in the Axis (if the Diameter of the Lens C D be but ſmall (by 1336) whoſe Diſtance V *a* may be found (by 1377 or 1378) *and this Point (a) will be the Repreſentation or Image of the Point* A *in the Object* (1305.)

1383. Let O be a Point in the extreme Part of the Object which ſends a Pencil of Rays D O C to the Lens C D; among theſe one Ray O V will be refracted thro' the Center of the Lens, and therefore the Poſition of the refracted Part V M will be ſimilar or parallel to the incident Ray O V (as will appear from 1332) and when the Thickneſs of the Lens C D is inconſiderable (as in moſt Optical Caſes it is) then O V and V M may without ſenſible Error be eſteemed *one right Line*; and therefore the Axis of that Pencil of Rays, in which at the Point M they will all be united after Refraction, *wherefore* M *will be the Image of the* Point O in the Object.

1384. And in like Manner I will be the Image of the Point B in the Object, and ſo the whole *Object* O B *will be repreſented in its Image* I M. And by calculating by the above Rules it will be found that the Image I *a* M will be *curvelineal*, more or leſs, as the *Aperture* C D of the Lens is greater or ſmaller.

1385. The *Poſition of the Image*, with reſpect to the ſimilar Parts of the Object, is *inverted*. The Reaſon of this is evident by Inſpection of the Figure, ſince the Axis of the Pencils from each extreme Part of the Object croſs each other in the Center of the Lens. 1386. *The*

† See my NEW PRINCIPLES of OPTICS, lately publiſhed.

1386. *The Object and Image ſubtend equal Angles at the Center of the Lens.* For ſince O M and B I may be conſidered as ſtrait Lines, the Angle B V O, under which the Object appears, will be equal to the Angle I V M, under which the Image is ſeen from the Center or Vertex V of the Lens.

1387. *The lineal Dimenſions of the Object and Image are as their Diſtances from the Lens reſpectively*; for ſince the Triangles O V B and I V M are ſimilar, we have O B, the Length of the Object, to I M the Length of the Image, in the ſame Proportion as O V or A V the Diſtance of the Object, to I V or *a* V the Diſtance of the Image. Hence their *Superficies* will be as the *Squares*, and their *Solidities* as the *Cubes* of the Diſtances from the Lens.

1388. *The Object and Image are reciprocal*; for if I M be conſidered as an Object, then will O B be the Image thereof; and A, *a*, are called the *conjugate or reſpective Focuſſes*, for thoſe Diſtances of the Object and Image.

1389. Hence it appears, that the nearer the Object is to the Lens, the farther off the Image will be formed; and when the Object comes to the Focus of the Lens, the Image will then be at an infinite Diſtance, ſince the Rays from every Point will after Refraction be parallel among themſelves (1373.) Laſtly, if the Object be nearer the Lens than the Focus, the Rays after Refraction will diverge, and in this Caſe no real Image can be formed of that Object at all by the ſingle Lens.

1390. There is another Form of a Lens called a *Meniſcus*, which is concave on one Side, and convex on the other (Fig. 16.) But tho' it was in great Uſe and Eſteem ſome Years ago, it is now entirely uſeleſs, as the *Plano-convex Lens* is known now much to exceed it in thoſe Properties for which it was then ſo much valued.

SCHOLIUM.

1391. The Lenſes hitherto conſidered are of a ſpherical Form, and will not admit of a *Geometrical Focus*, not even of Rays flowing from a Point in the Axis (1348, *&c.*) and therefore it is impoſſible that any ſuch Lens ſhould form a perfect Image of any Object, not ſo much as in a ſingle Point thereof. We have ſhewn indeed from the Theory, (1361) that a Lens

may

1381. In all that has been hitherto ſaid, it is ſuppoſed that the Lenſes are of Glaſs, and that the Sine of Incidence is to that of Refraction in Glaſs as 3 to 2; agreeable to (1347.) But as there is a conſiderable Difference in the refractive Powers of different Kinds of Glaſs, if we have Regard to that (as in ſome Caſes will be neceſſary) then the Rules above will be ſomewhat more complicated, and we muſt operate according to the Theorems referred to, and take in the Value of (*a*) as it is found by Experiment, in each particular Sort of Glaſs. †

1382. Let O B be any Object placed at a Diſtance from the convex Lens C D (Fig. 15.) Then it is evident that a Pencil of Rays C A D which flow from the Point A in the Axis will all be converged to another Point (*a*) in the Axis (if the Diameter of the Lens C D be but ſmall (by 1336) whoſe Diſtance V *a* may be found (by 1377 or 1378) *and this Point (a) will be the Repreſentation or Image of the Point* A *in the Object* (1305.)

1383. Let O be a Point in the extreme Part of the Object which ſends a Pencil of Rays D O C to the Lens C D; among theſe one Ray O V will be refracted thro' the Center of the Lens, and therefore the Poſition of the refracted Part V M will be ſimilar or parallel to the incident Ray O V (as will appear from 1332) and when the Thickneſs of the Lens C D is inconſiderable (as in moſt Optical Caſes it is) then O V and V M may without ſenſible Error be eſteemed *one right Line*; and therefore the Axis of that Pencil of Rays, in which at the Point M they will all be united after Refraction, *wherefore* M *will be the Image of the* Point O in the Object.

1384. And in like Manner I will be the Image of the Point B in the Object, and ſo the whole *Object* O B *will be repreſented in its Image* I M. And by calculating by the above Rules it will be found that the Image I *a* M will be *curvelineal*, more or leſs, as the *Aperture* C D of the Lens is greater or ſmaller.

1385. The *Poſition of the Image*, with reſpect to the ſimilar Parts of the Object, is *inverted*. The Reaſon of this is evident by Inſpection of the Figure, ſince the Axis of the Pencils from each extreme Part of the Object croſs each other in the Center of the Lens. 1386. *The*

† See my NEW PRINCIPLES of OPTICS, lately publiſhed.

1386. *The Object and Image ſubtend equal Angles at the Center of the Lens.* For ſince O M and B I may be conſidered as ſtrait Lines, the Angle B V O, under which the Object appears, will be equal to the Angle I V M, under which the Image is ſeen from the Center or Vertex V of the Lens.

1387. *The lineal Dimenſions of the Object and Image are as their Diſtances from the Lens reſpectively*; for ſince the Triangles O V B and I V M are ſimilar, we have O B, the Length of the Object, to I M the Length of the Image, in the ſame Proportion as O V or A V the Diſtance of the Object, to I V or *a* V the Diſtance of the Image. Hence their *Superficies* will be as the *Squares*, and their *Solidities* as the *Cubes* of the Diſtances from the Lens.

1388. *The Object and Image are reciprocal*; for if I M be conſidered as an Object, then will O B be the Image thereof; and A, *a*, are called the *conjugate or reſpective Focuſſes*, for thoſe Diſtances of the Object and Image.

1389. Hence it appears, that the nearer the Object is to the Lens, the farther off the Image will be formed; and when the Object comes to the Focus of the Lens, the Image will then be at an infinite Diſtance, ſince the Rays from every Point will after Refraction be parallel among themſelves (1373.) Laſtly, if the Object be nearer the Lens than the Focus, the Rays after Refraction will diverge, and in this Caſe no real Image can be formed of that Object at all by the ſingle Lens.

1390. There is another Form of a Lens called a *Meniſcus*, which is concave on one Side, and convex on the other (Fig. 16.) But tho' it was in great Uſe and Eſteem ſome Years ago, it is now entirely uſeleſs, as the *Plano-convex Lens* is known now much to exceed it in thoſe Properties for which it was then ſo much valued.

SCHOLIUM.

1391. The Lenſes hitherto conſidered are of a ſpherical Form, and will not admit of a *Geometrical Focus*, not even of Rays flowing from a Point in the Axis (1348, *&c.*) and therefore it is impoſſible that any ſuch Lens ſhould form a perfect Image of any Object, not ſo much as in a ſingle Point thereof. We have ſhewn indeed from the Theory, (1361) that a Lens

may

may be formed that ſhall have a Geometrical Focus for any radiant Point of its Axis, but not for any other; and therefore it is plain, *no Image of an Object can be formed in any Degree perfect by any Lens whatſoever*; and conſequently, *that refracting Teleſcopes will ever be, in their own Nature, imperfect, if conſtructed with a ſingle convex object Lens.* But this Imperfection from the Figure will admit of a little Correction from the Addition of *another convex Lens*, but will be increaſed by joining therewith a *concave one*, as I have largely ſhewn in my NEW PRINCIPLES of OPTICS. *

CHAP. VI.

Of the NATURE *and* STRUCTURE *of the* EYE; *and the* ELEMENTS *of* VISION *explained from the foregoing* THEORY.

1392. FROM the Elements delivered in the preceding Chapter we are enabled to explain the *true* THEORY OF VISION, as it is performed by the moſt exquiſite of all Dioptric Inſtruments, the EYE. For which Purpoſe it will be neceſſary

* Before we leave this Subject it may be neceſſary to obſerve, that the Hypotheſis of the Paſſage of a Ray of Light by Reflection, from A to C and D (ſee 1282) being a *Minimum*, or the leaſt Poſſible, is not an arbitrary Poſition; for if we conſider the Reflection of Light as an *Operation of Nature*, and that perfect Wiſdom has eſtabliſhed the Oeconomy of Nature's Laws, it is neceſſary to conclude, that every Thing is done in the moſt direct and ſimple Manner, and therefore that the Paſſage of the Ray A C + C D is the leaſt Poſſible from A to D, by Reflection from the given Plane B E; for this cannot be denied, without aſſerting, *that the Author of Nature has not taken the moſt direct and ready Way of doing Things*; which is inconſiſtent with our natural Notion of Deity, and conſequently is irreligious, as well as abſurd. Therefore it ought to be eſteemed an *Axiom*, or a primary Poſtulate in Optics, on which the whole Science depends. And notwithſtanding it is a ſelf-evident Principle, it is at the ſame Time the Reſult of the ſtricteſt *Mathematical Theory*; it is alſo demonſtrable on the *Principles of Mechanics*; and *all Experiments teſtify* that the Angles of Incidence and Reflection are equal, and therefore that the Paſſage of the Rays of Light from one given Point to another, either by *Reflection* or *Refraction*, muſt be the leaſt poſſible.

neceſſary to deſcribe the ſeveral Parts that are immediately concerned in producing this wonderful Effect.

1393. And here it muſt be obſerved, that Viſion is effected by a Refraction of Light thro' the Humours of the Eye to the Bottom or *Fundus*, where the Images of external Objects are formed on a fine Expanſion of the Optic Nerve called the *Retina*, and therefore the anterior Part of the Eye muſt neceſſarily be of a *convex Figure*, and of ſuch a preciſe Degree of Convexity as the particular refractive Power of the ſeveral Humours require for forming the Image of an Object at a given focal Diſtance, *viz.* the *Diameter of the Eye.*

1394. Hence we find, *Firſt*; The external Part of the Eye-ball C D (Fig. 17.) is a Pellucid, properly convex, and ſtrong Subſtance, which, when dried, has ſome Reſemblance to a Piece of tranſparent *Horn*, and is therefore called the *Cornea*, or *horney Coat* of the Eye.

1395. *Secondly*; Immediately behind this Coat there is a fine clear Humour which from its Likeneſs to *Water* (in a general View) is called the *Aqueous* or *watery Humour*, and is contained in the Space between C D and G F E.

1396. *Thirdly*; In this Space there is a Membrane or Diaphragm, called the *Uvea*, with a Perforation or Hole in the Middle as at F, of a muſcular Contexture for altering the Dimenſions of that Hole (or *Pupil*) for the adjuſting a due Quantity of Light.

1397. *Fourthly*; Juſt behind this Diaphragm is placed a lenticular Subſtance G E, called from its Tranſparency, the *Cryſtaline Humour*, tho' it be not a fluid Body, but of a conſiderable Conſiſtence. It is contained in a fine Tunic called the *Arachnoides*, and is ſuſpended in the Middle of the Eye by an *Annulus* of muſcular Fibres called the *Ligamentum Ciliare*, as at G and E. By this Means it is capable of being *moved a little nearer to, or farther from the Bottom of the Eye.*

1398. *Fifthly*; All the remaining interior Part of the Eye is made up of a large Quantity of a jelly-like Subſtance called the *Vitreous* or *glaſſy Humour*, tho' there is not the leaſt Likeneſs to Glaſs in it, except its Tranſparency; it being moſt like the *White of an Egg* of any Thing.

1399. *Sixthly*; On one Side of the hinder Part of the Eye as at K, the Optic Nerve enters it from the Brain, and is expanded

panded over all the interior Part of the Eye to G and E all around. This delicate Part is by Nature appointed the *immediate Organ of* SIGHT. On this wonderful Membrane the Image I M of every external Object O B is formed according to the *Optic Laws of Nature*, in the following Manner.

1400. Let O B be any Object, placed at a great Distance A L, from the Eye. Then a Pencil of Rays proceeding from any Point L will fall on the *Cornea* D C, and be refracted by the *Aqueous Humour* under it to a Point in the Axis of that Pencil continued out. Then supposing the *Radius of Convexity* of the Cornea to be 3,3 Tenths of an Inch; and the Sine of Incidence in Air to that of Refraction in the Aqueous Humour to be as 4 to 3 (as it is nearly) then if the Object be infinitely distant, or the Rays parallel, we shall find (per Theorem 1327) that the *focal Distance* after the first Refraction will be 13,3 Tenths of an Inch from the *Cornea*.

1401. The Rays thus refracted by the *Cornea*, fall converging on the *Crystaline Humour*, and tend to a Point 12,28 Tenths of an Inch behind it; also the *Radii* of Convexity in the said Humour are 3,3 and 2,5 Tenths respectively; and the Sine of Incidence is to that Refraction of the *Aqueous* into the *Crystaline Humour* as 13 to 12. Therefore (per Theorem 1328) the *focal Distance* after Refraction in the *Crystaline* will be 10,6 Tenths of an Inch from the fore Part thereof.‡

1402. The Rays now pass from the *Crystaline* to the *Vitreous Humour* still in a *converging State*, and the Sines of Incidence and Refraction being here as 12 to 13 (as found by Experiment); and since the Surface of the vitreous Humour is *Concave* which receives the Rays, and is the same with the *Convexity* of the posterior Surface of the *Crystaline*, the Radius will be the same, *viz.* 2,5 Tenths of an Inch. Then the focal Distance after this third Refraction will be found (by 1331) to be 6,1 Tenths from the hinder Surface of the *Cornea*.

1403. Now the Distance from the hinder Part of the Crystaline to the Bottom of the Eye, or *Retina*, is nearly equal to that focal Distance; and *therefore all Objects at a great Distance have their Images formed on the Retina in the Fund of the Eye*, and

‡ In 1328 for diverging, read converging.

and thereby *diſtinct Viſion* is produced by this *Organ of Optic Senſation.*

1404. When the Diſtance of Objects is not very great, the focal Diſtance, after the laſt Refraction in the *Vitreous Humour*, will be a little increaſed, and to do this we can move the Cryſtalline a little nearer the *Cornea* by Means of the *Ligamentum Ciliare* (1397) and thus on all Occaſions it may be adjuſted for a due focal Diſtance for every Diſtance of Objects, excepting that which is leſs than ſix or ſeven Inches, in good Eyes. Many, I know, are of Opinion, that this is effected by a Power in the Eye to *alter the Convexity* of the Cryſtalline Humour as Occaſion requires, but this does not very eaſily appear.

1405. By what has been ſaid, it appears that Rays of Light flowing from every Part of an Object O B, placed at a proper Diſtance from the Eye, will have an Image I M formed thereby on the *Retina* in the Bottom of the Eye; and ſince the Rays O M, B I, which come from the extreme Parts of the Object, croſs each other in the Middle of the Pupil, the Poſition of the Image I M will be contrary to that of the Object, or *inverted*, as in the Caſe of a Lens (1385.)

1406. The *apparent Place* of any Part of an Object is in the Axis and conjugate Focus of that Pencil of Rays by which that Part or Point is formed the Image. Thus O M is the Axis, and O the Focus proper to the Rays by which the Point M in the Image is made; therefore the *Senſation of the Place* of that Part will be conceived in the Mind to be at O; in like Manner the *Idea of Place* belonging to the Point I, will be referred, in the Axis I B, to the proper Focus B, therefore the *apparent Place* of the whole Image I M will be conceived in the Mind to occupy all the Space between O, B, and at the Diſtance A L from the Eye.

1407. Hence likewiſe appears the Reaſon *why we ſee an Object upright by Means of an inverted Image*; for ſince the apparent Place of every Point M will be in the Axis M O at O; and this Axis croſſing the Axis of the Eye H L in the Pupil, it follows, that the ſenſible Place O of that Point will lie, without the Eye, on the contrary Side of the Axis of the Eye to that of the Point in the Eye; and ſince this is true of all other Parts or Points in the Image, 'tis evident the Poſition of every Part of the Object

will be on the contrary Side of the Axis to every correſponding Part in the Image, and therefore the whole Object O B will have a *contrary Poſition to that of the Image* I M, *or appear upright.*

1408. The *Dimenſions, or Magnitude*, of an Object O B, we judge of by the Quantity of the Angle O A B which it ſubtends at the Eye. For if the ſame Object be placed at two different Diſtances L and N, the Angles O A B and *o* A *b*, which in theſe two Places it ſubtends at the Eye, will be of different Magnitude; and the *lineal Dimenſions* (*viz. Length* and *Breadth*) will be at N and at L as the Angle *o* A *b* is to the Angle O A B. And the *Surfaces* and *Solidities* of the Objects will be as the *Squares* and *Cubes* of thoſe Angles (670, 675, 1387).

1409. It is found by Experience, that two Points O, L, in any Object will not be diſtinctly ſeen by the Eye till they are near enough to ſubtend an Angle O A L of *one Minute*. Hence when Objects, however large in themſelves, are ſo remote as not to be ſeen under an Angle of *one Minute*, they cannot properly be ſaid to have any apparent Dimenſions or Magnitude at all; ſuch as is the Caſe of the large Bodies of the Planets, Comets, and fixed Stars. But the *Optic Science* has ſupplied Means of enlarging this natural ſmall Angle under which moſt diſtant Objects appear, and thereby encreaſing their apparent Magnitudes to a very ſurpriſing and delightful Degree in that noble Inſtrument we call a TELESCOPE, as we ſhall elſewhere explain.*

1410. On the other hand, we find in the Creation an Infinity of Objects, whoſe Bulks are ſo ſmall, that they will not ſubtend the requiſite Angle (1409) if brought to the neareſt Limits of *diſtinct Viſion*, *viz.* 6, 7, or 8 Inches from the Eye, as found by Experience; and therefore in order to render them viſible at a very near Diſtance, we have a Variety of Glaſſes, and Inſtruments of different Conſtructions, which we uſually call MICROSCOPES, by which thoſe minute Objects appear many Thouſands, yea Millions of Times larger than to the naked Eye; and thereby enrich the Mind with Diſcoveries of the ſublimeſt Nature, in regard to *creating Power, Wiſdom, and Oeconomy.* 1411. If

* The practical Part of Optics containing the Deſcription and Uſe of TELESCOPES, MICROSCOPES, and other Optical Inſtruments, the Reader will hereafter find in the Young *Gentleman* and *Lady*'s *Philoſophy.*

1411. If the Convexity of the Cornea C D happens not exactly to correſpond to the Diameter of the Eye, conſidered as the natural focal Diſtance, then the Image will not be formed on the *Retina*, and conſequently *no diſtinct Viſion can be effected in ſuch an Eye.*

1412. If the *Cornea* be too *convex*, the focal Diſtance in the Eye will be leſs than its Diameter, and the Image will be formed ſhort of the *Retina*. Hence the Reaſon why People having ſuch Eyes are obliged to hold Things *very near* to them, to lengthen the focal Diſtances (1340) and alſo why they uſe *concave Glaſſes* to counter-act or remedy the Exceſs of Convexity, *in order to view diſtant Objects diſtinctly.*

1413. If the Eye has *leſs than a juſt Degree of Convexity*, or is *too flat*, as is generally the Caſe with old Eyes, by a natural Deficiency of the Aqueous Humour, then the Rays tend to a Point or Focus beyond the *Retina* or Bottom of the Eye; and to ſupply this Want of Convexity in the Cornea, we uſe convex Lenſes in thoſe Frames we call *Spectacles*, or VISUAL GLASSES.†

1414. Since the Rays of Light O A, B A, which conſtitute the viſual Angle O A B, will, when they are intercepted by a Lens, be refracted ſooner to the Axis, (1380) the ſaid Angle will thereby be enlarged, and the Object of Courſe become magnified; which is the Reaſon why thoſe Lenſes are called *Magnifiers*, or READING GLASSES.

† Theſe *Viſual Glaſſes* are very different from Spectacles in two Particulars; for (1.) They are made with proper Apertures to admit of no more Light than what is requiſite. (2.) They are bent to an Angle, that the Rays may fall directly and not obliquely on the Eye; both which Precautions are neceſſary for eaſy and diſtinct Viſion.

1422. It muſt ſuffice therefore to know, *that Light is the Cauſe of Colour*; and that where there is *no Light* there can be *no Colour*; and as *Darkneſs*, or *total Shadow*, is nothing more than the *Abſence*, or *Privation of Light*, ſo BLACKNESS is no other Thing *in itſelf* than a *Want* of the *natural Operation of Light*; and with reſpect to *us*, it is the *Want of all Colour* in Bodies. Hence *Black* is, properly ſpeaking, *no Colour at all.*

1423. Since all the various *colour-making Rays r* E, *o* E, *y* E, *&c.* before they are ſeparated by the Priſm A B, compound one common Beam of Light, whoſe Colour is *White*; we may eaſily thence infer that WHITENESS is not a ſimple Colour, but only the Reſult or Compound of all the ſimple Colours before mentioned (1419) blended together.

1424. Therefore ſuch Bodies which imbibe all the Light incident upon them, or reflect none, will appear abſolutely *black*, or colourleſs. And thoſe which reflect all the Light which falls on them, will appear *White*; and the ſame in regard to Refraction.

1425. But ſuch Bodies as reflect or refract one ſimple Sort of Light only, will appear of the Colour peculiar to that homogeneous Ray; thus if any Object reflects or refracts only the Rays E *r*, its Colour will be *Red*; if the Rays E *g*, it will be *Green*; and the Ray E *v* will, when reflected or refracted alone, ſhew the Object of a *Violet-colour* (1419) and ſo of the reſt.

1426. Again, if Bodies reflect one ſimple Colour, and refract another, they will appear of one Colour by Reflection, and another by Refraction; as is the Caſe of *Leaf-gold*, Decoction of *Lignum Nephriticum*, *&c.*

1427. Rays of Light are alſo *differently reflexible*, and thoſe which are *moſt or leaſt refrangible*, are alſo *moſt or leaſt reflexible*; therefore Bodies will appear of *different Colours in the ſame Part*, if made to receive the Beam of Light under ſuch Angles of Incidence as are proper to each reſpective Sort of Rays, for a given Poſition of the Object and the Eye.

1428. Thoſe Objects which reflect or refract two or more of the homogeneal Rays will appear of a Colour compounded of them; and it is obſervable, that of three different Rays (next to each other) the two extreme ones produce nearly the Colour of the middle One; thus *Red* and *Yellow* made an *Orange*; *Yellow* and *Blue* make a *Green*; *Blue* and *Purple* make an *Indigo*-

colour;

colour; and from hence all the *Phænomena* of *Colours* in natural Bodies ariſe, and are eaſily confirmed by Experiments.

1429. The Image of any Object, formed by reflected or refracted Light, muſt be of the ſame Colour in every Part with the Object; for the Rays of Light which proceed from the ſeveral Parts of the Object are not altered or changed in their Nature by Reflection or Refraction; and therefore whatever Colour they excite in the Object, the ſame muſt they ſhew in the correſponding Part of the Image; the whole Image will therefore be variegated and painted with the ſame Colours in every Reſpect as we view in the Object itſelf.

1430. I think, then, it deſerves to be conſidered, that every PICTURE formed by an *Optic Glaſs*, ought to be looked upon as the *Portrait of Nature itſelf*, and conſequently deſerves a much greater Regard than we uſually pay to it. ——— The Performance of *Titian*'s Pencil being as much inferior to the PAINTINGS of NATURE, as a *created Being* is below the CREATOR.

1431. For the ſame Reaſon that a Priſm ſeparates the Beam of Light into its original or ſimple Rays, ſo likewiſe does a *Lens*, *viz.* becauſe its Sides are inclined to each other (1369) and therefore the Image in the Focus of a ſingle Lens muſt be as compounded as Light itſelf, and conſequently in ſome Meaſure confuſed; for each particular Species of Rays does in Reality form a diſtinct Image in its own peculiar Focus. And of Courſe, when this compound Image is viewed with a deep Magnifier, it will appear both coloured and confuſed, as we find by Experience in all our Microſcopes, Teleſcopes, *&c.* of the *refracting Sort*.

1432. But Images formed by *reflected Light* are not ſubject to either of thoſe Imperfections, becauſe there is no different Reflection of Light while the Angles of Incidence are the ſame; and therefore only one ſimple Image is formed in the Focus of a *Speculum*; and ſo perfect, that it will bear to be magnified a *ſecond Time* with ſufficient Diſtinctneſs; and conſequently a *double Power of magnifying* in a REFLECTING TELESCOPE will have the ſame Effect in a ſmall Length as we have in a very great Length by Refraction; and this is the Reaſon of that noble Invention.

1433. There

1433. There have been Methods invented to remedy, or rather to palliate, the Imperfections of a refracting Teleſcope, which I have conſidered at large in my *New Elements of Optics*, to which I refer the *Reader*, as being a Subject too prolix for a *Syſtem of elementary Principles* only. In that Treatiſe I preſume, it is demonſtrated, that as there are two Defects in refracting Teleſcopes, *viz.* one from the different Refrangibility of Rays, and the other from the Figure of Glaſſes, ſo the very Means of correcting the former will inevitably augment the latter; ‖ and in thoſe very Teleſcopes where this Correction has been applied, by *joining a Concave with a convex Object Glaſs*, the Rays are afterwards made to paſs thro' a ſingle convex Lens before the Image is formed, and therefore if the Colours were taken away by the two Glaſſes, they muſt be again produced by the firſt of the five next the Eye. And therefore we have not yet any ſuch Thing as an *achromatic* Refractor, or one that will ſhew Objects *entirely free from Colours*.

1434. But as in the abovementioned Treatiſe I had omitted one or two Particulars, relative to this new Refractor, and alſo for the Sake of the Inquiſitive, I ſhall here give the following Diſſection of the compound Object Glaſs as I found it in one of thoſe Teleſcopes I purchaſed for 3 ½ Guineas, and was three Feet long.

1435. The *convex Lens* was of *Crown-glaſs*, doubly and equally convex on both Sides; and its ſolar focal Diſtance was preciſely 9 ½ Inches. The *concave Lens* was of *White-flint*, or Cryſtal; it was a Plano-concave, and its focal Diſtance by Reflection was four Inches from its Surface. The focal Diſtance of both theſe combined together, was juſt 29 ½ Inches.

1436. The Radius of the convex Lens was 10,1 Inches, as will appear from (1340,) for in Crown-glaſs, $a = 0{,}532$, and $2a = 1{,}064$; then $1 : 2a :: f : r$, or $1 : 1{,}064 :: 9{,}5 : 10{,}1$. But with regard to the Plano-concave of White-flint, ſince the Radius is double the ſolar Focus by Reflection, (1291) it is in that 8 Inches. The two Radii, therefore, in theſe two Object Glaſſes, are as 10,1 to 8, or as 160 to 127 nearly: That is

‖ This is to be underſtood of a convex and a concave Lens of the *ſame Sort* of Glaſs; and how little the Caſe will be altered, by having one of *Cryſtal* and the other of *Crown-glaſs*, will appear bye and bye.

is (if R be put for the Radius of the Plano-concave) $r : R :: 160 : 127$.

1437. Now it is known by Experience, and is by all confeſs'd, that two Lenſes, one a Plano-convex of *Crown-glaſs*, and the other a Plano-concave of *Flint*, muſt have their Radii of Sphericity very nearly as 2 to 3, in order to prevent the Error of Refraction ariſing from the different Refrangibility of the ſame Beam of Rays, and that in ſuch a Caſe, we have $R : r :: 3 : 2 :: 8 : 5{,}35$; therefore in a double and equally convex Lens of Crown-glaſs to produce the ſame Effect, the Radius muſt be $r = 10{,}7$; but that in the Teleſcope is only 10,1 and therefore too convex to prevent a *coloured Image*, when compounded with a Plano-concave of Flint whoſe Radius is 8 Inches.

1438. Then if the Sine of Incidence be to the Sine of Refraction (of the ſame Ray) out of Crown-glaſs into Air as n to m, and out of White-flint into Air as n to M; it is demonſtrated by Sir *Iſaac Newton* * the Error of Refraction ariſing from the Figure of the Lens is $\frac{m^2 y^3}{4 n^2 r^2}$ in Crown-glaſs, or $\frac{M^2 y^3}{4 n^2 R^2}$ in White-flint, in a Lens of a *Plano-convex* Form. In theſe Expreſſions, (y) is the Semi-aperture of the Lens.

1439. Becauſe there is the ſame Refraction, and Error from thence ariſing, in an equally Plano-concave Lens, and being made the contrary Way, therefore when the Errors are equal in a Plano-convex of Crown-glaſs, and a Plano-concave of White-flint, they will deſtroy each other; or when two ſuch Lenſes are combined together they will correct each other, and prevent any Aberration of Rays from the Figure.

1440. Therefore let $\frac{m^2 y^3}{4 n^2 r^2} = \frac{M^2 y^3}{4 n^2 R^2}$, from whence we have $\frac{m^2}{r^2} = \frac{M^2}{R}$, or this Analogy $M : R :: m : r$; but by Experiments it appears, that $M : m :: 160 : 153$ in Crown and Cryſtal. And therefore ſuppoſing, the Radius R of a Plano-concave to be 8 Inches, (as in the abovementioned Teleſcope) then $160 : 153 :: 8 : 7\frac{3}{4} = r$, the Radius of a Plano-convex Crown-lens that combined with the other, ſhall prevent any Error from the ſpherical Figure.

* See alſo *Philoſophia Britannica*, 2d Edit.

1441. Now ſince the Effect or Refraction is the ſame in a double and equally convex Lens, of a double Radius, *viz.* $r = 15\frac{1}{2}$ Inches; therefore ſuch a Lens only, combined with a Plano-concave of White-flint, whoſe Radius is 8 Inches, can cauſe the Aberration of Rays from the Figure of the Glaſſes to vaniſh. But that in the Teleſcope has a Radius of $10\frac{1}{10}$ Inches only; whoſe Sphericity is therefore much too great to anſwer this Purpoſe.

1442. Therefore ſince in a Plano-convex of Crown-glaſs, and a Plano-concave of White-flint, the *Radii* for preventing the Error from the different Refrangiblity of Rays muſt have the Ratio following, *viz.* $R : r :: 3 : 2 :: 160 : 107$; and for annihilating the Error from the Figure, the Ratio muſt be $R : r :: 160 : 153$; it appears to be impoſſible, that the ſame two Glaſſes, *viz.* Crown and Cryſtal, which correct one Error, ſhould at the ſame Time correct the other; and farther it appears, that the Ratio of the Radii of the Object Glaſſes in the Refractor under Conſideration, being in a Ratio different from either of theſe (1436), can correct neither of the Errors.

1443. But we are told, "*That the Surfaces of ſpherical Glaſſes admit of great Variations tho' their focal Diſtances be limited.* With regard to parallel Rays on a Plano-convex, or Plano-concave, we have ſhewn (1341, 1345.) the focal Diſtance is $f = \frac{r}{a}$; and ſo $r = f\,a$; but (f) is limited or given by Suppoſition, and (a) is the Refraction of the particular Species of Glaſs (1347) and therefore is given of Courſe; how then does it appear, that the Radius (r) or ſpherical Surface of the Glaſs can admit of ſuch *great* (or indeed any) Variation at all? 'Tis true, the ſame Degree of Sphericity may be divided and variouſly proportioned between the two Surfaces, but while the focal Diſtance is limited, the Refraction, and the Error occaſioned thereby, will be ſtill the ſame.

1444. Therefore the "*Poſſibility of making the Aberrations of any two Glaſſes equal*, is a Thing that does not appear;" no more than how, a "*Perfect Theory for making Object Glaſſes can be obtained*, from any Principles of Optics hitherto publiſhed." A Demonſtration of theſe great Poſitions is a Satisfaction we have

yet

yet to come, and which the Public has long, and with great Impatience, waited for.

1445. In the mean Time it may not be unacceptable to many Perſons to be informed and directed how to make this compound Object Lens for their own Uſe; and ſo procure at an eaſy Rate a Teleſcope, which they have been told is of *infinite Service to Mankind.* For this Purpoſe, *take* a *double Convex* and a *Plano-convex* of Crown-glaſs, and a *double Concave of White-flint*, all ground upon the ſame Tool, and let the Concave be put between the two Convexes, and place them in the End of the Teleſcope with the Plano-convex outwards, and they make the *triple compound Lens for taking away Colours* in Refractors of a ſmall Length.

1446. But if the Teleſcope is to exceed the Length of 18 or 24 Inches, then two Glaſſes will do, *viz.* one Convex of Crown-glaſs, and the other Concave of Flint, whoſe ſolar focal Diſtances are to each other as 3 to 4 † very nicely; and they will form an Image without Colours, for the Teleſcope propoſed. If theſe two Lenſes are held together in the Sun-beams, they will converge them to a Focus, and thereby ſhew the focal Diſtance of the compound Object Glaſs, and conſequently of the Teleſcope itſelf.

1447. But as to the Error from the Figure of the Glaſſes, I confeſs it is not in my Power to give any Directions for preventing or extenuating the ſame in any great Degree. If any Perſon can find among the different Sorts of Glaſs, or any tranſparent Mediums, any two, which being formed into Priſms, *ſhall have their refracting Angles, which take away Colours, reciprocally proportioned to their Sines of Refraction into Air, reſpectively*; then he may be aſſured of a *perfect Theory of making Object Glaſſes.* —— And he that ſhall do this, *erit mihi* pluſquam *Magnus Apollo.*

† For let $\frac{r}{a} = f$ in a Plano-convex of Crown-glaſs, and $\frac{R}{b} = F$, in the Plano-concave of White-flint. Then $f : F :: \frac{r}{a} : \frac{R}{b} :: \frac{2}{0{,}53} : \frac{3}{0{,}6} :: 0{,}376 : 0{,}5 :: 3 : 4$, as in the Preſcript above.

INSTITUTIONS OF PERSPECTIVE:

CONTAINING

The *Mathematical* THEORY thereof deduced from OPTICAL PRINCIPLES, and applied to a *General* PRAXIS.

CHAP. I.

The Optical ELEMENTS *of* Lineal PERSPECTIVE.

1448. WE are now prepared to treat of the *Elements*, or *firſt Principles* of PERSPECTIVE, the moſt delightful and neceſſary of all the Mathematical Sciences. Theſe Elements are immediately derived from *Optics*; for *the Art of delineating Objects as they appear on a given tranſparent Plane, or Superficies, to an Eye at a given Height and Diſtance*, is the true DEFINITION of PERSPECTIVE.

1449. The Want of theſe preliminary Principles has rendered the Treatiſes on this Subject defective in the moſt eſſential Part, and the Art of Perſpective itſelf very difficult to be underſtood: In ſhort, it would be abſurd to ſuppoſe any Man can underſtand Perſpective, without being acquainted at leaſt with as much of the *Optic Theory* as we have premiſed, and ſhall here ſuperadd in this Chapter.

1450. It appears by the above Definition, that in order to delineate the true Appearance of an Object on a given Plane, it will be firſt neceſſary to know the Law according to which the apparent linear Dimenſions of Object increaſe or decreaſe; and here we muſt obſerve (1.) That the viſual Angle, or the apparent Magnitude of a Line will be leſs at a *greater Diſtance*; and *vice verſa*. (1408) (2.) That it will be leſs as the ſaid Line is viewed

viewed *more obliquely*. (3.) Therefore the Law of Diminution will be nearly *in Proportion to the Diſtance and Obliquity of the View conjointly*.

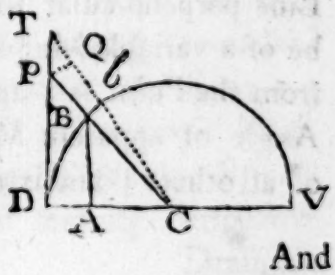
1451. It is true, the Diminution of any Line viewed directly increaſes with the Diſtance, but not exactly in Proportion thereto, unleſs the Diſtances be very great. Let O B (Fig. 19) be an Object viewed directly by an Eye at C. On the Center C, with the Radius C B deſcribe the Semi-circle A D B, and join C O; then will O C B be the Angle under which the Object O B appears at C. Again, ſuppoſe the Eye at A, and draw A O; then the Angle O A B is the viſual Angle under which it appears at A. Now A B is double the Diſtance B C, but the Angle O A B is *more than half* the Angle O C B; for the Angle O A B or E A B is juſt half the Angle O C B or E C B (642.) But at great Diſtances, the differential Angle E A O will become inſenſible; and therefore in ſuch Caſes, *the Diminution will be directly as the Diſtance of the Object*.

1452. When any Object as B D (Fig. 23) is viewed obliquely (that is, when the Angles A D B and A B D are not equal) then a Diminution of its apparent Magnitude will enſue; for in all ſuch Caſes, the Length B D is reduced to B O, which ſubtends the viſual Angle D A B at the Diſtance A B. Therefore the Propoſition is evident. (1408)

1453. It is further evident, that if the ſame Object B D were removed to twice the Diſtance of A B, then the Angle of apparent Magnitude O A B would be but one half ſo large (642); and therefore the apparent Magnitude of any Object on Account of its Obliquity, is alſo diminiſhed in Proportion to its Diſtance; and therefore the whole Diminution of Magnitude on Account of its Diſtance and oblique Poſition jointly, is proportional to the *Square of the Diſtance*.

1454. But we have a more direct Demonſtration of this fundamental Propoſition in Perſpective, in the Diagram of (823) which we ſhall here infert.

If T P be any Object viewed obliquely by an Eye at C, at the Height of C D above it, then the viſual Angle is P C T, and its Meaſure the Arch B *b* to the Radius C B, or P Q to the Radius C P.



And

And becauſe TP : QP :: CP : CD; and PQ : B*b* :: CP : CB = CD; therefore (*ex Equo* 652.) it is TP : B*b* :: CP^2 : CD^2. But when TP is a given Quantity, then (ſince Radius CD = 1) we have B*b* : $\frac{1}{CP^2}$ or *the viſual Angle* B*b* *of apparent Magnitude is ever in the inverſe Proportion of the Square of the Diſtance* CP.

1455. Hence it appears, that in any Line AF (Fig. 20.) equal Parts AB, BC, CD, DE, EF, are ſeen by an Eye at C under ſuch Angles, whoſe Meaſures *gh*, *hi*, *ik*, *kl*, *lm*, are reciprocally as the Squares of the Diſtances AC, BC, CC, CD, CE. Therefore the Circle IK may be called *the Line of Meaſures in Optical Perſpective*, or ſuch *Delineations as are made on ſpherical and cylindrical Surfaces.*

1456. If the right Line GH be parallel to the Line AF, then will the viſual Rays AC, BC, *&c.* divide the Line GH in a ſimilar Manner to that in which they divide the Line AF; that is, if the Diviſions in AF are equal, they will alſo be equal in GH; but if they are unequal in AF, they will have the very ſame Ratio of Inequality in GH. For the Triangles FCE and *f*C*e*, alſo ECD and *e*C*d*, are ſimilar; which give theſe Analogies *ef* : EF :: *e*C : EC; and *ed* : ED :: *e*C : EC; conſequently *ef* : *ed* :: EF : ED; and ſo of the Reſt.

1457. If any Objects AB, CD, (Fig. 21.) are ſeen by the Eye at C under equal Angles *ab*, and *cd*, *then will they be as the Squares of their Diſtances from the Eye directly.* For in the foregoing Figure to (1454) we had TP : B*b* :: CP^2 : CD^2; and therefore ſince in this Caſe B*b* and CD are conſtant Quantities, TP will be directly as CP^2; that is (in Fig. 21.) AB : CD :: BC^2 : DC^2.

1458. The Object BD (Fig. 22.) of a given Length, placed at a given Height AB will be ſeen by the Eye at C in a right Line perpendicular to AB, under an Angle BCD, which will be of a variable Magnitude as the Eye approaches to, or recedes from the Point A; and there is one Diſtance AC where that Angle of apparent Magnitude will be a *Maximum*, or greateſt of all others; and is determined in the following Manner.

1459. Every

1459. Every Angle P C Q (ſee Fig. to 1454) is as the Arch P Q directly, and the Radius P C inverſely or as $\frac{PQ}{PC}$. But $TC : CD :: TP : PQ = \frac{TP \times CD}{TC}$; alſo we have $PC = \sqrt{CD^2 + DP^2}$. Therefore the viſual Angle P C Q is as $\frac{TP \times CD}{TC \times \sqrt{CD^2 + DP^2}}$. That is, if (in Fig. 22) we put $Ab = a$, $BD = b$, $AD = a + b = c$, and $AC = x$; then will the Angle B C D be as $\frac{bx}{\sqrt{a^2 + x^2} \times \sqrt{c^2 + x^2}}$; then by taking the Fluxion thereof (800) and making it equal to *Nothing* (818) we ſhall get $x^2 = ac$, whence $a : x :: x : c$, or the Diſtance A C is a *geometrical Mean* between A B and A C, when the viſual Angle B C D is the greateſt poſſible.

1460. If any Line (Fig. 25) O B appears to an Eye at A, under the ſame Angle with another Line C D, then at any other Point E, the two Lines will have the ſame apparent Magnitude (however the viſual Angle C A D may vary) if the Diſtances of the Eye from each Line preſerve the ſame Ratio. For let *o b* be equal and parallel to O B, and draw C E and D E; then by ſimilar Triangles, we have A H : A K :: C H : O K; and E H : E L :: C H : *o* L = O K; therefore A H : A K :: E H : E L. Alſo a right Line paſſing from the Eye to thoſe Lines divides them in the ſame Ratio; for it is G H : H D :: O K : K B :: *o* L : L *b*.

1461. The Appearance of any diſtant Line or Object O B upon another Line or Plane C E given in Poſition, will increaſe or decreaſe with the Diſtance from the ſaid Line or Plane, tho' not in the ſame Ratio; for at the greater Diſtance A it will occupy the Length C E, but in a leſs or nearer Diſtance at D, it occupies only the Space *c e*, much leſs than before, as is evident by Inſpection.

1462. Having thus premiſed ſuch Optical Principles of *lineal Perſpective*, or the *Appearance* of LINES among themſelves with regard to their Poſitions, and Diſtance of the Eye; we ſhall now proceed to *a generally* THEORY of *univerſal* PERSPECTIVE, and demonſtrate the ſame from its genuine Principles. And tho' the common Methods of making Perſpective Draughts

Draughts are very eaſy, and in every ones Hands, yet as the judicious *Architect*, *Deſigner*, *Painter*, &c. will find it no eaſy Matter to come at a conciſe and plain *Theory*, or eaſy *Rationale* of this neceſſary or fundamental Part of his Art, in any of the Books hitherto publiſhed; it is preſumed, the following new Method of demonſtrating the Reaſon of ſuch uſeful and common Rules will not be unacceptable to the ingenious Artiſts of all ſuch Profeſſions, as require the Aſſiſtance of this excellent Science.

CHAP. II.

The THEORY *of* PERSPECTIVE *demonſtrated from the* PRINCIPLES *of* OPTICS *and* GEOMETRY.

[PLATE II. *of* PERSPECTIVE.]

1463. FROM the Definition of Perſpective, (1448) it appears, that in order to give a *Rationale* of the common Practice, we muſt premiſe a *Theory of the Viſion of Objects* on a tranſparent Plane at a given Diſtance and Height of the Eye. Thus ſuppoſe the Eye of the Spectator at I, and E F G H the Plane on which it obſerves the Appearance of Objects, (Fig. 1.) This is called the *Perſpective Plane*.

1464. As the Appearance of Objects will be variable according to the Situation of the Planes in which they are poſited in regard to the *Perſpective Plane*; theſe particular Situations of Object-planes are principally to be conſidered. And, firſt, it is evident, that any Plane paſſing thro' the Eye cannot be ſeen as a *Plane*, but as a *Line* only on the perſpective Plane; for the Eye having no Elevation above ſuch a Plane, can ſee no Part of its Surface; the Edge or bounding Line of ſuch a Plane being all the Appearance it can have to the Eye.

1465. Of theſe *Ocular Planes*, there are two of principal Note, *viz.* the *horizontal Plane* O K L M parallel to the Horizon; and the *vertical Plane* N R Q P, which is perpendicular thereto. The firſt interſects the perſpective Plane in the Line I K,

The THEORY of PERSPECTIVE

Plate II.

Fig. 1

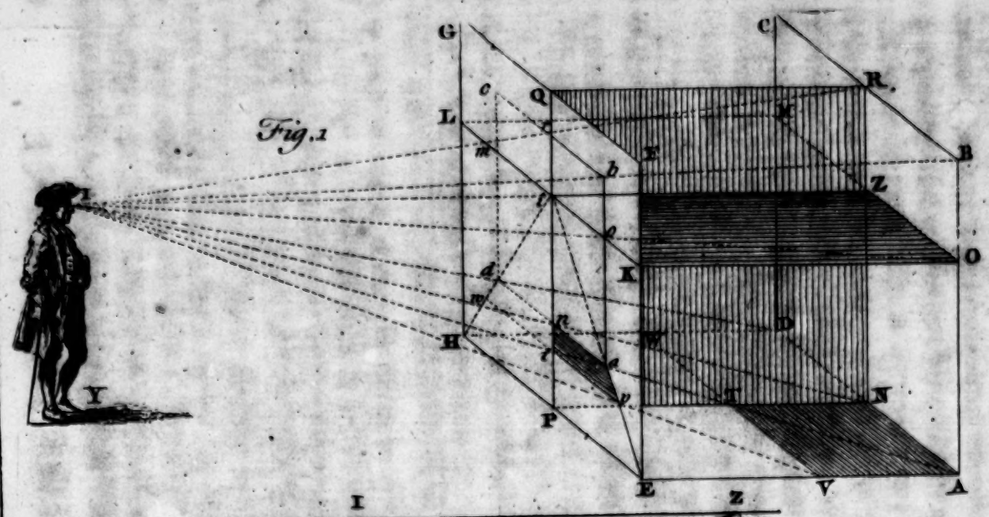


Fig. 2

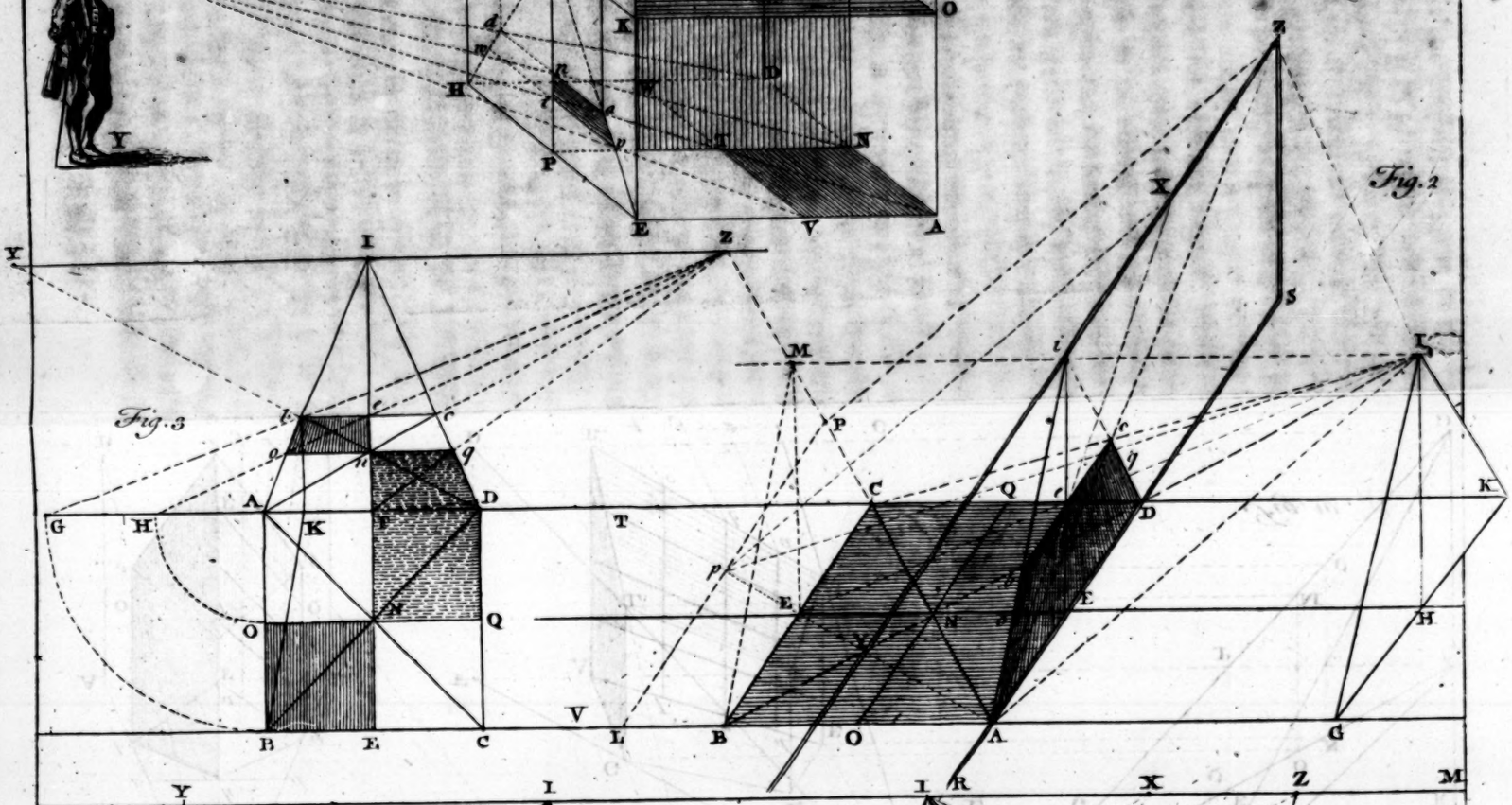


Fig. 3

Fig. 4

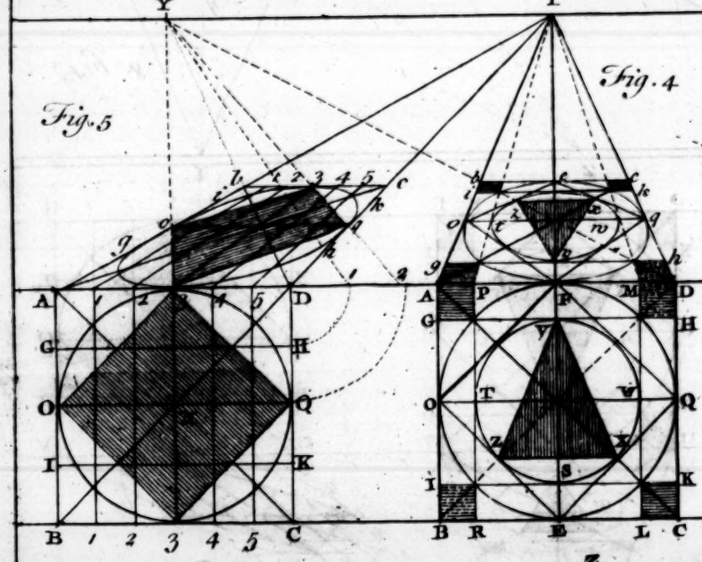


Fig. 5

Fig. 6.

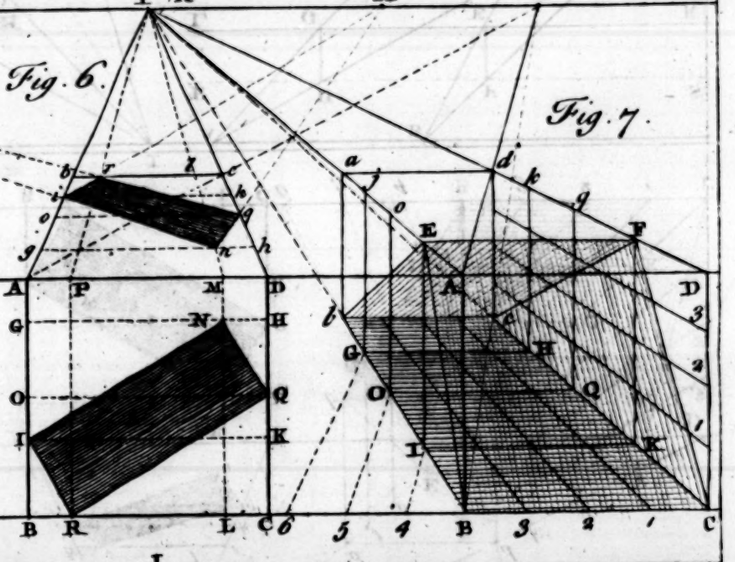


Fig. 7.

Fig. 8.

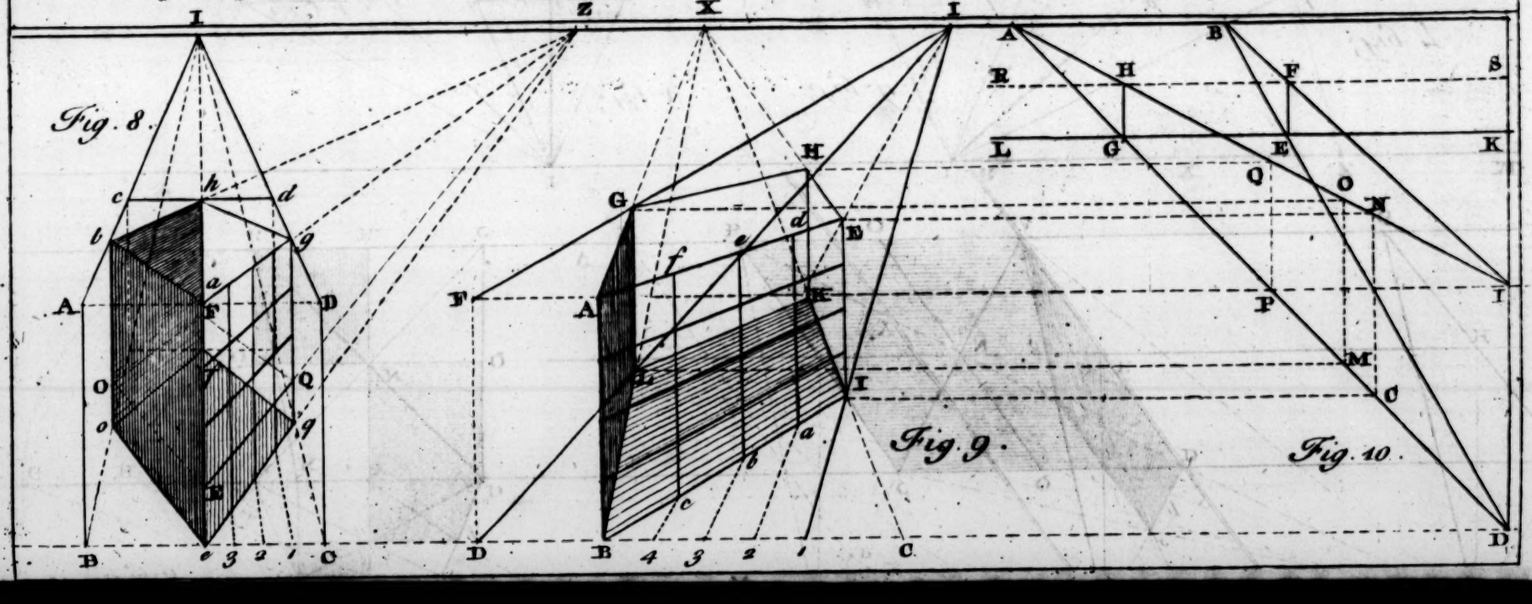


Fig. 9.

Fig. 10.

LK, which is therefore called the *horizontal Line*; and the other, in the Line QP, which is called the *vertical Line*, on the perſpective Plane.

1466. Secondly; thoſe Planes which do not paſs thro' the Eye, will have a *direct or an oblique Situation* with reſpect thereto. If it be a direct Situation, it will be parallel to the perſpective Plane which is ſuppoſed to be placed directly before the Eye. Thus the Plane A B C D is a direct one, and parallel to the perſpective Plane H F. And among all Planes ſituated obliquely to the Eye, that which is in the Plane of the Horizon, as A E H D, is the moſt conſiderable; and is called the *Ground Plane*. This is perpendicular to the perſpective Plane H F.

1467. From what has been ſaid, it appears, that Objects in the Surfaces of the *horizontal and vertical* Planes cannot be ſeen at all by the Eye at I; and therefore they are not to be regarded in Perſpective. We muſt therefore conſider of Objects in and upon the direct and oblique Planes, and the Appearance they make on the perſpective Plane.

1468. Thus let O B be an Object in the direct Plane; and from the extreme Points O and B draw the viſual Rays O I, B I, to the Eye at I. They will paſs thro' the perſpective Plane in the Points *o* and *b*; and by joining thoſe Points with the right Line *o b*, that Line will be the Picture of the Line or Object O B, upon the perſpective Plane. Thus alſo the Perſpective of the Line O A is *o a*; and the Proportion of Objects and their perſpective Appearances is the ſame in this direct View, *viz.* that of their Diſtances from the Eye; for O B : *o b* : : O A : *o a* : : O I : *o* I, by ſimilar Triangles.

1469. Thus (*b r*) and (*a n*) will be the Perſpectives of the Lines B R and A N in the object Plane; and *r b a n* will be the Perſpect of R B A N of half the Plane A C, and *a b c d* the Perſpective of the whole Plane A B C D. And all Lines parallel to A B or C B in the object Plane, will have their perſpective Lines parallel to *a b* and *c d* in the Picture on the perſpective Plane. And in what Manner ſoever the object Plane A C is divided by Lines drawn upon it, their Repreſentatives will divide the Picture (*a c*) in a ſimilar Manner. (1456.)

1470. Any Point B in a direct Plane has the ſame Ratio of Diſtance from the *horizontal and vertical Planes*, as its perſpec-

tive (*b*) has from the *horizontal and vertical Lines*, *viz.* that of the Diſtances of the Planes from the Eye. For B R : *br* : : B O : *bo* : : I O : I*o* : : I*z* : I*i*, from the Nature of ſimilar Triangles; hence it is eaſy to delineate the Appearance of any Objects, or form their Pictures on the perſpective Plane when they are preſented in a direct View; as in the Front of Buildings, *&c.* as will be exemplified hereafter.

1471. The laſt Sort of Plane, in and upon which we are ſuppoſed to view Objects, is that of the *Horizon* itſelf, as A D H E, above which the Eye has an Elevation, more or leſs, as *i* P = I Y. This is therefore called the *ground Plane*, and its Interſection H E with the perſpective Plane, is called, the *ground Line*. And this is the moſt conſiderable of all others, as being the common Table or Plan of all perſpective Views, Landſcapes, and pictureſque Draughts of every Kind.

1472. With regard to this horizontal Plane, it has been ſhewn (1469) that the two remote Angles thereof A and D are repreſented by *a* and *d*, in the perſpective Plane; and the other two Angles E and H, are in the ſaid Plane alſo, as being common to both; therefore by drawing the Lines *a* E and *d* H, there will be formed the Figure E and H on the perſpective Plane which will be the true perſpective Appearance of the *ground Plane* A D H E.

1473. Hence it follows, that *a* E is the Perſpective of A E, *n* P of N P, and *d* H of D H. Whence it appears that Lines which are parallel in the ground Plane, and perpendicular to the perſpective Plane, are not ſo in their perſpective Picture, but they all converge to a Point *i*, which is called by the *Point of Sight* in the perſpective Plane as being exactly oppoſite to the Eye, or that Point in which a Perpendicular from the Eye falls on the Plane.

1474. In the ground Plane draw V W parallel to A D; its perſpective *v w* will be parallel to *a d*, in the Picture; and *a d w v* will be the Perſpective of the Part A D W V in the original Plane. But for a Demonſtration of what relates to forming the Picture or perſpective Appearance of the ground Plane and Objects upon it, and conſequently, of the common Practice of Perſpective, we muſt have Recourſe to the following Method of Repreſentation, which is the moſt natural, conciſe, and perſpicuous, of any I have been able to think of.

1475. Let

1475. Let A B C D be a right-lined Figure in the ground Plane V G K T, contiguous to and at right Angles with the perſpective Plane Y Z S R; F H the Diſtance of the Plane; and H I the Height of the Eye at I. H E is parallel to G B or C K and biſects A D and B C in the Points F and E. On the Point E raiſe the perpendicular E M = H I, and draw the Lines B M, C M, G I, K I.

1476. Draw the viſual Lines I A, I B, and I M, which is called the *principal Ray*, and is perpendicular to the perſpective Plane in the Point *i*. Then it is evident the Plane I G B M interſects the perſpective Plane in the Line A *i*, and the Ray B I being in the ſaid Plane G M, muſt interſect the Line A *i* in ſome Point (*b*) which is therefore the Perſpective of the Point B; and of Courſe A *b* is the Perſpective of the Line A B.

1477. In like Manner it is ſhewn, that as the Plane I K C M interſects the perſpective Plane R Z in the Line D I; and the Ray I C being in that Plane and interſecting the Line I D in the Point (*c*), that Point (*c*) will be the Perſpective of the Point C; and D *c* the Perſpective of the Line D C. And joining the Points *b*, *c*, the Line *b c* will be the Perſpective of the Line B C in the ground Plane.

1478. Let A B = D C, then B C will be parallel to A D, and ſince in this Caſe A *b* = D *c*, therefore *b c* will be parallel to A D alſo. And hence it appears, *that all right Lines, as* B C, *in the ground Plane which are parallel to the ground Line* A D, *will alſo be parallel to the ſame in their Picture on the perſpective Plane.*

1479. Hence alſo it is evident, *that the Perſpectives* A *b*, F *e*, D *c*, *of all Lines* A B, F E, D C, *which are perpendicular to the ground Line* A D, *do converge or tend to the Point of Sight* (*i*) *in the perſpective Plane.*

1480. If the Line A B be continued out *in Infinitum*, towards V, then ſuppoſing the Point B to move along that Line continually, the viſual Ray B I, will keep riſing on the Plane I G B M towards I M making the Angle B I M ſtill leſs and leſs, till the Point B arrives to an infinite Diſtance, and then the Ray I B will coincide with I M; and conſequently the Line A *i* will be the Perſpective of A B continued to an infinite Length. Thus alſo D *i* will be the Perſpective of the Line D C infinitely continued towards T. *And therefore the Triangle* A *i* D *will*

on the perſpective Plane be the Picture, or true Perſpective, of the Plane A B C D *continued out upon the Plane of the Horizon to an infinite Length.*

1481. *Hence the Line* Y *i* Z, *is the Perſpective of the Horizon, or Boundary of the Sight at an infinite Diſtance*; and therefore all Objects on the Plane of the Horizon, will, in their perſpective Appearance, or *Landſcape*, keep riſing from the ground Line, or Baſe A D, towards the Point of Sight (*i*); and leſſen in their Dimenſions as they are more remote, till at laſt they vaniſh in the *horizontal Line* Y Z.

1482. We have ſeen how Lines parallel or perpendicular to the ground Line A D are to be delineated or drawn in Perſpective; and we are next to ſhew how thoſe right Lines appear, or are to be drawn upon the perſpective Plane, which lie oblique to, or make any Angle with the ground Line A D, or any other parallel to it. In order to this, make A L = A G = I *i*, and draw A *p* to make any Angle *p* A R or *p* A D with the Baſe A D, acute or obtuſe. Then in the horizontal Line Y Z, take *i* X = L *p*; and draw *p* X and I X; and the Plane I X *p* A will interſect the perſpective Plane in the Line A X. Draw the viſual Ray I *p* which as it is in the Plane I X *p* A muſt go thro' the perſpective Plane ſomewhere in the Line A X, which ſuppoſe at (*r*), then is the Point (*r*) the Perſpective of (*p*); and ſince while the Point *p* is ſuppoſed to paſs from A to *p* in deſcribing the Line A *p*, its Perſpective (*r*) will move in the Plane A Z from A to *r*, and deſcribe the Line A *r*; which therefore will be the Perſpective of the Line A *p*.

1483. If the Line A *p* were continued out to an infinite Length, and the Point (*p*) ſuppoſed to move conſtantly therein, its perſpective (*r*) will appear to move towards X, till at length the Point (*p*) being at an infinite Diſtance, the Point (*r*) arrives at, and coincides with X, in the horizontal Line; *the Line* A X *is therefore the Perſpective of the Line* A *p infinitely continued*; and X is called the *accidental Point, to which the Perſpectives of all Lines parallel to* A *p tend.*

1484. Let L P be taken equal to A L; and *i* Z equal to *i* I, and then joining A, P, and I, Z; we have the Triangles A P L, and *i* Z I, equal to each other; then will the Plain I A P Z interſect the preſpective Plane in the Line A Z, which

will

will be the Perſpective of the Line A P continued out to an infinite Diſtance.

1485. But ſince A L = L P, and L P is parallel to A D; therefore A P is the *Diagonal of a Square*, and contains an Angle D A P of 45 Degrees with the ground Line A D; *therefore the Point of Diſtance* Z *is that to which all Rays parallel to* A P *tend in the perſpective Plane.*

1486. Let A B = A D, then is A B C D a geometrical Square, and its Diagonal A C, of which the Perſpective is A *c*; and the Point (*c*) is therefore that in which the perſpective diagonal A Z interſects the Ray or radial Line *i* D. Make *i* Y = *i* Z, = *i* I; and join D Y, then will that Line D Y be the perſpective Diagonal of D B, (the other Diagonal of the Square A C) infinitely continued, and D *b* the Perſpective of the Diagonal D B determined by the Interſection of the Lines D Y and A *i*, as before.

1487. Thus it is demonſtrated that A *b c* D on the perſpective Plane A S Z Y is the true *Picture or perſpective Delineation* of the *original Square* A B C D, on the ground Plane, as required. And from hence is deduced the *Rationale* of the common Method of drawing the Perſpective of a given Square or any Figure inſcribed therein; as alſo of a ſtill more univerſal Method of aſſigning the Perſpective of any Point, Line, Superficies, or Solid, on the Plane or perſpective Table, which we ſhall next proceed to explain.

CHAP. III.

The RATIONALE *of the* common METHODS *of drawing the* PERSPECTIVES *of* OBJECTS, *explained from the preceding Theory.*

[PLATE II. *of* PERSPECTIVE.]

1488. THE common Methods of *Drawing in* PERSPECTIVE are much better known than the Reaſon of them; we ſhall here give both together, and then our Endeavour will at leaſt have the Face of Novelty. Let T G (Fig. 3.) be

be the *Baſe* or *ground Line*, in any *perſpective Plane* or *Table* T G Y Z. The Method of drawing the *perſpective Appearance* of any Object on this Table is as follows.

1489. *Let it be required to draw the Perſpective of the given* SQUARE A B C D *and all its Parts, in a front View.* The Square is ſuppoſed to be contiguous to the Perſpective Table, and therefore its Side A D will be in the ground Line. As the View is direct, or in Front, the Eye muſt be ſuppoſed directly againſt the Middle of the Side A D; and therefore in the Perpendicular E F continued out, take F I equal to the given Height of the Eye, and (I) will be the *Point of Sight* (1476) ſet off the given Diſtance of the Eye from the Table each Way from I, to Y and Z; and theſe will be the *Points of Diſtance* (1486) in the horizontal Line Y Z.

1490. From the Points A and D draw the *Radials* I A, I D; and the *Diagonals* A Z, and D Y, interſecting the *Radials* in the Points *b*, and *c*; then draw *b c*; and the Figure A *b c* D is the Perſpective of the given Square A B C D as required. For A*c* is the Perſpective of the Diagonal A C, and D *b* that of D *b* (by 1486); therefore A *b* and D *c* are the Perſpectives of the Sides A B and D C (1479) and *b c* of the remote Side B C (1477.) Alſo the Point *n* is the Perſpective of N the Interſection of the Diagonals in the Square; and *o q*, F *e* are the Perſpectives of O Q and F E which biſect the Square orthogonally, (1478, 1479.)

1491. It is not neceſſary to make uſe of more than one Point of Diſtance, Z; becauſe one Diagonal A Z, gives the Point (*c*) in the Radial I D which determines the Perſpective of the Diagonal A C; and then by drawing thro' the Point (*c*) the Line *b c* parallel to the ground Line A D, the Perſpective of the Square A B C D is compleated as before (1477.)

1492. From hence we may derive a Demonſtration of an *univerſal Rule* for putting all Objects into Perſpective from given Point therein. The Rule is this.

From the given Point in the ground Plane, let fall a Perpendicular on the ground Line, from whence draw a Radial to the Point of Sight; then from the Radial ſet off that perpendicular Diſtance in the ground Line, and from thence draw a Diagonal to the Point of Diſtance, the Interſection of the Radial and Diagonal will give the Seat

or

or Place of the Perſpective of the given Point. And thus the Perſpectives of any Number of Points may be found, which being connected will form the Perſpective of the given Figure which they terminate.

1493. To exemplify this Rule; let C be any given Point in the ground Plane, from which let fall the Perpendicular C D to the ground Line T G; and from the Point D draw the Radial D I to the given Point of Sight I. Then in the ground Line make D A = D C, and from the Point A draw the Diagonal A Z to the given Point of Diſtance Z, which will croſs the Radial in the Point (*c*) the Perſpective of the Point C as required. This is evident from the common Method beforegoing (1476), becauſe A C is the Diagonal of a Square.

1494. Suppoſe B any other given Point; let fall the Perdicular B A, and draw the Radial A I; then make A G = A B, and drawn the Diagonal G Z, it will alſo interſect the Radial in the Point (*b*) the Perſpective of B as required. To prove which, let us ſuppoſe B C = A D = D C = A B. Then joining (*b c*) that muſt be the Perſpective of B C, the ſame as before determined by the commen Method; and ſo it will prove. For in the ſimilar Triangles A I D, *b* I *c*, we have A D : *b c* :: I A : I *b* (656,) according to the common Method; alſo from the ſimilar Triangles Z G A, Z *b c*, we have Z G : Z *b* :: A G : *b c*, according to the Rule; but from the ſimilar Triangles *b* G A and *b* Z I, we have I A : I *b* :: Z G : Z *b*; and therefore it is A D : *b c* :: A G : *b c*; but A D = A G, by ſuppoſition; therefore the Line (*b c*) is the ſame in both Caſes.

1495. Hence we have eaſy Methods of drawing the Perſpectives of any ſuperficial Figures whatſoever; for if they are circumſcribed with a Square, and Lines drawn thro' their terminating Points and Angles to the Sides of the Squares, then by taking the Perſpective of the Square you will at the ſame Time have the Seat of every Point or Line therein which bounds the Figure propoſed, and theſe being connected, the Perſpective of the Figure is formed as required.

1496. Thus ſuppoſe any Line B K were to be put into Perſpective, then one Extreme being placed on the ground Line at K, let the Angle B of the Square be placed on the other; then the Points K and (*b*) being connected in the Perſpective, will

give

give K *b*, for the Perſpective of the Line K B required. Or if from the Point B you let fall the Perpendicular B A, and make A G = A B; then drawing the Radial A I, and Diagonal G Z, they will interſect each other in the Point (*b*), which therefore is the Seat of the Point B, and ſo K *b* is of Courſe the Seat of the Line K B, according to the Rule (1492.)

1497. Since any right Line is ſo eaſily put into Perſpective, there can be no Difficulty in finding the perſpective Appearance of any given right-lined Figure. Thus the Perſpective of the Parallelogram C E F D is *c e* F D; the Perſpective of the Square D F N Q is D F *n q*, and the Perſpective of the remote Square C E N Q is *c e n q*, all determined by inſcribing them in the geometrical Square A B C D.

1498. If the Figure be not right-lined, but bounded by a Circle or ſome other Curve, yet if they are properly circumſcribed by a Square, the Interſections of the Diagonals and other Lines with one another in the ſeveral Parts of the Perimeter of the Curve, will give a ſufficient Number of Points in the Perſpective Table, thro' which to draw the Perſpective of the Circle or Curve propoſed. Thus for Example (Fig. 4), if in the Square A B C D you draw the Circles O F Q E, and T V W S; then by drawing the Diagonals of the Square and other proper Lines G H, I K, L M, R P, O Q, E F, *&c.* their Interſections in the Periphery of the Circle, being all found in the Perſpective Table, and a curve Line drawn thro' them; ſuch curve Lines are the Perſpectives of the Circles, as required.

1499. The Square O F Q E with an Angle F in Front, has its Perſpective F *o e q* at the ſame Time determined. Alſo the Triangle X V Z, is projected by the viſual Rays into its Perſpective *x v z* on the Table; and the Perſpectives of its ſeveral Parts (which are ſix *Triangles* and a *Trapezium*) are formed by the Lines which divide it; all which is evident by Inſpection of of the Figure.

1500. If it be required to delineate the Perſpective of a Square A B C D viewed obliquely (as in Fig. 5.) then the given Obliquity of the View will determine the Poſition of the Point of Sight I in the horizontal Line, and the given Diſtance being ſet off from I to Y; then if the Radials I A, I D, are drawn, and one Diagonal D Y to interſect the Radial I A in (*b*)

that

that will be the Perſpective of the Point B; *b c* drawn parallel to A D, will be the Perſpective of B C; and A *b c* D will be the Perſpective of the Square A B C D as required. The Theory (1475) being in this Caſe the ſame as in a direct View, the Practice muſt be conducted by the ſame Rules of Courſe.

1501. We muſt not quit this Subject without a Remark on the Abſurdity of ſeveral Writers on Perſpective, *viz. that tho' they give the practical Rule of drawing the Perſpective of any Superficies right, yet they place the Perſpective itſelf wrong.* Or they place that Part of it neareſt the ground Line, which is in Reality moſt remote from it; for Example (in Fig. 4.) the Perſpective *v z x* of the Triangle V Z X, they place contrary to its Poſition here, *viz.* with its Baſe *z x* towards the ground Line A D, and its Vertex *v* towards the Point of Sight I. In ſhort, they give the ſame Poſition to all the Parts of the Perſpective, which the Originals themſelves have in the ground Plane, contrary to what really appears to the Eye viewing thoſe Objects thro' a tranſparent Plane, as we have ſhewn. This erroneous Repreſentation of Gardens, Fortifications, *&c.* gives ſuch an unnatural Idea thereof as muſt be very diſagreeable to a nice Genius. Many Inſtances of this you will find in the JESUITS PERSPECTIVE. And even POZZO himſelf has puzzled his Readers with this Piece of Nonſence.

CHAP. IV.

The THEORY *of the* PERSPECTIVE *of* CIRCLES, *and* circular AREAS, *demonſtrated; with the Solution of ſome very uſeful* PROBLEMS *relative thereto.*

(PLATE I. *of* OPTICS *and* PERSPECTIVE.)

1502. AS there is ſomething very curious in the Speculation of the *Perſpective of a Circle*, and very little known, as being rarely found in Treatiſes on this Subject; I preſume it will be agreeable to the ingenious Reader, if I employ a Page or two, in order to ſet that Matter in a clear Light.

1503. In the firſt Place, then, it is to be conſidered, that the Rays which proceed from a Circle to the Eye (in any Elevation above the Plane in which it lies) do form a *Cone*; whoſe Baſe is the Circle, and whoſe Vertex is in the Eye. (625.)

1504. If this Circle be viewed thro' a tranſparent Plane parallel to the ground Plane of the Circle, then its Perſpective will be a CIRCLE alſo; becauſe in this Caſe, the viſual Cone is cut by a Plane parallel to its Baſe, and therefore the Figure of the Section will be the ſame as that of the Baſe, in any Poſition of the Eye above the perſpective Plane.

1505. If the perſpective Plane paſs thro' any Part of the Circle or Baſe of the Cone, and the Poſition be parallel to either Side of the Cone, then that Part of the Circle's Periphery which is ſeen through, or under the Plane, is projected on the ſaid Plane in the Figure of a PARABOLA, (by what has been ſaid of the Geneſis of that Curve in (740); for the Eye at F will view the Point *e* at R on the Plane, and the circular Periphery B *e* B will by the viſual Rays of the Cone be depicted in the Curve of the *Parabola* B R B, which will therefore be the true Perſpective of that Part of the Circle in this Caſe. *

1506. If the perſpective Plane paſſes thro' the circular Baſe of the viſual Cone, in a Poſition not parallel to either Side, and yet ſo as not to paſs thro' both in the ſame Cone if they were continued out beyond the Circle or Baſe, then ſo much of the Circle as could be ſeen thro' the Plane would be projected thereon in the Form of an HYPERBOLA (by 765); for the Eye at C will project the Point *e* on the Plane at V, and the circular Periphery B *e* B into the *hyperbolic Curve* B V B on the Plane, which therefore will be its perſpective Appearance.

1507. In all other Poſitions of the perſpective Plane with regard to the Sides of the Cone of Rays, the Perſpective of the Circle will be an *Ellipſis* therein (by 763.) For ſuppoſe the Plane to cut the Cone in the Line T V, then to an Eye at the Vertex L, the Point F will appear at T, and the Point P at V on the ſaid Plane; and of Courſe the whole circular Baſe will be projected in the Ellipſis T B V B, which is therefore its perſpective Appearance.

1508. If a Circle E D be placed on an horizontal Plane A E, and viewed thro' a perpendicular tranſparent Plane D G (Fig. 28.) by

* See the Figures of the conic Sections here referred to.

by an Eye at F at the Diſtance D B, and Height B F; then its Perſpective D C will be an *Ellipſis* alſo. But becauſe the viſual Cone E F D is in this Caſe in an oblique Poſition (and what they call a *ſcalenous Cone*), and as the Obliquity of the Cone, or its Inclination to the Horizon, will vary with the Diſtance and Height of the Eye, ſo it will be cut in a various Manner by the Plane D G, producing as many different Ellipſes by thoſe Sections in ſome of which the Diameter E D of the Circle will be the longer Diameter of the Ellipſe, in others it will be the ſhorteſt; conſequently between both there will be one Caſe were the *Ellipſe* will become a *Circle*, or have all its Diameters equal on the perſpective Plane.

1509. Now this will happen when the Cone E F D is cut in a ſimilar Manner by both the Planes, *viz.* the ground Plane A E, and the perſpective Plane D G, or when the Angle G D F is equal to the Angle D E F; for if we ſuppoſe the Eye to move thro' the Arch of a Circle G F A from the vertical to the horizontal Plane, it is evident, when the Eye is at G the Angle D E C will be of ſome Quantity leſs than a right Angle; and the Angle F D C will be Nothing. But as the Eye departs from G, the ſaid Angle F D C begins, and encreaſes to a right Angle when the Eye arrives at A; on the other hand, during all that Time the Angle D E C decreaſes, and at laſt vaniſhes; and therefore at ſome Moment of that Interval the two Angles muſt be equal, which let us ſuppoſe to happen when the Eye is in the Poſition F.

1510. Then are the Triangles E F D and D C F ſimilar; for the Angle C D F = D E C, and the Angle C F D is common to both; therefore the Angle E D F = D C F, and conſequently the viſual Rays of the Cone make *ſubcontrarily* the ſame Angles with each Plane A E and D G; therefore the Section of the Cone made by each Plane muſt be of the ſame Figure, and conſequently ſince E D is a *Circle* its Perſpective D C will be a Circle alſo.

1511. From hence ſome curious Problems will ariſe of the laſt Importance in Perſpective. The firſt is, *To determine the Poſition of the Point* F, *or Point of Sight, from the given Diameter of the original Circle* D E, *and Diſtance from the Table* D B. In order to this, let D E = D, D B = *s*; alſo put the Height

of the Eye B F = h; the Diameter of the perſpective Circle D C = d; and let E F = e, and D F = f. Then D : e :: d : f (1510) let D + s = r; then $e = \sqrt{r^2 + h^2}$ and $f = \sqrt{h^2 + s^2}$ (636) alſo D : d :: r : h, therefore $d = \frac{Dh}{r}$. Whence D : $\frac{Dh}{r}$ :: 1 : $\frac{h}{r}$ :: $\sqrt{r^2 + h^2}$: $\sqrt{h^2 + s^2}$; whence we get $h^2 = rs = \overline{D + s} \times s$; and therefore $h = \sqrt{\overline{D + s} \times s}$ = B F, the Height of the Eye at F required.

1512. The above Equation gives this Analogy, D + s : h :: h : s, whence it appears (645, 660) that the *Locus* of the Point F is the Circle E F A. Hence A B = B D = s; A F = D F; the Diameter A E = D + 2s; and the Radius A N = $\frac{D + 2s}{2} = \frac{DE + 2DB}{2}$; from whence the Conſtruction of the Problem is evident.

1513. The *ſecond Problem* is, the *Diameter of the Circle* D E, *and Height of the Eye* B F *being given, to find the Diſtance* D B *from the perſpective Table.* The Solution is thus; $h^2 = Ds + ss$, (1511); let the Square be compleated (338) then $h^2 + \frac{1}{4}D = ss + Ds + \frac{1}{4}D$, and therefore $s + \frac{1}{2}D = \sqrt{h^2 + \frac{1}{4}D}$; conſequently $s = \sqrt{h^2 + \frac{1}{4}D} - \frac{1}{2}D$ = D B the Diſtance required.

1514. A *third Problem*, is *to determine the Diſtance* D B, *at which the Perſpective* D C *of a given Circle* D E *ſhall be a Circle, and its Area be to the Area of the given Circle in any given Proportion of* z *to* y. To ſolve this Problem, it muſt be conſidered that ſince D B = B A, (1512) we have D F = F A; and ſince the Areas of Circles are as the Squares of their Diameters (840) we have D E^2 : D C^2 :: E F^2 : D F^2 = F A^2 (1512) but E F^2 : F A^2 :: E B : A B, (660) therefore D E^2 : D C^2 :: (y : z ::) E B : A B (:: D + s : s) whence we have $ys = zD + zs$; and $ys - zs = zD$, therefore $y - z$: z :: D : $s = \frac{zD}{y - z}$ = D B, the Diſtance required.

1515. The Center of the *original Circle* will not be projected into the Center of the *perſpective Circle*; neither will the Diameters

Diameters of the latter be the Perſpectives of the Diameters of the former. Nor will any right-lined Figure (as a *Square*, *Triangle*, *Polygon*, &c.) inſcribed or circumſcribed about the original Circle have its Perſpective in, or about the perſpective Circle of the ſame Kind or ſimilar to it; for that of a *Square* will be a *Trapezium*; that of a *Triangle* a *diſſimilar Triangle*, that of a *Pentagon* will be an *irregular Polygon*; and ſo of others. All which the Reader will be eaſily aſſured of by drawing this Caſe in the uſual Manner and Form as directed (1492.)

CHAP. V.

The Theory of CATOPTRIC PERSPECTIVE; *or the perſpective Appearance of Objects on a* REFLECTING PLANE.

[PLATE I. *of* OPTICS *and* PERSPECTIVE.]

1516. WE have conſidered the *Dioptric* THEORY of PERSPECTIVE ſo far as it relates to *Lines* and *Superficies*; and before we go farther it will be proper to conſider the perſpective Appearance of the ſame Objects by *Light reflected from a poliſhed Plane*; in as much as we ſhall find the *Perſpective* of any *Line* or *Superficies* is identically the ſame on the *Catoptric* or reflecting Plane as it is on the *Dioptric* or refracting one.

1517. Preparatory to a Demonſtration of this Poſition, let HD (Fig. 26) be a Line or Section of the Speculum, perpendicular to the ground Line A F, and B the Place of the Eye at the given Diſtance A D, and Height A B; from any Point E in the Line A D, let a Ray of Light proceed to the Speculum in the Point C, and be reflected from it to the Eye at B; then continuing the reflected Ray out beyond the Speculum, it will interſect the Line A F in the Point F, making the Line DF = DE; and conſequently, the *Point* E *by reflected Light will appear upon the Speculum in the ſame Point* C *with another Point* F *viewed*

viewed by the Eye thro' a tranſparent Plane H D, *and placed at the ſame Diſtance behind it.*

1518. The Demonſtration is eaſy; let C G be perpendicular to the Plane H in the Point C, it therefore will alſo be parallel to the Line A F; then becauſe the Angle G C B = G C E (1282) and G C B = C F D, and G C E = C E D (631) therefore we have the Angle C E D = C F D; and therefore the Line C E = C F, and conſequently the Line D E = D F. Wherefore *the apparent Place of the Point* F, *by Reflection, is the Point* F *at an equal Diſtance beyond the Speculum.*

1519. Proceeding in the ſame Steps of Demonſtration here as in Dioptric Perſpective (1475), we ſhall find the ſame Appearance alſo of Lines and Superficies on a reflecting Plane as we did there on the refracting one. Let A Y Z D (Fig. 27.) be the Speculum, A *B C* D be a Square placed before, and contiguous to it; extend the Sides A *B*, and D *C* indefinitely on each Side, and draw E H between and equidiſtant from both. Let I be the Place of the Eye at the Diſtance F H, and Height H I; take F E = F *E*; and thro' E and H, draw B C and G K; alſo on the Point E erect the Perpendicular E M = H I, and draw I M paſſing thro' the Speculum in the Point *i*; and join I G, I K; M B, M C.

1520. Then becauſe the Point *B* of the Square is in the Plane I G B M which interſects the reflecting Plane A Z in the Line A *i*; and becauſe the incident and reflected Rays are in the ſame Plane (1282) therefore the Ray of Light which proceeds from the Point *B* and is by the Speculum reflected to the Eye at I, muſt imping on the Speculum in ſome Point (*b*) in the Line A *i*; and the reflected Ray will be *b* I. If this Ray be continued out, it will meet the Side of the Square A *B* produced in B, making A B = A *B*. (1518)

1521. Now becauſe the ſame Thing is demonſtrable of every other Point in the Side of the Rectangle A *B*, *therefore the Perſpective of the Line* A *B by Reflection is the Line* A *b*, *or that very Part of the Radial* A *i which is the Perſpective of the equal Line* A B *by Refraction thro' a tranſparent Plane* (1476).

1522. In like Manner it is demonſtrated, that the Perſpective of the Side D *C* by Reflection is D *c*, the ſame Part of the Radial as is that of the equal Line D C by tranſmitted Rays.

By

By joining *b c*, the Line (*b c*) is the Perſpective equally of *B C* by reflected, and B C by refracted Light. And conſequently, A *b c* D *is the perſpective Appearance of the Rectangle* A *B C* D *by Reflection, and every Way the very ſame as it was ſhewn to be of an equal Rectangle* A B C D *beyond the pellucid Plane* (1487).

1523. Therefore whatever was demonſtrated relative to the Perſpective of Lines and Superficies any how poſited or formed with reſpect to a common tranſparent Plane in the Sequel of the Dioptric Theory, will hold equally true here in Catoptric Perſpective. And tho' this Part of the Science has been the leaſt of all conſidered by Writers on this Subject, yet when compared with common Perſpective it will be found not only more delicate in its Nature, but alſo more adapted to Uſe. This we ſhall illuſtrate more particularly when we come to treat of *Picturesque Perſpective*, or that Part of the Science which treats of the PRINCIPLES of PICTURES or LANDSCAPES formed by OPTICAL GLASSES both of the *reflecting*, and *refracting* Sort.

CHAP. VI.

Of INVERSE PERSPECTIVE; *where the Nature and Principles* of the ANAMORPHOSIS *or* Deformation *of* Figures, *and their Rectification, are explained.*

1524. THE *inverſe Method of Perſpective* is not leſs pleaſant than uſeful, nor is it in the leaſt Degree difficult, as it is only going backwards in the ſame Steps we took in the direct Method; and it muſt be obſerved, that with regard to any Science, unleſs the Student can go backwards as well as forwards, he makes but an imperfect Proficiency.

1525. *Inverſe Perſpective* ſhews how from a given Piece of Perſpective to determine the *Original* or *Prototype* thereof under the Circumſtances of *Size, Situation, Diſtance, &c.* And here, as in the direct Method, we muſt begin with finding the Original of a *Point*, then of a *Line*, and laſtly of a *Superficies*; and

and being able to determine theſe, there will remain no Difficulty with reſpect to *Solids*, becauſe the Perſpective of a Solid is only the complex Perſpective of its ſuperficial Parts; and for this Reaſon, we think this the moſt proper Place to treat of the Inverſion of a perſpective Piece, and the Nature of *Anamorphoſis* or the *Deformation of Pictures*, which naturally reſults from thence.

1526. For by the direct Method it appears, that the Perſpective is very diſſimilar to its Original, and therefore if the latter be formoſe and regular, the former will be diform and irregular, or in the uſual Phraſe, *deformed* and *diſtorted*. So on the other Hand, if any natural and well-formed *Picture* be conſidered as a *Piece of Perſpective*, formed on a perſpective Plane, then when it is reſolved into its *Prototype*, that muſt be a very unnatural, miſhapen, and ill-proportioned Figure; all which will be evident from the following Proceſs.

1527. Let T V W X be a Perſpective Table (Fig. 29.) ſtanding on the ground Line S V. Alſo let I be the Point of Sight, and Y the Diagonal Point of Diſtance; and then ſuppoſe it required to find in the ground Plane the Seat of the Original of a given Point P in the Perſpective Table. You proceed thus; from the Point of Sight I, draw thro' P the Radial I K, and on K in the ground Line erect the Perpendicular K G; then from Y draw thro' the given Point P, the Diagonal Y L to meet the ground Line in L. Laſtly, on the Point K, with the Diſtance K L, deſcribe the Arch I H to interſect K G in H; and the Point H is the Original of the Point P, as required (1476, 1493.)

1528. In the ſame Manner any other Point Q in the Table will have its Prototype determined in C; and the Diſtances of any of thoſe original Points H, and C, from the ground Line S V will always be equal to the Diſtances K L and L S contained in the ground Line between the Interſections of the reſpective Radials and Diagonals.

1529. To find the Original of a given Line P Q in the Perſpective Table; nothing is required but to determine the Original Seats H and C of its two extreme Points P and Q (1494) then by connecting the Points H, C, the Line H C is formed on the ground Plane and is the Original of the Perſpective

tive PQ, as required. The Diſtance of this Original at each Extreme from the ground Line, as alſo it's Magnitude or Length, is meaſured on the ſaid Line SV, in *Inches*, *Feet*, &c. according to the Meaſure by which that Line is divided.

1530. In the ſame Manner any other Perſpective Line OP is found to have it Prototype BC, on the ground Plane; and thus any *Perſpective Angle* OPQ on the Table VX, will have its *original Angle* BCH determined on the ground Plane; whoſe Quantity is there alſo meaſured in the uſual Manner.

1531. Therefore, *to find the Original or Prototype of any given Perſpective Figure or Superficies*, OPQR *on the Table*, nothing more is neceſſary than to find the original Seats E, B, C, H, of it's angular Points O, P, Q, R, (1493) for theſe being connected, from the Figure EBCH, which is the true Original of the Perſpective given; whoſe Diſtance and Dimenſions will then be eaſily known by common Geometry.

1532. As in the *direct Method*, the Perſpective of every regular Figure is irregular and diſſimilar, (1487) ſo in the *Inverſe Method*, the *Original* of every regular Piece of Perſpective, muſt itſelf be irregular and of a different Form; and therefore, if the Perſpective Figure OPQR be a *Square*, the Original EBCH is a *Trapezium*, (1531) and if that Square be divided into equal Parts, by Lines drawn thro' the equidiſtant Points *f*, *e*, *d*, the Original will be divided very unequally by correſpondent original Lines drawn thro' the original Points *a*, *b*, *c*, at unequal Diſtances in the Line HC, the Original of PO.

1533. Therefore it will follow, that if any *Portrait*, or other *Picture* be drawn on the Perſpective Square, it will, when projected by the viſual Rays in the original Trapezium, be there deformed, or appear of a monſtrous Shape. For the natural Form and juſt Proportion of Parts in the Portrait, which depend on equal and ſimilar Spaces which they fill in the Square, will be all deſtroyed, when projected on the Trapezium into unequal and very diſſimilar Spaces. And this Diſtortion of the Figure is called, an *Anamorphoſis*, *Deformation*, or *monſtrous Projection*.

1534. The *Anamorphoſis* is eaſily effected in any Degree by the Rules of Art. For if the Perſpective Square PR be divided

into a Number of ſmall Squares, or other equal or ſimilar Spaces; then ſince by the Rules above the Original of the Square and all its Parts are to be found, and drawn on the ground Plane, it remains only to draw the ſame Parts of the Picture in each irregular Space of the Trapezium, as you ſee in the correſponding Part of the Perſpective Square; and the Deformation will be compleated, as required.

1535. The Diſtortion or Deformation will be in Proportion to the Height and Diſtance of the Eye from a given Perſpective Figure O P Q R. For if the Diſtance of the Eye I Y continue the ſame, the Breadth of the Projection at each End, *viz.* E H and B C will leſſen as the Height of the Eye W V increaſes, and conſequently, the Deformity of the Picture will be the greater, as its Length continues the ſame nearly.

1536. Alſo, if the Height of the Eye be the ſame, but the Diſtance greater, then will the Deformation or Trapezium vaſtly encreaſe, both in Length and Breadth, in all the remote Parts towards B C, in Compariſon of thoſe towards H E; and therefore the Deformation becomes greatly augmented in this reſpect alſo; and, indeed, much more than by altering the Height of the Eye at the ſame Diſtance.

1537. Hence it appears, that there is a certain Poſition of the Eye, in which it will view any given artificial Deformation or monſtrous Picture E B C H, ſo that it ſhall appear perfectly natural, or in a juſt Proportion and Symetry of all the Parts of the Object it is intended to repreſent. For it is evident, from the foregoing Theory, if the Deformation E B C D be placed on the Horizon, at the Diſtance M E, from the Perſpective Table V X, then an Eye placed at I, at the Diſtance I Y, and Height T X, will *view that monſtrous Projection on the ground Plane, as a regular and perfect Square on the Table; and the deformed Image contained in it; as a well-proportioned and natural Portrait or Picture.*

1538. Now, becauſe the Image in the Eye is every Way ſimilar to the Object it views on a Plane placed before it, and parallel to the *Fundus* or Bottom of the Eye; therefore, ſince this Image on the Retina in the Eye is the ſame as would be formed of a Picture well drawn or deſigned on the Plane or Table V X, and ſince it does no ways depend on the

ſaid

ſaid Table, but on the peculiar Pencil of Rays paſſing through it only, which is ſtill the ſame when the Table V X is removed; it follows, that the Eye in the Poſition aſſigned at I, will view, by itſelf alone, the Deformation contained in B E H C, as a juſt and well drawn Picture.

1539. This Invention has given Riſe to thoſe common *Anamorphoſes of King* Charles's *Head*, *St.* George *and the Dragon*, &c. on long Slips of Paper (like Fig. 30.) ſold at the Shops, which being held in a Poſition parallel to the Horizon, and viewed at the proper Diſtance E D, and ſmall Height of the Eye at I, do very agreeable ſurprize the Spectator (inconſcious of the Deſign) with a regular and beautiful View of thoſe Objects, of which in the Deformation they ſaw ſcarce any Appearance.

1540. From what we delivered in the Theory of Catoptric Perſpective, it is evident, that if T V W X be conſidered as a *reflecting Speculum*, and E B C H a Deformation placed before it, then its Appearance O P Q R, in the *Speculum*, will be a regular Picture; for that will be the true Perſpective of the Figure on the ground Plane, in the ſame Manner by a reflected, as by a tranſmitted Pencil of Rays (1522.)

1541. Since the Law of Reflection is the ſame in Surfaces of every Figure (1300) it will follow, that if we conceive the *Speculum* T V W X to be pliable and formed into a Cylindrical Surface, as A B C D (Fig. 31.) There will ſtill be a regular Perſpective Picture formed by Reflection from a proper Deformation E F G H; the greater Divergency of Rays in this Caſe, cauſing only a greater Diſtortion in the *Anamorphoſis*, and a Diminution of the Picture in the Speculum.

1542. Hence ariſe all thoſe Experiments of *poliſhed Cylinders*, *Cones*, and other figured *Speculums*, which rectify the Appearance of thoſe ſeemingly unmeaning and ludicrous Deformations painted round about them, on the Planes on which they are placed; any of which are eaſily drawn by any Perſon ſkilled in the Theory of Catoptrics; but thoſe who are not, may very ſucceſsfully uſe the following practical Method which is univerſal for all *Speculums*.

1543. Let a ſquare Hole be cut in a Piece of Paper, Paſtboard, Vellum, &c. and fine Threads paſted on the Paper over the Hole, horizontally and perpendicularly, ſo as to divide

the ſquare Hole into a proper Number of leſſer Squares; then paſt this Paper with the *Lattice-Square* on the Surface of your poliſh Cone, Cylinder, Looking-Glaſs, *&c.* and place a Candle at a proper Diſtance and Height (ſuch as you intend the Eye ſhall have to view the Picture) then will that polliſhed Lattice be reflected on the Table in the Form of the required Anamorphoſis, with the Shadow of the Threads, dividing it in a proper Manner for drawing the Deformation of any propoſed Object, whoſe Picture is drawn in a Square, of the ſame Size with that on the *Speculum*.

CHAP. VII.

Of SCENOGRAPHIC PERSPECTIVE; *or the* METHOD *of drawing* FIGURES *of* THREE DIMENSIONS *in* PERSPECTIVE.

1544. THE Superficies which terminate or bound a Solid, conſtitute its Form, or external Appearance; and therefore the PERSPECTIVE *of a* SOLID, is nothing more than the *complex Perſpective of all its ſuperficial Parts*; ſo that, properly ſpeaking, *there is nothing beyond ſuperficial Perſpective in Nature*, for that rightly applied, gives the Perſpective of every Figure or Form of Solids, or rather Figures of three Dimenſions *; as will appear in the following Articles.

1546. We ſhall begin with the moſt regular and ſimple Form of a Solid, *viz.* that of a CUBE, whoſe Sides are ſix in Number, and all equal; but if a Cube be placed direct before the Eye, only one or two of the ſix Sides can then appear. For if Fig. 1. (Plate II.) be conſidered as a Cube, then, becauſe its Height

* We have thought proper here to make a Diſtinction between a *Solid*, and a Figure of three Dimenſions; for though every Solid has always three Dimenſions, yet a Figure of three Dimenſions may not be a *Solid*, but a hollow or concave Body; thus there may be a *ſuperficial Cubic Inch*, as well as a *Solid one*, but the Perſpective of both is the ſame; for this Art is not concerned with the Solidity or internal Parts of Bodies that do not appear to the Eye.

Height QP exceeds the Height of the Eye IY, it will be impoſſible for the Spectator to ſee more than the Side in Front, EG. For Rays of Light proceeding from any Point in the two ſide Planes EB, HC, and that on the Top GB, can never come to the Eye at I; and the Bottom ED, and fartheſt Side DB, are precluded from the Sight by the Solidity of the Cube, ſo that one Side only can in this Caſe appear.

1546. But if the Height of the Eye IE (Fig. 4.) exceeds the Height of the Cube AB, then not only the Front-ſide ABCD, but likewiſe the Top or uppermoſt Side will appear to the Eye. And only theſe two can appear for the Reaſons before-mentioned (1545.) The Perſpective of the Side ABCD of the Cube in Front, will be ſimilar to the Side of the original Cube, and therefore a *Square*, as is evident from (1468, 1469, 1470.)

1547. But ſince the Top, or upper Side of the Cube is below the Eye, and parallel to the Horizon, it may be conceived as placed on a ground Plane at the Height of the Cube above it; and its Perſpective on the Table, will therefore be the ſame as of any Square ABCD for the Height IF, and Diſtance IY of the Eye, (1487) that is, A*b c* D will be the Perſpective of the upper Side of the Cube; and conſequently, the Perſpective of the whole Cube, viewed in Front, will conſiſt only of thoſe two Sides, as in the Figure.

1548. A Cube of a leſs Height than that of the Eye, placed with one Side parallel to the perſpective Table, but removed from the direct Ray IF towards either Side, to more than half it's Width AF or FD, will diſcover a third Side to the View; and in a ſolid Cube, three Sides are the moſt that ever can be ſeen; this is called a View in *Profile*.

1549. Therefore to give the moſt general and eaſy Idea of the Perſpective of a Cube (or any ſolid Body) it will be beſt to diveſt it (as it were) of its Solidity, and repreſent it as hollow, or conſiſting only in Form, or the external Sides or Superficies.

1550. Wherefore (Fig. 7.) let ABCD be the Side of a hollow Cube, placed contiguous the perſpective Table, and viewed in Profile. Then will the Perſpective of the Baſe be B*b c*C (by 1487.) This is called ICHNOGRAPHY or *perſpective Plan* of the Cube. The Perſpective A*a d*D of the Top of the Cube

Cube is determined in the ſame Manner, by ſuppoſing A D the ground Line in regard thereto. The Perſpective of the Front A B C D, as it coincides with the Table, is the very Side itſelf. And the Perſpective of the Oppoſite or remote Side, as it is parallel to that Front and to the Table, will be of the ſame Figure, *viz.* a Square *a b c d* (1468, 1469.)

1551. Therefore the ſame Part A *a* of the Radial A I which bounds the Perſpective of the Top of the Cube, terminates alſo the upper Part of that Side of the Cube which is in View, as being the common Interſection of both thoſe Sides or Planes. For the ſame Reaſon B *b* terminates the lower Part of the viſible Side of the Cube on the Table; therefore the Perſpective of that Side of the Cube is A *a b* B. And ſince the Cube is hollow, and the Sides open, the other Side will appear internally; and its perſpective Profile will be D *d c* C.

1552. All we have ſaid is evident from the Conſideration of the mutual Interſection of Planes, as indeed from the Figure itſelf; for if we turn the Figure Side-ways, and look upon C D continued out to M as a ground Line, then will the two Sides, which are now perpendicular to the Horizon, be parallel to it, or they will be the Bottom and Top of the Cube, and the Height of the Eye above thoſe Sides or Planes will be I M, as it is now C M. And C *c d* D, A *a b* B, will be their Perſpectives, drawn by the common Rules (1489, 1492.) as B *b c* C, A *a d* D, are for the lower and upper Sides in the preſent Poſition of the Cube.

1553. If the Baſe of the Cube be divided by Lines parallel to the ground Line, like the Square in Fig. 5. then by ſetting off thoſe equal Diviſions in the ground Line from B to 4, 5, 6; and from them drawing Diagonals to the Point of Diſtance Z, they will cut the Side B *b* in the Points I, O, G, thro' which the Lines I K, O Q, G H, are to be drawn for their Perſpectives in the ichnographic Plan B *b c* C (1478, 1479.) and they will be all parallel to the ground Line B C, as in the original Baſe.

1554. In the ſame Manner, if C D be conſidered as a ground Line, and the equal Diviſions in the Side C D, be ſet off from D towards M, then Diagonals drawn from them to the proper Point of Diſtance in this Caſe, will interſect the Side D *d* in the Points

Points F, *q*, *k*, through which the Lines FK, *q*Q, H*k*, being drawn parallel to DC (1552) will be true Perſpectives of the original Lines in the Side of the Cube.

1555. Hence we ſee that the perſpective Parallels IK, OQ, GH, on one Side, contain right Angles with the perſpective Parallels KF, Q*q*, HK, on the other adjacent Side, and are therefore perpendicular to each other, as in the original Cube. And conſequently, the perſpective Sections of the Cube through thoſe Lines, *viz.* EIKF, *o*OQ*q*, *l*GH*k*, will be all of them *Squares*, and parallel to the Side ABCD, as they are in the Cube itſelf.

1556. If each Side CB, CD, be divided by perpendicular Lines in the Points 1, 2, 3; then if from thoſe Points, Radials are drawn to the Point of Sight I, they will in the Plains B*bc*C, C*cd*D, be the Perſpectives of the original Lines in the Sides of the Cube reſpectively (1552). And thus it appears, that to put a Cube into a Perſpective, is nothing more than to find the Perſpective of a Square for its ſeveral Sides ſeparately, and that thoſe ſix perſpective Plans together compleat the *Perſpective of the Cube*, which is uſually called the SCENOGRAPHY of it.

1557. The Cube having an Angle placed in Front, or in ſuch Manner that a Diagonal of its Baſe may be perpendicular to the ground Line, is put into Perſpective by the ſame Rules, and with the ſame Eaſe, as the other; in this Caſe we ſee the upper Side of the Cube, and have an equal Profile of two of its Sides. Thus Fig. 8. is the *Scenography* of a Cube, whoſe Side is equal to the Square EOFQ in Fig. 4. Then *abbg* is the Perſpective of the upper Surface (the ſame as F*oeq* in Fig. 4.) and EOFQ is the *Ichnography* of its Baſe, found as directed (1485) for the Height EI, and Diſtance IZ, of the Eye. Having theſe two Plans, the Perſpective of the Cube is compleated by drawing the Lines *a*E, *b*O, *b*F, *g*Q.

1558. Becauſe the Sides of the Cube, or parallelopiped in this Caſe, make an Angle of 45 Degrees with the ground Line, therefore their Perſpectives *ag*, *eq*, will converge to the *Point of Diſtance* Z, (as in the other Caſe they converged, to the *Point of Sight* I.) As will all other Lines drawn parallel to them on the Sides of the original Solid. (1485)

1559.

1559. If on any one Part, the Solidity of the Cube be ſuppoſed to be uniformly continued for any given Length E*e*, then the Cube becomes a *Parallelopiped*, the Perſpective of which is ſtill determined by the ſame Rules, as is evident from the Figure.

1560. And if any Side of the Parallelopiped be divided by parallel Lines perpendicular to the Baſe; then by ſetting off the Diſtance of thoſe Lines from *e* to C in the ground Line to the Points 3, 2, 1, reſpectively; if a Ruler be laid from each of thoſe Points, to the Point of Sight I, it will give the Points in the Line *eq* of the perſpective Baſe, through which, if Lines are drawn parallel to *ae*, the Side of the Perſpective of the Cube or Parallelopiped, will be divided in the ſame Manner as the Prototype. All which is evident from the Figure, and what we have before taught (1552). For the *Radials* divide the Baſe Lines in this Caſe, as *Diagonals* did in Fig. 7. where the Cube was viewed by the Side. But this Diviſion is to be made equally by Radials or Diagonals at Pleaſure, as the Reader will eaſily underſtand from the Theory.

CHAP. VIII.

The RATIONALE *of the practical* METHODS *of* SCENOGRAPHIC PERSPECTIVE, *with the* THEORY *of Perſpective* MEASURES *for that Purpoſe.*

1561. AS in ſuperficial Perſpective there are two practical Methods of putting any plane Figure in Perſpective, *viz.* one by inſcribing them in a Rectangle, (1488) the other by given Points (1492). So in like Manner we here proceed to draw the Perſpective of any Upright, Solid, or ſuperficial Figure of three Dimenſions, two different Ways, *viz.* The *firſt* by inſcribing the given Figure, if regular, in a Cube, or proper Parallelopiped; the *Second*, is by finding the Perſpective of the ſeveral Lines which are formed by the Interſection of Planes, compoſing

poſing the Superficies of the given Figure; and theſe Lines connected, from the *Scenography* thereof. This ſecond Method is general for all figured Bodies, regular, or irregular.

1562. As a Specimen of the firſt Method, let it be required to put a PRISM into Perſpective; to do this, we may conceive it circumſcribed by a hollow Parallelopiped, whoſe Baſe and Height are equal to thoſe of the given Priſm; then draw the Scenography or Perſpective of this Parallelopiped, which let be A B *b a d c* C D, the Baſe of which B *b c* C is alſo the Perſpective of the Baſe of the *Priſm*; and ſince the Edge of the Priſm touches the Surface of the Parallelopiped, let the Diſtance thereof be meaſured, and ſet off on the ground Line from B to 4; then a Ruler laid from Z to the Point 4, will cut the Side B *b* in I: Draw I E parallel to A B, and it will be the Perſpective of the Perpendicular let fall from the Edge or Angle of the Priſm on the Baſe. Therefore drawing E F parallel to A D, it ſhall be the Perſpective of the Edge of the Priſm. Then drawing the Lines B E, *b* E, and C F, *c* F, they compleat the Perſpective of the Priſm in the Figure B E *b c* F C, which is ſhaded for the Sake of Diſtinctneſs.

1563. After the ſame Manner, a *Pyramid*, *Cone*, *Globe*, &c. may be drawn in Perſpective, which cannot be difficult to thoſe who underſtand the foregoing Principles. What I here ſpeak of relates to Matters *merely Perſpective*; but to give a natural *Relievo* to *conical* and *ſpherical Figures*, whether Convex or Concave, requires more a *ſkilful Management and Diſpoſition of Lights and Shadows*, as we ſhall hereafter ſhew.

1564. The *ſecond Method* for determining the *Scenography* of Solids, is by finding the Perſpective of Lines which are formed by their interſecting Planes. To underſtand the Reaſon of this Method, the following *Theorem* muſt be premiſed. Let I D *be a given Line, placed on the ground Line* C D (*Fig.* 10.) *then from any two Points* A, B, *in the horizontal Line, draw the Radials* A D, A I; B D, B I; *and parallel to* C D *draw any Line* L K, *cutting the Radials* A D, B D, *in* G *and* E; *from thoſe Points draw* G H, E F, *cutting the Radials* A I, B I, *in* H *and* F. *This done, I ſay, the Line* G H *is equal to the Line* E F. The Demonſtration is from ſimilar Triangles (656.)

For AG : AD :: GH : DI.
And BE : BD :: EF : DI.
Alſo AD : BD :: AG : BE.
Therefore AD × BE = BD × AG.
Therefore alſo DI × EF = DI × GH. (652.)
Conſequently EF = GH. *Q. E. D.*

1565. If FD (in Fig. 9.) be made equal and parallel to ID (in Fig. 10.) and from the Point of Sight I, the Radials IF, ID, be drawn; then if any Line LM be drawn parallel to the common ground Line DC D, interſecting the Radials ID, AD, in the Points L and M; and LG and MO be drawn parallel to FD or ID, they will be equal to each other alſo, as is demonſtrable in the ſame Manner as the above Theorem.

1566. Therefore let BLKI be the Perſpective of the Baſe of the Solid (Fig. 9.) drawn for an oblique View; and neither Side or Angle in Front. Then when the oppoſite Sides of the Solid are equal and parallel, and one Angle placed in the ground Line at B; if AB be the Height of the Solid, then through L draw the Radial ILD, meeting the ground Line in D; on the Point D draw DF equal and parallel to AB, and join IF; on the Point L erect the Line LG parallel to DF interſecting the Radial IF in G; and join AG; then is ABLG the Perſpective of one Side of the Solid. In like Manner you find the perſpective Lines IE, KH; and drawing AE, EH, and GH, the Perſpective of the whole Solid is compleated.

1567. Or thus more univerſally, let ID be the Meaſure of the Solid's Height, and placed on the ground Line at D (Fig. 10.) and make any Triangle AID at Pleaſure, whoſe Vertex A is in the horizontal Line; then from the Angles of the perſpective Baſe I, L, K, (Fig. 9.) draw Lines IC, LM, KP, interſecting the Radial AD in C, M, and P, on which Points erect the Lines CN, MO, PO, and they will be the perſpective Heights of the Solid at the Angles I, L, K; that is, CN = IE, MO = LG, and PQ = KH; (by 1564) which Heights are all determined by the parallel Lines EN, GO, and HQ, (in Fig. 9, 10.)

1568. Hence it appears, that if any Line be perpendicular to the ground Plane, and its Diſtance from the ground Line, and Height,

Height, be given; then its Perſpective may be eaſily drawn. For the perſpective Seat of any given Point on the ground Plane is found by (1492) and the Perſpective of any given Height or Line is found for that Seat or Point, by (1494.)

1569. If the given Line be perpendicular to the ground Plane, but placed above it, at a given Height; then that Height is to be added to the given Length of the Line, and both conſidered as one Line; whoſe Perſpective is to be found for the Whole firſt, and then for the lower Part or Height above the Plane; then this latter ſubducted from the Whole, leaves the Perſpective of the given Line.

1570. A Line of a given Length, Poſition, and Inclination to the ground Plane, if placed upon it, is thus put into Perſpective. Firſt let the Perſpective of its Seat on the ground Plane be found (1492) then from the elevated End let fall a Perpendicular to the ground Plane, and find the Perſpective of that Perpendicular (1494). Laſtly, draw a Line from the perſpective Seat to the Top of the perſpective Perpendicular, and that will be the Perſpective of the given Line.

1571. For Example; ſuppoſe (*c*) the Seat of the Line in the perſpective Plane (Fig. 7.) and K the Seat of the Perpendicular, then through K draw the Radial I C, cutting the ground Line in C; upon the Point C erect C D equal to the Perpendicular, and draw I D; and from the Point K draw K F parallel to C D, and it will be the Perſpective of the ſaid Perpendicular; then join *c* F, and it will be the Perſpective of the Oblique Line, as required.

1572. If the Line be wholly elevated above the ground Plane, then its Perſpective may be drawn two Ways; for, firſt, you may continue the given Line to the ground Plane; and then finding the Perſpective of the Whole, and its Parts, that of the given Line will be known of Courſe. Or, ſecondly; from each End of the elevated Line let fall a Perpendicular to the ground Plane, and then find the Perſpective of thoſe Perpendiculars, the Summits of which being joined, give the Perſpective of the Line, as required.

1573. Having thus ſhewn how all Lines may be drawn in Perſpective of any given Length, Poſition, or Elevation on, or above the ground Plane, it follows, that all Bodies of what Form

or Figure ſoever they may be, are drawn in Perſpective at the ſame Time, and by the ſame Rules. Since the Lines formed by their various interſecting Planes will be given in *Length*, *Poſition*, and *Inclination* from the Nature of the Solid; their Perſpectives therefore are eaſily drawn by the Rules above (1492 to 1495) and the Extremities of thoſe perſpective Lines being joined, give the Scenography or Perſpective of the Body required.

1574. It farther appears how, by the *menſural Triangle* AID (Fig. 10.) the perſpective Heights of Men, Houſes, Trees, Beaſts, Birds on the Wing, *&c.* may all be duely adjuſted and meaſured; and their proper Diminutions proportioned to their Diſtances from the ground Line in *Landſcapes*, *Prints*, and all *perſpective Views*. Thus ſuppoſe I D were a Tree on the firſt Ground of a Landſcape, then the Radials A I and A D determine the Diminution of Trees of the ſame Height in all remote Diſtances from the ground Line. So that at C the Height is C N; at P, it is P Q; at G, it is G H, and ſo on, till it vaniſhes in the Horizon at A.

1575. But by ſimilar Triangles we have C N : I D :: A C : A D; and P Q : I D :: A P : A D; and G H : I D :: A G : A D, *&c.* Therefore C N : G H :: A C : A G; or the *Linear Dimenſions* of Objects, in a Piece of Perſpective or Landſcape, are *directly as the Diſtances from the Point of Sight.*

1576. Again, the Diminutions of the Heights of Objects in a Landſcape, will always be as the Diſtances from the fore Ground, or ground Line; for A D — A C : I D — N C :: A D — A G : I D — H G (649.) That is, I D — N C : I D — G H :: C D : D G.

1577. Hence it appears, that while Objects are at the ſame perpendicular Diſtance from the perſpective Plane, their perſpective Appearance or Magnitude will always be the ſame; and therefore if a Bird, or any other Body aſcend or deſcend in a perpendicular Direction through any Height, it will ſtill have the ſame Dimenſions in every Part of the Aſcent or Deſcent on the perſpective Plane, and muſt be ſo repreſented in Drawing, as we obſerved before. And thus we have premiſed every Thing neceſſary in the THEORY and PRACTICE of *common* PERSPECTIVE: We now proceed to illuſtrate the ſame by a

Variety

Fig. 1

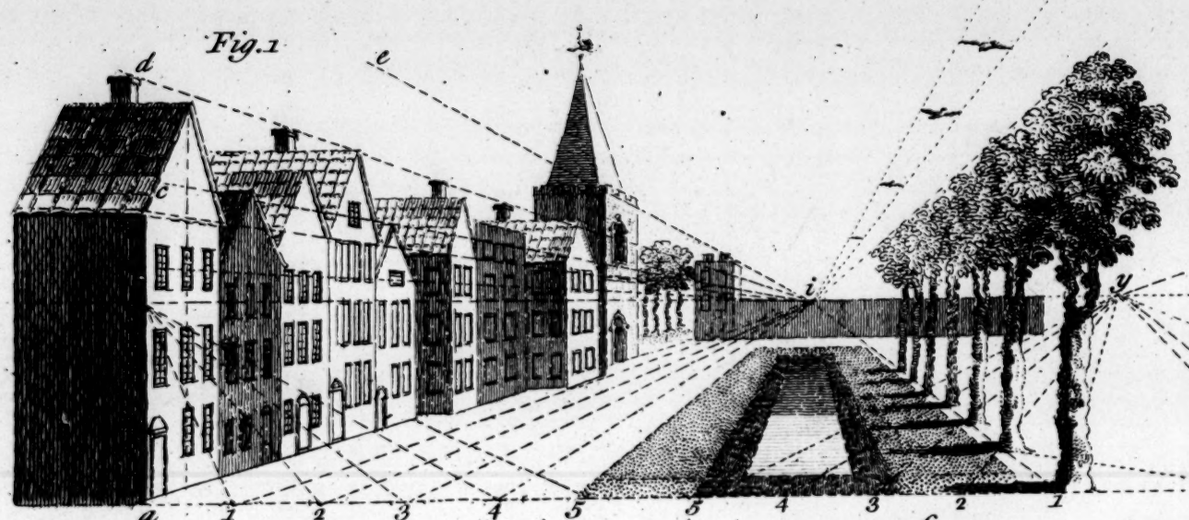


Fig. 2

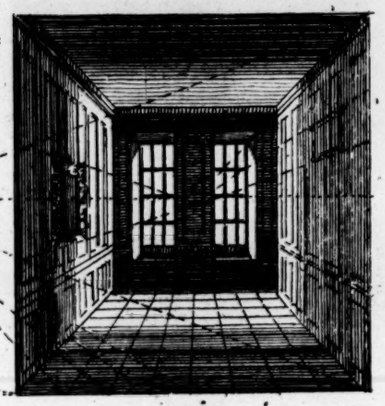


Fig. 3

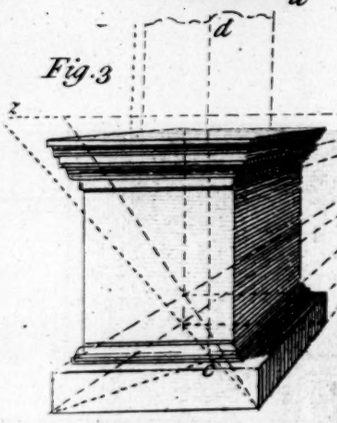


Fig. 4

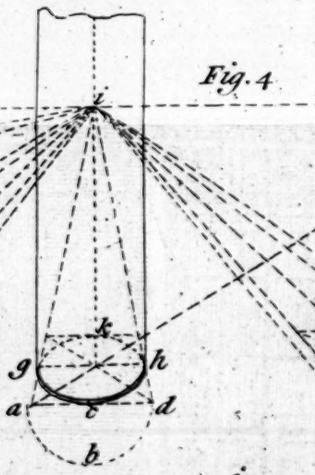




Fig. 5

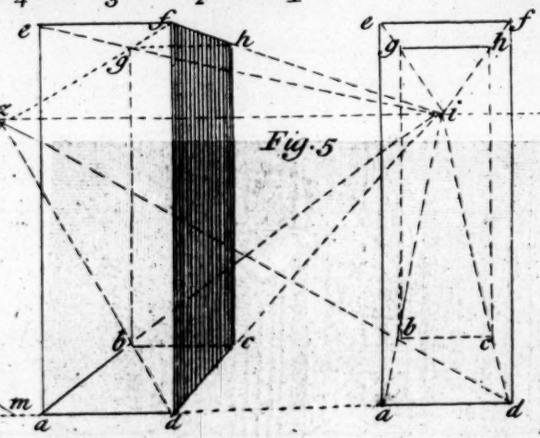


Fig. 6

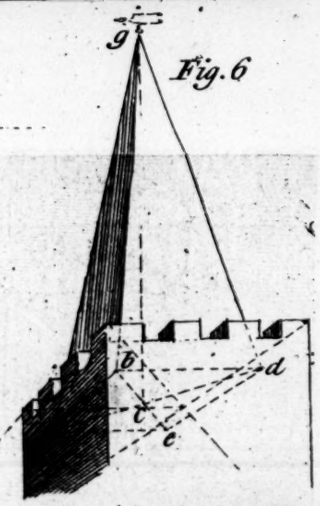


Fig. 7

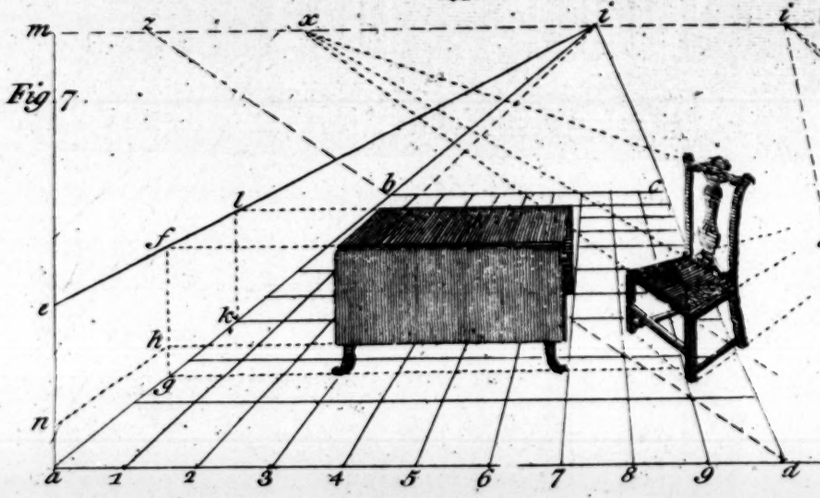


Fig. 8

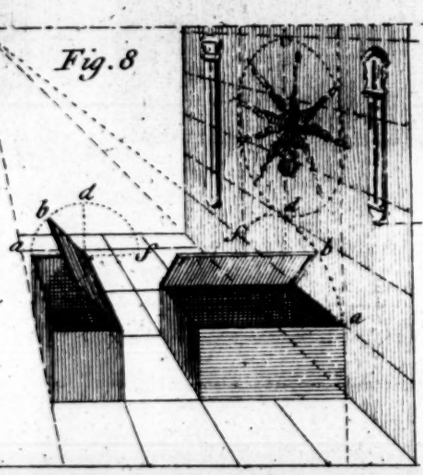


Fig. 9

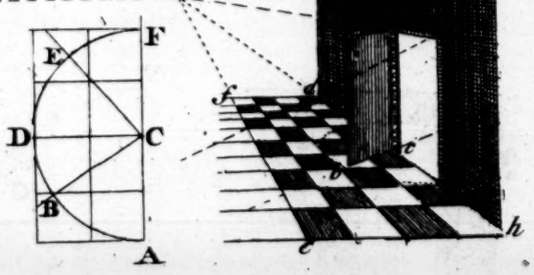


Fig. 10

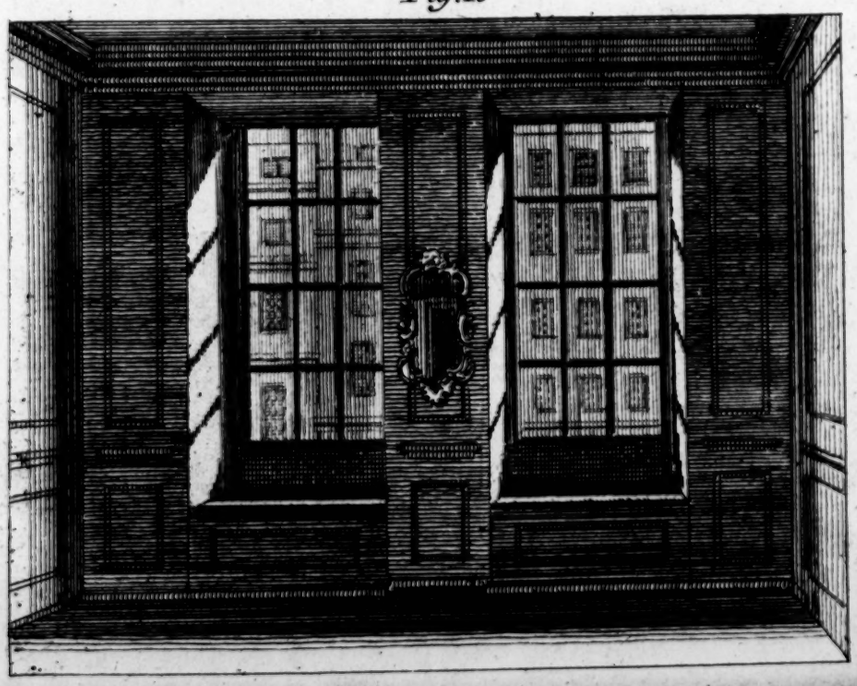


Fig. 11



Variety of uſeful Examples; with Obſervations thereon, as Occaſion requires.

CHAP. IX.

The RULES *of* PERSPECTIVE, *illuſtrated by a Variety of uſeful* EXAMPLES, *with proper Obſervations thereon relative to a* general PRAXIS.

[PLATE III. *Of* PERSPECTIVE.]

1578. AS we have given the fundamental Rules for drawing all Kinds of Superficies and Solids in Perſpective; it remains now, that we illuſtrate the ſame by proper Examples, with ſuch farther Obſervations and Inſtructions as may tend to facilitate the Practice of thoſe Rules to the young Deſigner.

1579. In Fig. 1. we have Inſtances of a *long Parade*, *gravel Walks*, *graſs Plats*, a *Canal*, &c. all put into Perſpective according to the Rules laid down (Chap. III.) by drawing the proper Radials and Diagonals from the Point of Sight (*i*) and Diſtance (*z*); as alſo the *Scenography* of a whole Row or *Street of Buildings* on one Side, and of Trees on the other, by the Rules contained in Chap. VII. Here we may obſerve, that in order to preſerve Uniformity, and Identity of Meaſures, in any Perſpective or Landſcape, there ought to be but *one Point of Sight, and Diſtance*; for if the Breadth or Diſtances of the Houſes were determined from the Point of Diſtance (*z*) and the Diſtances of the Trees from another (*y*) there could be no Meaſure found, nor any Judgment formed, of any Diſtances at all in ſuch a Piece.

1580. To the ſame Point of Sight (*i*) the ſame Rules of Perſpective are applicable for Objects on all Planes in any Situation about the Eye; for the *Elevation of the Plane above the Eye*, and *the Elevation of the Eye above a Plane*, are the *ſame Thing* in Perſpective; and conſequently the Diminution of Objects in the *Air* follow that ſame Rules as thoſe on the *ground Plane*. Thus

Birds

Birds (Fig. 1.) diminiſh as they recede from the fore Ground of the Sky (if we may uſe that Expreſſion) to remoter Diſtances, and at Length vaniſh at the Point of Sight in the horizontal Line.

1581. Thus alſo (in Fig. 2.) which is a direct perſpective View of the Inſide of a ROOM, the CIELING above, the *Floor* below, and each Side, are to be conſidered as ſo many Planes put into Perſpective from the given *perpendicular Diſtance* of the Eye from each (called the *Height* of the Eye) and the given Diſtance (*i y*) from the Front of the Room. For ſince the Interſections of the ſide Planes with the *Cieling* above, and the *Floor* below, are ſuppoſed perpendicular to the ground Line, or perſpective Plane, they will converge to the Point of Sight (*i*) according to (1479) and thereby terminate the Perſpective of the whole inſide View; which, indeed, cannot be difficult, as it contains the plaineſt Caſes of *direct*, *ſuperficial*, and *Scenographic Perſpective*.

1582. In Fig. 3. you have a ſide View of the PEDESTAL of a *Pillar*, whoſe *Front* being formed by the Meaſures proper to the Order, you proceed to form the Perſpective of its Baſe and Capital for the given Height and Diſtance of the Eye, as in the Figure. Then the Scenography of the Whole is finiſhed by the Rules in Chap. VII. and VIII.

1583. This Figure is added to ſhew how the Pillar is to be placed on its Pedeſtal in Perſpective; for it is eaſy to obſerve, that the Axis or central Line of the Pillar muſt be perpendicularly over the perſpective Center (*c*) of the Baſe, and conſequently the Perpendicular (*c d*) is that Axis or Line which determines the perſpective Poſition of the Pillar on its Pedeſtal.

1584. Fig. 4. contains the Perſpective of two PILLARS or COLUMNS in Front; one in a *Direct*, the other in a *Side View*; the *geometrical circular Baſe* (*a b d*) being given, the *perſpective elliptic Baſe* of each is determined by (1498) and then the Pillars themſelves are drawn or defined by the two parallel Lines which are perpendicular to the ground Line, and touch the Ellipſis on either Side.

1585. Hence it appears, that a *Colonade* of Pillars, viewed in Front, will not have all an equal perſpective Magnitude, or they muſt

muſt not be drawn all of an equal Size; but that which is neareſt to, or directly before the Eye, has the leaſt perſpective Magnitude; and others appear larger on the perſpective Plane, as they are farther removed from the Eye or Line of direct View. However paradoxical this may ſeem, it is thus eaſily demonſtrated — The Ellipſis (*g c h k*) of the ſame Circle on the ſame ground Line is greater or leſs in Proportion to the Obliquity or Diſtance of the View, as is evident from the Figure here, but more ſo in Fig. 4, 5. of Plate II. And ſince the Sides of the Pillars are Tangents to thoſe Ellipſes, and perpendicular to the ground Line, they muſt be at a greater Diſtance in an oblique than in a direct View, and therefore give a greater perſpective Magnitude of the Pillar.

1586. It may be proper alſo to give an optical Demonſtration of the Truth of this Poſition; let (*a d*) be the Diameter of the geometrical Baſe; this in the direct View ſubtends the greateſt Angle (*a i d*) under which the Baſe of the Pillar (*g c h k*) can appear, and the perſpective Diameter is *g h*; but *a c* and *g h* are the ſame in both Views, and a Pillar on that Diameter would be the ſame in both Caſes. But the apparent Magnitude of a Pillar is in Proportion to the Angle which is ſubtended by that Diameter that is perpendicular to the viſual Ray (*i c.*) Suppoſe (*e f*) to be that Diameter, and draw the Rays (*i e*) and (*i f*) which produce to the ground Line at (*m*) and let (*i e*) croſs it in (*l*) then it is evident the Diameter (*e f*) will eclipſe from the Eye a Part of the ground Line equal to (*l m*) in the oblique View; and therefore the perſpective Magnitude of the Pillar will be as much greater here than it is in the direct View, as (*l m*) is greater than (*a d.*)

1587. Hence it appears, that though it is always the Practice of *Architects* to repreſent all Pillars in a front View of an equal Size, yet the Abſurdity of ſo doing is as great as it would be to give the ſame Objects the ſame Heights at different Diſtances from the ground Line. And thoſe who vindicate ſuch a Proceedure, muſt be looked upon as either very ignorant, or elſe prevaricating with the Rules of Art in a moſt licentious Degree.

1588. The Axis of a Pillar in a direct View is equidiſtant for either Side; but it is not ſo in the oblique View; for as it riſes

riſes from the Center of the elliptic Baſe, it will, in that Caſe, be nearer to that Side next the Eye than to the other, as is evident by Inſpection of the Figures. The ſame Thing is to be underſtood of any other regular Body in an oblique View.

1589. In Fig. 5. we have two PARALLELOPIPEDS, or *ſquare Columns*, put into Perſpective, on their proper Baſes (*a b c d*, *e f g h*) which are ſuppoſed to be ſquare; one in a direct View, in which only one Side can be ſeen; the other in an oblique View, ſhews the ſame equal Side in Front, and a Profile of the other Side (*d c f h*) next the Eye at (*i*) the Point of Diſtance being (*z*). Theſe are Caſes ſimilar to thoſe before obſerved of the Cube (1546 to 1549) in the ſame Points of View.

1590. The Manner of placing a SPIRE or STEEPLE on the Top of the *Tower* of a Church, *&c.* in Perſpective, is ſhewn in Fig. 6. The Spire is ſuppoſed to be a *ſquare Pyramid*, the Perſpective of whoſe Baſe is (*a b d e*) (being a ſide View, like that in Fig. 1.) Here alſo you can ſee only two triangular Sides (*viz. b g d* and *a g b*) ſo far as they riſe above the Top of the Tower. If from the Center (*c*) of the perſpective Baſe a Perpendicular be raiſed, and its perſpective Height (*c g*) proper to the perpendicular Height of the Pyramid, than (*g*) will be the Vertex of the Spire, which will be much nearer to that Side of the Tower next the Eye, than to the other, for the Reaſon before-mentioned.

1591. To place a TABLE, CHAIR, *&c.* on a Floor, as (*a b c d*,) in true Perſpective, is no difficult Thing by the Rules preſcribed. For Example, let *a d* be 10 Feet, and ſuppoſe the Table 4 Feet long, and to ſtand 3 Feet from the Side of the Floor (*a b*) and 1 $\frac{3}{4}$ Foot from the ground Line (*a d*) as in Fig. 7. is evident; let the Height of the Table be 2 $\frac{1}{2}$ Feet, and its Width 16 Inches, then on (*a*) erect the Perpendicular (*a m*) and from (*a*) to (*e*) ſet off 2 $\frac{1}{2}$, the Height of the Table, and draw (*e i*) to conſtitute the *menſural Angle* (*e i a*). Then draw Parallels to the ground Line at the Diſtance 1 $\frac{3}{4}$, and 3 Feet; they will cut the Radial (*i a*) in (*g* and *h*) then the Perpendiculars *g f*, *k l*, will be the perſpective Heights of each Side of the Table; all which, with every other Circumſtance, is directed by the Rules in Chap. VIII.

1592.

1592. As to the CHAIR, if the Perſpective of its Baſe be found, then the Height of the Seat and Back are in the ſame Manner determined by an Angle of Meaſures, or by the common Rules. The Bottom or Seat of the Chair not being a *Square* but a *Trapezium*, will have its Perſpective determined by the Point of Sight (*i*) and accidental Point *x* (1483) with regard to the two Sides which are *parallel*; but the other two which are inclined converge to a Point before they arrive at the horizontal Line, which is ſuppoſed at an infinite Diſtance from the Eye, and where only parallel Lines can meet, nor they neither but in Perſpective.

1593. If a *Box* is to be opened in Perſpective, then the Poſition with regard to the ground Line is to be conſidered. If the End be parallel thereto, then the Corner of the Lid, in opening, will deſcribe a Circle in Perſpective, as (*a d f*) Fig. 8. (1469, 1504) and therefore if a Circle be deſcribed on each End of the Box, with a Radius equal to the perſpective Width of the Box, and the given Angle to which the Lid is opened, be ſet off from (*a*) to (*b*); then you will have the four angular Points of the Lid by which it may be drawn as required.

1594. But if the Side of the Box be placed parallel to the ground Line, then the angular Point of the Lid, in opening, will deſcribe a Semi-ellipſis in Appearance on the perſpective Plane, as (*a b d f*), ſuppoſing the End of the Box contiguous to the Side of a Room. Here the elliptic Angle (*a c b*) is determined from the given circular Angle in which the Lid of the Box ſtands open, as directed (1498), or if the Angle (*b*) has its perpendicular Height meaſured, and its Perſpective found (1570) the Perſpective of the Lid or Cover is thereby determined.

1595. Again, if Doors, Windows, *&c.* are to be put into Perſpective as they ſtand open in any given Angle; the Rules for doing it are ſtill the ſame. Thus ſuppoſe the Door be 2 Feet wide = A C (Fig. 9.) then on the Center C deſcribe the Semicircle A D F, upon which the angular Point of the Door moves; ſuppoſe the given Angle to which it is opened be A C B; then if on the perſpective Floor (*b e f d*) you find the Perſpective of the Semicircle A D F; and therein make the elliptic Angle (*a c b*) correſpondent to the given Angle A C B in the Prototype; you will have the Poſition of the Point (*b*) on which you

raiſe the perpendicular Perſpective (*b o*) of the given Height of the Door (1568) and then, laſtly, by forming the Perſpective of the Door-place or Aperture (*a m n c*) in the Side of the Room, you will have the four angular Points of the Door determined, by which it may be readily drawn in the Poſition required.

1596. In the ſame Manner the Windows on the upper Part of the Room may be drawn. But it is to be obſerved, that ſince the Angle is given in which the Door or Window is opened, and alſo the Diſtance on, and Height, above the ground Plane, the Angles of the Door, or Caſement, may be moſt eaſily determined by the Method of Perpendiculars as directed in (1567.)

1597. If a Perſon in a Room, on one Side of a Street, views through the Windows the Buildings on the other Side, the Window then becomes the perſpective Plane or Table for the orthographic Delineation of whatever appears in Front on the other Side of the Way, (1649,) as is here repreſented in Fig. 10.

1598. The whole *Scenography* or perſpective View of the Inſide of a Shop is preſented in Figure 11. in order to compare it with the Picture made thereof by a Speculum or Lens which we ſhall hereafter have Occaſion to take Notice of.

CHAP. X.

The PRINCIPLES *of* Catoptric *and* Dioptric PERSPECTIVE *conſidered with regard to* VIEWS, PICTURES, LANDSCAPES, *&c. formed by* MIRRORS *and* LENSES; *with the* RULES *for exhibiting them* in any required Proportion *to the* OBJECTS.

1599. WE have already conſidered ſo much of *Catoptric Perſpective* as relates to *reflecting Planes*; but it was neceſſary to treat of *Scenographic Perſpective* before we could properly handle the Doctrine of *Optical Perſpective* univerſally as conçern'd in the Formation of *Images*, *Pictures*, *Landſcapes*, and all Kind

Kind of *Views* by reflected and refracted Light from and through Convex and Concave Glaſſes of every Kind.

1600. The Subject we are now entering upon, though the moſt eſſential and exquiſite Part of the Science, has not (that we know of) been touched upon by any Author on Perſpective, and therefore we ſhall be the more particular and explicit on this Head. In order therefore to determine how far the *Picturesque* IMAGES of OPTICAL GLASSES agree and coincide with the *Drawings made* by the *Rules of common Perſpective*, it muſt be remembered, that the Linear Dimenſions of an Object and its Perſpective are proportioned to their Diſtance from the Eye (1468) that is (in Fig. 1. Plate II.) R N : *r n* :: I Z : I i :: A B : *a b* :: A D : *a d*.

1601. Now ſuppoſe A G were an open Cube or other rectangular Body whoſe oppoſite Sides are equal, and let the Side E G in Front be called the *Proſcene*; the hinder Side A C the *Poſtſcene*; the Sides E B, H C, the *Lateral Scenes*; the ground Plane A H, the *Primary* SCENE, and C B the *Ærial Scene*, if placed above the Eye; or *Secondary Scene*, if below it. Then conceive a perſpective Plane placed contiguous to the Proſcene, on which the whole *Scenography* of the Body will be viewed in Perſpective by the Eye at I.

1602. Put the linear Dimenſion of the *Proſcene* E H or E F = P, and the Perſpective of the *Poſtſcene a d* or *a b* = *p*; alſo let their Diſtances from the Eye be I i = D, and I Z = *d*. Then we have P : *p* :: *d* : D. And this Analogy will hold for any Situation of the perſpective Plane, between the Eye and the given Cube.

1603. If inſtead of the Eye, a *Convex* or *Concave* Speculum were placed at I, then an Image will be formed of the ſaid Cube or Body, and the linear Dimenſions of every Part will be proportioned to the Diſtance from the Glaſs, as we have ſhewn (1310) Then, if we call A B or A D = an Object, and put it equal to O, and its Image = *i*; we ſhall have O : *i* :: *d* : *f* = focal Diſtance of the Image; and therefore $\frac{di}{O} = f$ in all Kinds of Speculums.

1604. In a CONVEX MIRROR, we had $\frac{dr}{2d+r} = f$ (1291) therefore $\frac{dr}{2d+r} = \frac{id}{O}$; from whence we get $Or = 2id + ir$, which gives this Analogy, $O : i :: 2d + r : r$. Or, *the Proportion of the Length or Breadth of the Poſtſcene is to that of the Image thereof, as twice the Diſtance added to the Radius, is to the Radius of the Speculum.*

1605. The Proſcene being equal to the Poſtſcene (1601) its Length or Breadth EF or EH will ſtill be denoted by O, but its Diſtance being leſs, let that be $= D$; and becauſe the Image will be larger in Proportion, put it $= I$, and proceed as before; we have $Or = 2ID + Ir$, and the ſame Analogy $O : I :: 2D + r : r$.

1606. But becauſe in both Caſes Or is the ſame, therefore we have $2id + ir = 2ID + Ir$, and therefore $I : i :: 2d + r : 2D + r$. From hence we have the Ratio of the linear Dimenſions of the Proſcene to that of the Poſtſcene; and conſequently the *Optic Scenography* of the Solid is thereby determined, and its Difference from that in common Perſpective.

1607. In the CONCAVE *Speculum*, becauſe $\frac{dr}{2d-r} = f$, (1291) we have $Or = ir - 2di = Ir - 2DI$; whence $I : i :: r - 2d : r - 2D$. From whence it appears, that when the Radius of the Speculum exceeds twice the Diſtance of the Objects, then the Images will be *poſitive* or *behind* the Glaſs; but otherwiſe *negative* or before it, agreeable to (1315, 1316.)

1608. In Caſe of a LENS, it is every where $O : i :: d : f$, and $\frac{id}{O} = f$ (1340). Alſo for a *double and equally Convex Lens* we have $f = \frac{dr}{d-r} = \frac{id}{O}$ (1387) and therefore $Or = id - ir = ID - Ir$; whence we have $I : i :: d - r : D - r$; or in the *Scenography of the Image* of ſuch a Lens, the *linear Dimenſions of the Proſcene and Poſtſcene are as the Differences between the Diſtances and Radius of the Lens reſpectively.*

1609. After the ſame Manner the *Scenography of Images* for all different Forms of Lenſes may be found, and compared with thoſe

thoſe of common Perſpective. And ſince, in all, the Terms of Compariſon conſiſt of the Sums or Differences of the Radius and Equimultiples of the Diſtances; therefore *while the Radius of the Glaſs bears any conſiderable or ſenſible Proportion to the Diſtances of Object, the optical* SCENOGRAPHY *of their* IMAGES *will differ* more or leſs from the common Perſpective thereof.

1610. But it is evident, when the Diſtances are ſo great that the Radius of the Glaſs bears no ſenſible Proportion thereto, it will then vaniſh out of the Terms of the Compariſon, and then the Analogies

become in the { Convex Mirror, $I : i :: 2d : 2D :: d : D.$ / Concave Ditto, $I : i :: -2d : -2D :: d : D.$ / Convex Lens, $I : i :: d : D.$ }

1611. Conſequently, in all ſuch Caſes the optical Scenography of the Image is the very ſame with that of common Perſpective on the tranſparent Plane; for by all the Glaſſes, we have $I : i :: d : D$, and on the perſpective Table, we have $P : p :: d : D$ (1468) therefore $I : i :: P : p$. Whence it appears, *that all the Parts in the optical and perſpective Scenography are perfectly ſimilar, and that, therefore, the* IMAGES *of* OPTICAL GLASSES in ſuch a Caſe, are PICTURES, PROSPECTS, or LANDSCAPES in *true Perſpective.*

1612. Hence the excellent Uſe of a *Convex Speculum*, in exhibiting to the Eye a *genuine perſpective View* of all diſtant Objects, as a perfect Copy for the Artiſt to draw by. ——— It preſents him with an inſtantaneous Conſtruction of the perſpective Scenography of the interior Parts of a Room, and all its Furniture, Tables, Chairs, Scrutores, Book-caſes, Pier-glaſſes, Pictures, *&c.* juſt as they ought to be drawn. —— An Inſtance of its Uſe in this reſpect, is Fig. 11. Plate III. which is a *perſpective View* of a *Mathematical* SHOP, with its *Counters, Spheres, Globes, Air-pumps, Teleſcopes, Glaſs-caſes, &c.* actually delineated from the *perſpective* PICTURE of a CONVEX MIRROR (1598.)

1613. Another great Uſe of ſuch a *Speculum*, is to exhibit the true Perſpective of an Object diminiſhed in any given Degree. For, becauſe $Or = 2id + ir$ (1607) it is $Or - ir = 2id$,

$2id$, and therefore $d = \frac{Or - ir}{2i}$. For Example, ſuppoſe it required to draw a Microſcope, Air-pump, &c. 4 Times leſs than the Life; then $O : i :: 4 : 1$; and let the Radius of the Speculum be $r = 12$ Inches; then $d = \frac{48 - 12}{2} = 18$ Inches, the *Diſtance from the Mirror to make the Image 4 Times leſs than the Object.*

1614. Since the Image is larger in Proportion as the Radius of Convexity is ſo, it appears how excellently well adapted ſuch Mirrors are, when very large, for exhibiting the moſt perfect LANDSCAPES of diſtant SCENES, whether *Gardens*, *Fields*, *Lawns*, *Woods*, *Mountains*, *Vales*, *Rivers*, *Sea*, &c. with all the natural COLOURS, LIGHTS, and SHADES, MOTION, and every other Incident which can tend by this *perſpective Miniature*, to improve and out-vie even NATURE itſelf.

1615. The ſame Things may be ſaid of a *Concave Mirror*, with regard to the Form and Proportion of the ſeveral Parts of the Image, and the juſt Perſpective of the Whole, but then the *inverted Poſition* of the Picture before the Glaſs, and the Inconvenience of obſerving it, renders it not ſo uſeful in the Arts of drawing and deſigning, as that of the Convex Form.

1616. But then it has this moſt entertaining Property of reſolving the perſpective Picture, or Landſcape formed by the *Convex Mirror* into its *Original* or *Prototype*, and gives each Part Diſtances, Size, and Situation. In this Caſe it is ſuppoſed that the Radius in both Speculums is the ſame. Thus, for Illuſtration, ſuppoſe a *View of 7 Miles round St.* PAUL's were drawn from a Convex Mirror of 10 Feet Radius, that Drawing held at the ſame Diſtance before an *equal Concave Mirror* as it appeared to be behind the Convex One, the ſaid Concave would revert the Landſcape, and preſent the Eye with a *delightful View* of the large *City of London*, and County about it, as *large as the Life*, and at the *ſame Diſtance* in every Part, as appears to the naked Eye. The Reaſon of all which, is very evident from the Theory we have above premiſed. Hence *the great Uſe of Concave Mirrors in viewing perſpective Prints*, in the *portable* CAMERA OBSCURA.

1617.

1617. From the ſame Theory (1608) it appears, that a *Convex Lens* does alſo preſent us with the ſame *perſpective Figure* or *Image* of diſtant Objects in the Focus, which as the Radius is longer, will be larger in Proportion; and therefore, in a real *Camera Obſcura*, when ſuch a Lens is applied in a *ſcioptric Ball and Socket*, you view upon a Screen, at a proper Diſtance, the *Scenography* of *Buildings*, and a LANDSCAPE of every rural Scene, ſo heightened by Colours, and animated by Motion, as juſtly excite our Admiration, and we readily pronounce the Pencil of Nature perfect, and all her Paintings inimitable.

1618. Upon the Whole we may conclude, that as all artificial Paintings are but Copies of Nature, the more they approach to, and are regulated by the Art of Perſpective, the more natural and valuable they will be, and Beauty and Harmony will ſo much the more evidently appear; for we have ſhewn in the Theory (1292) that in all Nature's Painting the ſtricteſt *harmonical Proportion* is obſerved; and conſequently NATURE is all PERSPECTIVE and HARMONY.

CHAP. XI.

Of the PERSPECTIVE APPEARANCE *of* OBJECTS *on* INCLINED PLANES.

[PLATE IV. *Of* PERSPECTIVE.]

1619. WE have hitherto treated of ſuch perſpective Views as are ſuppoſed to be taken on Planes perpendicular to the ground Plane; but as it will, on many Occaſions, be neceſſary to conſider them on Planes or perſpective Tables inclined in any Angle thereto, I ſhall here deliver the Theory and Rules for that Purpoſe.

1620. Let A D N R be a perſpective Table inclined to the ground Plane B C G K in a given Angle (Fig. 1.) A B C D is a Rectangle viewed upon the Table by an Eye at I, at the Height I H, and Diſtance from the ground Line H F. Biſect B C in E,

E, and draw E H, and I *i* parallel thereto, then will (*i*) be the Point of Sight on the Table, and R Y, drawn through it parallel to A D, will be the horizontal Line, in the ſame Manner as in the upright Plane (1465).

1621. Produce C K and E F indefinitely, and make the Angle I M H = Angle of the Plane's Inclination; then will I M be parallel to *i* F. Draw the Radials *i* D, *i* A; make K L = H M, and join I L, which is then parallel to *i* D. From the Points B and C, draw the viſual Rays I B, I C; they will interſect the Radials *i* A, *i* D, in the Points S and P, then joining S P, the Area A S P D will be the Perſpective of the Rectangle A B C D; the Demonſtration is the ſame as was uſed for the upright Table (1475, *&c.*)

1622. But for a practical Method, the following Theorem in this Caſe may be preferable. In the ground Line A D produced, take D O = D C; and in the horizontal Line R Y, take *i* Y = *i* I, the Diſtance of the Eye from the Table; draw Y Q parallel to *i* D, and draw Y O, which will interſect the Radial *i* D in *p*, the ſame as the Point P found by the other Theory. For in the ſimilar Triangles L C I and D C P, we have I C : D C :: L I : D P; and in the ſimilar Triangles Y O Q and *p* O Q, we have Q O : D O :: Y Q : D *p*. But the three firſt Terms of each Analogy are ſeverally equal to each other; for L C = L D + D C = Q D + D O = Q O; and D C = D O, by Conſtruction; alſo L I = *i* D = Y Q. Therefore, alſo D P = D *p*, and conſequently the Points P and *p* coincide or are the ſame. *Q. E. D.*

1623. Whence the practical Rule for finding the Perſpective of a given Point C on the ground Plane, upon a given inclined Table is this. *From the given Point draw a perpendicular* C D *to the ground Line, and draw the Radial* i D; *in the ground Line* A D *produced, take* D O = D C, *and* i Y = *i* I, *and draw* Y O, *which will cut the Radial* i D *in the Point* P, *the Seat or Perſpective of the given Point* C, *as required.*

1624. The Diſtance of the Eye from the Plane is equal to its perpendicular Diſtance from the ground Line, together with the Cotangent of the Plane's Inclination; for *i* I = F M = F H + H M, but H M is the Tangent of the Angle H I M,

the

the Complement of the Inclination HMI to the Radius IH the Height of the Eye.

1625. Hence it will be eaſy to draw the Perſpective of any given Object upon an inclined Table BCTV (Fig. 2.) Let the Object be a hollow Cube whoſe Side is MNOP; ſuppoſe it to touch the Plane on its upper Part; and that the Inclination is equal to the Angle WCV. I is the Point of the Sight on the Plane, and X, Z, the Points of Diſtance, as in the upright Plane. Then, becauſe the Diſtance of the Cube from the Table at Bottom is CP, if we draw the Radials IC, IB, and the Diagonal YP, the Point of Interſection F will be the Seat of the Point P; and drawing the Line YO, it gives the Point H for the Seat of the Point O. Therefore drawing EF and GH parallel to the ground Line BC, the Area EFGH will be the Perſpective of the Square Baſe of the Cube, by the Theory (1487.)

1626. The inclined Table being juſt the Width of the Cube, we have AD = MN; continue AD to S, which is now to be conſidered as a ground Line, for the Top of the Cube; therefore taking AR = MN, and drawing RZ it will cut the Radial IA in K; and then drawing KL parallel to AD, it will give AKLD for the Perſpective of the Top of the Cube. Laſtly, by joining AE, FD, GK, and HL, the whole *Scenography* of the Cube will be compleated.

1627. From this Procedure for the Cube, I preſume the Method of finding the Perſpective of any *Points*, *Lines*, *Superficies* and *Solids*, will not be difficult to the young Drafts-man; for this one Example includes them all, either contiguous to, or at a Diſtance from the Table; and parallel and perpendicular to the ground Plane, ground Line, *&c.*

1628. I need not obſerve, that the Sides of the Cube in Front, are not *Squares* in the Perſpective, as they are in upright Planes, but *Trapeziums*; for EF is leſs than AD; and GH leſs than KL. And this will be the Caſe of all Lines and Superficies not parallel to the perſpective Table.

1629. The Inclination of the Plane may be ſuch that the *Perſpective of a Circle* on the ground Plane ſhall be a *Circle* alſo. Let DE (Fig. 3.) be the Diameter of the given Circle; HD the Diſtance, and HI the Height of the Eye; draw the Rays ID, IE; and let the Plane HF cut them *ſubcontrarily* in *e* and *d*;

(1510) then ſince all the Angles in the Triangles DHI, and EHI, are known; and by Suppoſition the Angle HD*e* = H*d*E, and the Angle HE*d* = H*e*D; therefore ſince H*e*D = IH*e* + HI*e*; we have H*e*D — HI*e* = IH*e*, the Inclination of the Plane HF required for that Purpoſe.

1630. Hence alſo it is eaſy to aſſign the Inclination of a Plane wherein the Seat or Perſpective of a given Point ſhall be the ſame as in the perpendicular Poſition. Thus (Fig. 2.) ſuppoſe CV the upright Plane, the Eye at Y views the Point O thereon at (*a*); on the Center (C) with the Diſtance C*a*, croſs the Ray YO in (*b*,) and through that Point draw CQ which is the Poſition of the Plane required, ſince C*b* = C*a*.

1631. Let CW be perpendicular to the Ray YO, then is the Angle VCW = *a*OC; and conſequently the Angle *a*C*b* = 2*a*OC.

1632. Since the right Line CO occupies the ſame Space C*b*, C*a*, on the inclined and upright Planes, it will follow that any Line CP at a leſs Diſtance will take up a leſs Space, and any Line CX at a greater Diſtance will take up a greater Space on the inclined Plane, than on the ſame in the erect Poſition. Much more might be ſaid on this Head, which we ſhall leave to the Reader's Invention, having premiſed the Principles, which is here all that is propoſed.

CHAP. XII.

Of Theatrical PERSPECTIVE; *or, the* RULES *of* PERSPECTIVE *applied to the* SCENERY *of a* THEATRE.

[PLATE IV. *Of* PERSPECTIVE.]

1633. THE whole Artifice or Deſign of a THEATRE, is founded in PERSPECTIVE. For as we are to be entertained with an Imitation of ſome memorable Actions paſſed, ſo in order to render the Performance more natural and pleaſing, the Scenes on which they were tranſacted ought to be repreſented as they appear to the Eye of a diſtant Spectator, that is, in *Perſpective*. For whether there be a tranſparent Plane, or not, be-

Plate IV.

Fig. 1

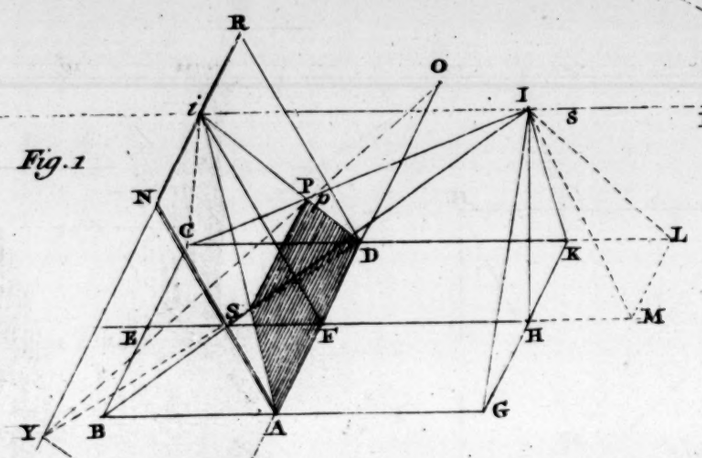


Fig. 2

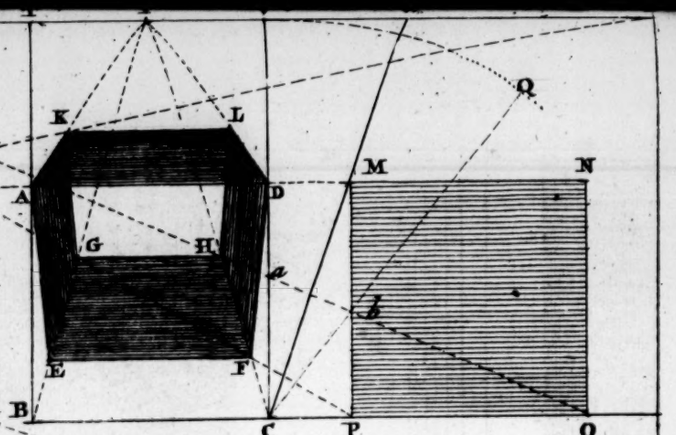
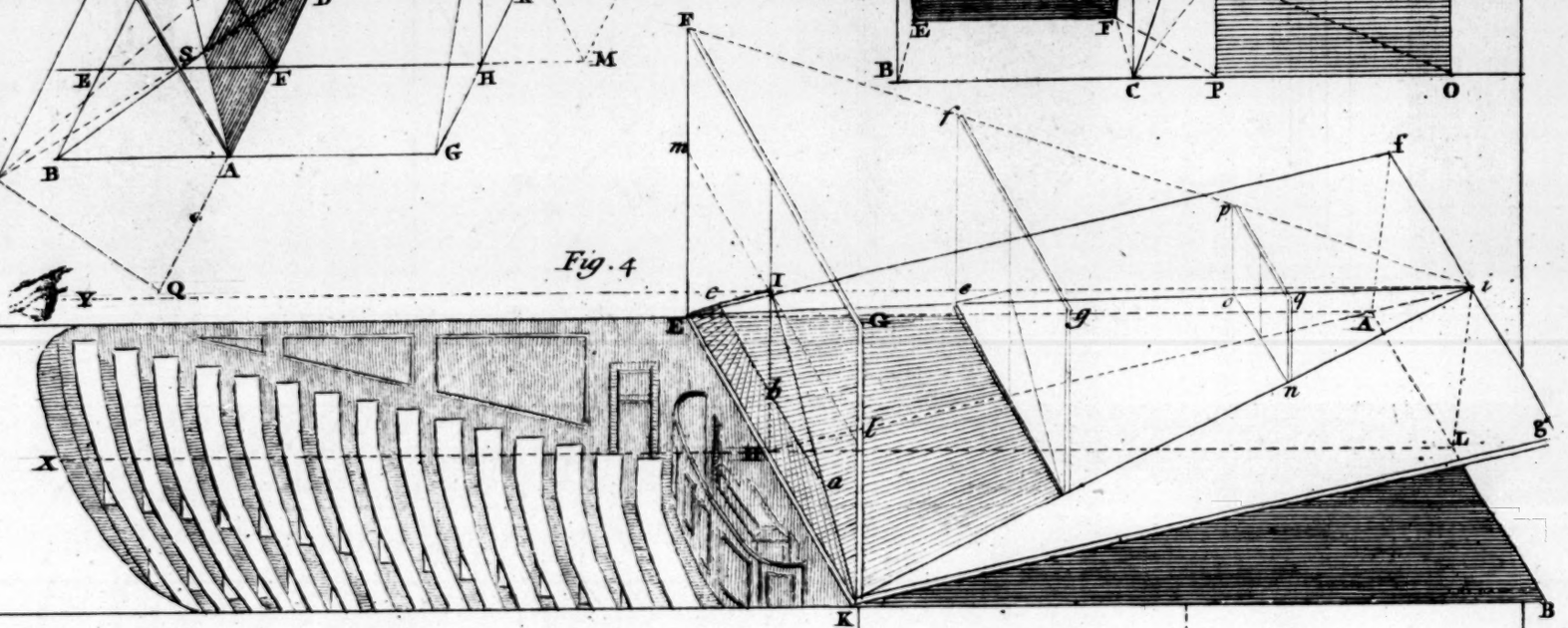
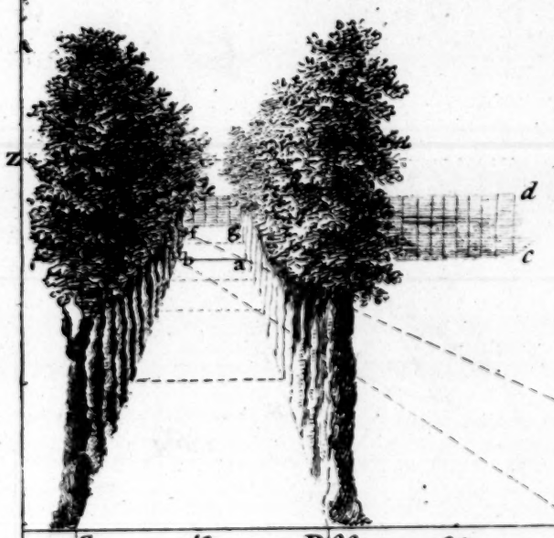


Fig. 4





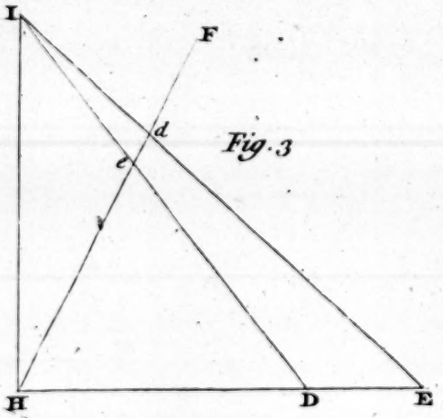


Fig. 5

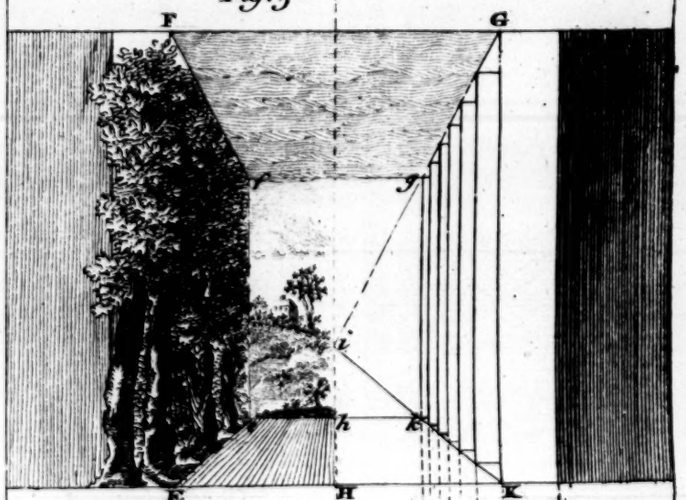


Fig. 7

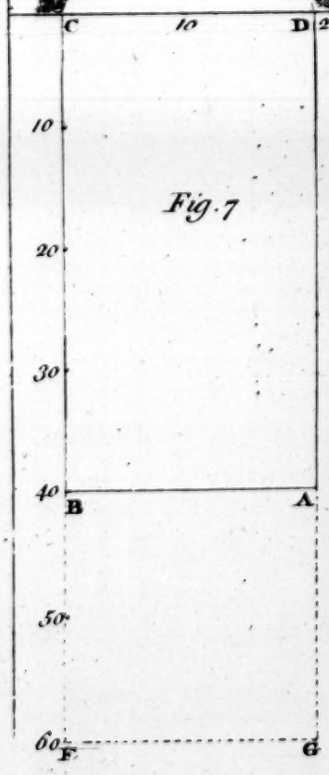


Fig. 8

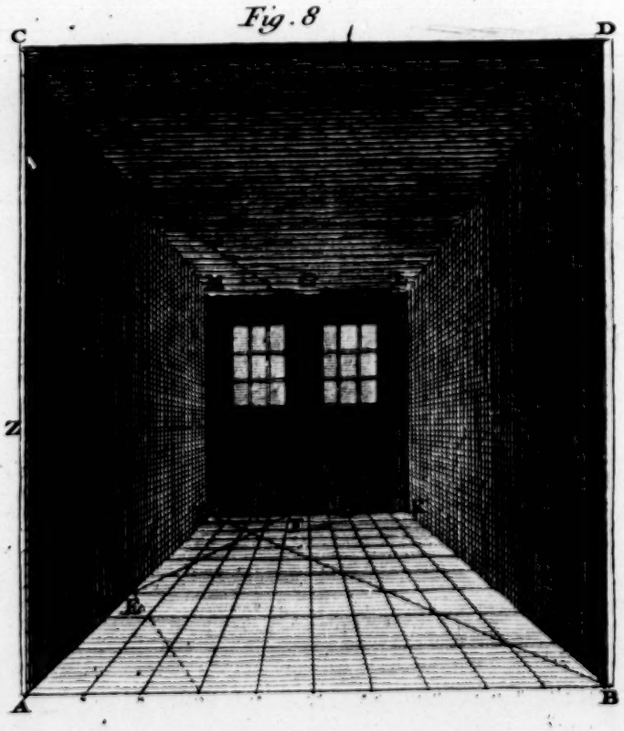


Fig. 6.

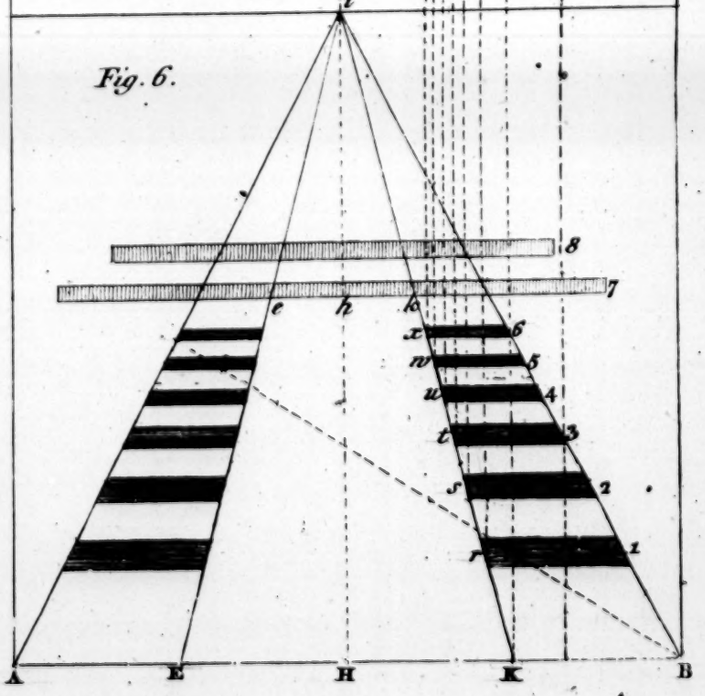


Plate V.
of
PERSPECTIVE.

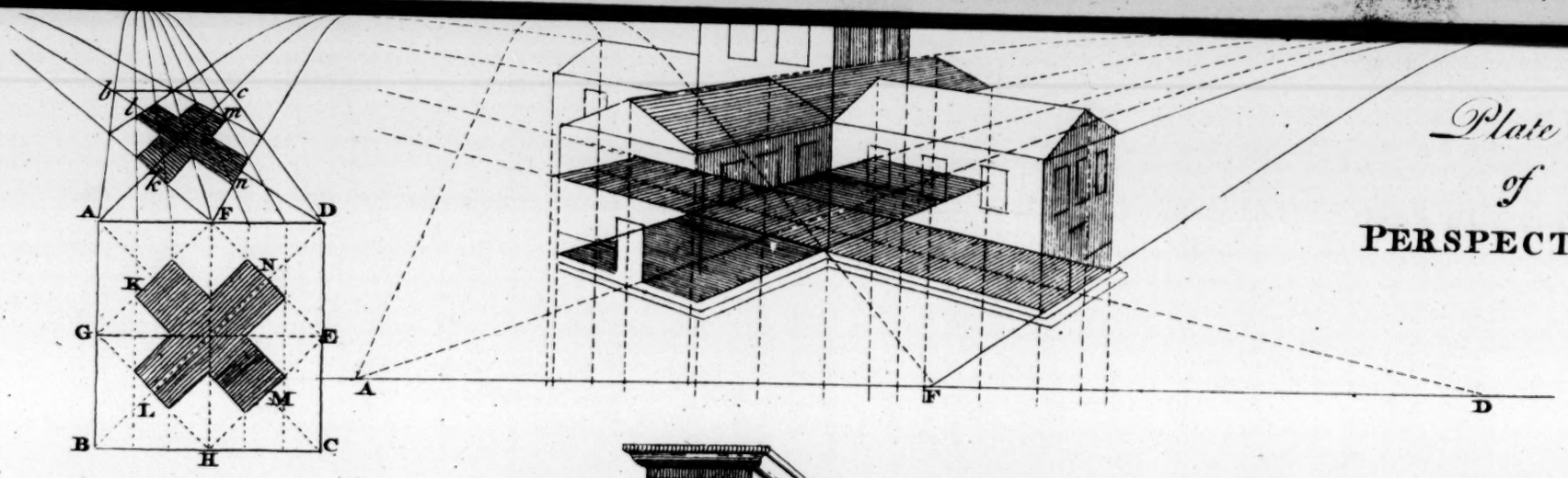


Fig. 3

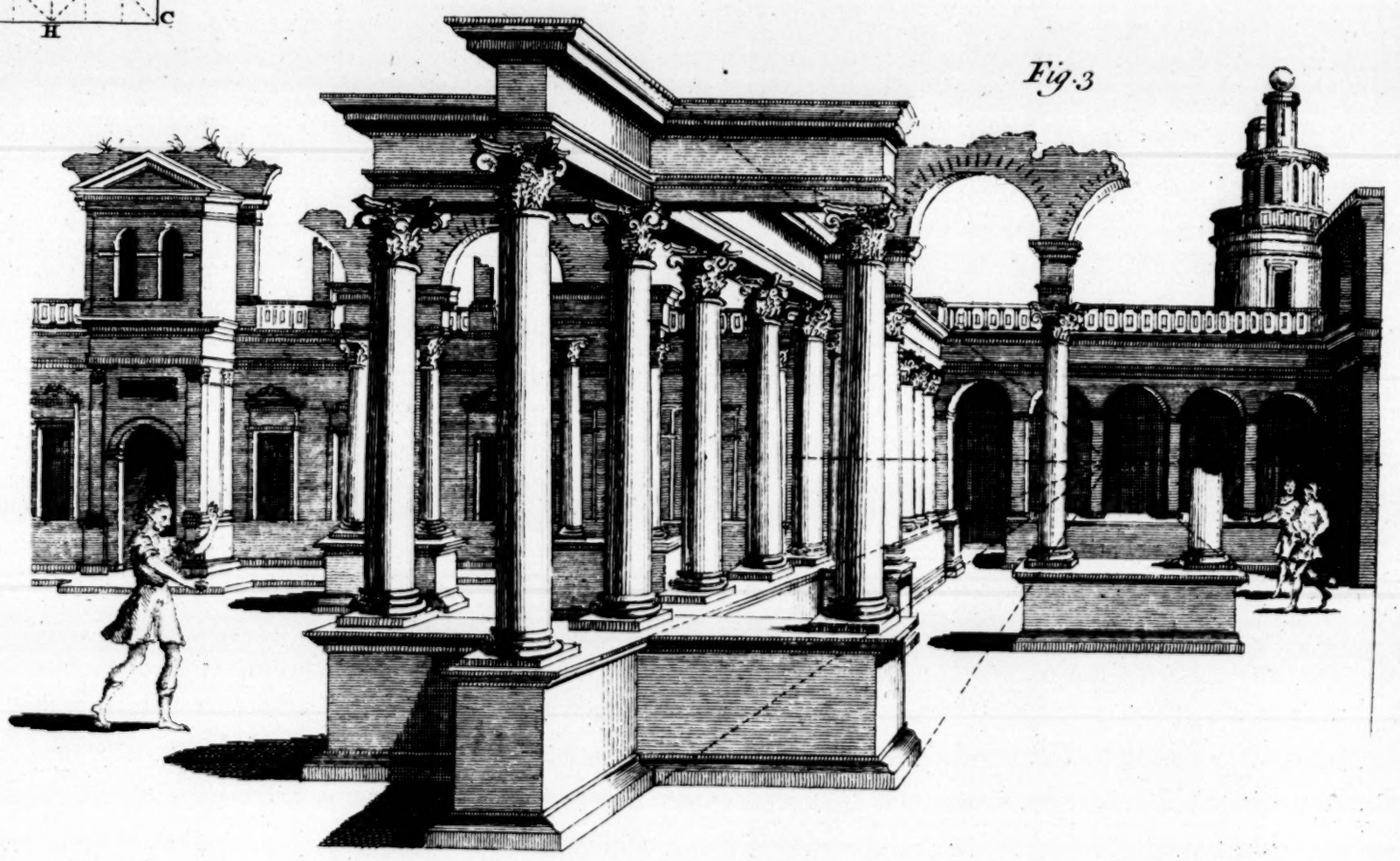


Fig. 4

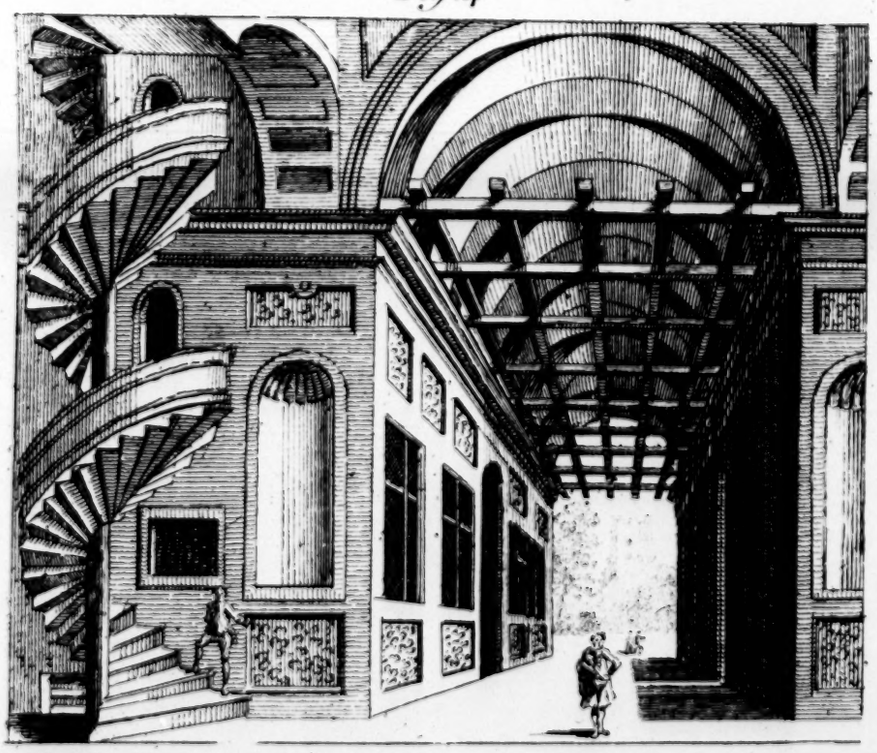


Fig. 5



between us and the Scene of Action, the Image in the Eye is equally in Perſpective, and excites the ſame Idea in the Mind, as if the whole Affair were viewed on a perſpective Plane.

1634. Hence it is, that the whole HOUSE, internally, is only one large Scenography, or perſpective View of an hollow *Parallelopiped*, or large *Room*, about three or four Times as long as it is Wide or Deep. Hence it is, that you ſee the Body of the Houſe ſo contrived, as to repreſent the Perſpective of the firſt Part of ſuch a large Room, the Ceiling, Sides, and Floor, all Converge, by the Rules of Perſpective, to a Point of Sight at the fartheſt End, which is the remoteſt Part of the *Stage*.

1635. For the STAGE is only the Floor of the other Part of the Scenography of the large Room, ſuppoſed to have been the Scene of Action; and therefore alſo the *Sides* and *Ceiling*, *Proſcenes*, and *Poſtſcenes*, are all in regular Perſpective. But in order to give a more exact Idea of this Matter, we muſt have Recourſe to perſpective Repreſentation alſo.

1636. Therefore let A B C D (Fig. 4.) repreſent the ground Plane or Floor of a long and large Room, on one Part of which toward A B, ſuppoſe ſome great Action or Converſation to have happened, and the ſame to have been obſerved by the Eye Y at the other End of the Room. Then it is evident, if in any intermediate Part a tranſparent Plane E F G K had been erected, the Obſerver would have viewed the Whole tranſacted perſpectively on the Plane E G, in that Part *ac* E K which includes the Scene of Action A E K B. (1487.)

1637. Now though we have ſhewn how ſuch an Event may be truly repreſented in a perſpective View, yet this is ſtill but a Picture of it; there, it is true, we ſee the *Place*, the *Perſonages*, the *Manner*, the *Time*, and ſome other concurring Circumſtances, that altogether, give ſome Idea of the Thing; but real *Life*, *Action*, *Voice*, and *Variety of Scenes*, *Attitudes*, *Paſſions*, *&c.* are wanting to animate and realize the perſpective View, or Picture.

1638. Now all this is effected by the ARTIFICE of a STAGE or THEATRE. For let Y X be the Height of the Eye placed over the middle Line of the Room, and drawing Y I parallel to H X, the Point of Sight will be I in the horizontal Line (*l m*) of he perſpective Plane. And now, inſtead of the upright Plane

E G, we ſuppoſe another Plane f E K g placed on the ſame Line E K, and ſo far inclined, that its Elevation i L be juſt equal to the Eye's Height Y X, it is evident, the perſpective Horizon (*l m*) will coincide with the End f g of the inclined Plane; the perſpective Scene *a c* E K will be projected into the large One *e* E K *k*, and all above it on any other perpendicular Plane at (*e k*) or (*n o*) behind.

1639. This new inclined perſpective Plane f E K g is the *Stage* or *Theatre* we ſpeak of, and is a Scene of real Action in a perſpective View at the ſame Time. The Area *e* E K *k* is here the perſpective Scene of Action (or *Scena Dramatis*) and inſtead of Men and Women in the Picture *a c* E K, we have ſufficient Room for real Perſons (*Perſonæ Dramatis*) to act the ſeveral Parts of the intended Repreſentation (or *Drama*); here now are the *Proſcene*, the *lateral Scenes*, the *Poſtſcene*, &c. all in Perſpective, and moveable to the moſt convenient Situations. Not only the whole Scenography of the Stage is by this Means formed and adjuſted by the Rules of Perſpective, but the very Actors themſelves, as they advance or retreat, have their Appearance encreaſed or diminiſhed to the View of the Spectators (eſpecially thoſe remote in the Pit and Gallaries) according to the Laws of *optical Perſpective*.

1640. On this inclined Plane, or perſpective Stage, the dramatic Scene may be encreaſed at Pleaſure, and thereby very diſtinct Views and Proſpects agreeably riſe to the Sight, by the artful Paintings and Deſigns on the Lateral and Poſtſcenes; thus *e* E K *k* may be enlarged to *o* E K *n*; and *f e k g* which was before the Poſtſcene, may now be conſidered as the Proſcene, to a diſtant Dramatic Scene (*o e k n*) whoſe Poſtſcene is (*n o p q.*) And thus the Stage, or Scene of the *Drama*, may be extended to comprehend the moſt diſtant View or Scenes of Action, even to the Horizon itſelf, for the remote End of the Stage, *viz.* the Line f g, is the horizontal Line in this theatrical perſpective Plane.

1641. As the *Dramatic Scene e* E K *k* is variable according to the Nature and Circumſtances of the particular Parts or Actions of the Play, ſo the Lateral and Poſtſcenes muſt be conformable thereto, and conſequently variable, and of different Sorts; thus if the Stage repreſent the Inſide of any particular Building, as a Ca-

Caſtle, Priſon, Church, *&c.* the Side and Back-ſcenes muſt correſpond to the ſame in all the Characters appropriate to ſuch Edifices. If the Stage be a Street, the lateral Scenes repreſent Houſes, *&c.* on each Side, and the Poſtſcene gives a perſpective View of all at the far End, or beyond it. If you ſuppoſe the Scene a *Viſta*, then Rows of Trees, *&c.* deſcend in Perſpective on either Side. If *Fields* or *Lawns* be the primary Scene, and Swains, Shepherdeſſes, *&c.* the Perſons of the Drama, then Trees, Woods, Cottages, *&c.* make the Perſpective of the Sides; and diſtant Views of Countries, ſunny Hills, and horizontal Clouds are portray'd on the Poſtſcene.

1642. With regard to the Scenes, they are variable and different in the different Parts of the Scenography; when the Stage is not open, the *Proſcene* is only a plain Curtain, to be drawn up and let down as Occaſion requires. The *Poſtſcene* is either a Curtain with Deſigns or Drawings in Perſpective; or elſe it conſiſts of two ſliding Parts, which being put together, from each Side the Stage, make one uniform perſpective Piece ſuitable to the Nature of the Part of the Play then acting.

1643. The lateral Scenes, and thoſe above the Stage, conſiſt of many different Parts; thoſe on the Sides are moveable in Grooves made in the Floor of the Stage; and may by this Contrivance be variouſly changed, and being placed oppoſitely one a little before the other, they make together but one united View or perſpective Scene, and by this Means produce an agreeable Deception, and at the ſame Time give an Opportunity for the Actors to come on, and go off, on any Part of the Stage, as the Circumſtances of the Action may require. In the ſame Manner the aerial Scene may be compoſed of ſeveral Parts of particular Curtains, or moveable Pieces, deſcending one below the other, and uniting in one perſpective Deſign.

1644. But to ſee more clearly how theſe Things are contrived and diſpoſed, let Fig. 5. repreſent a front View of the Stage in Perſpective. One Half of which is plain and naked, and the other Half diſguiſed and decorated with perſpective Paintings, *&c.* E F G K is the Front; E *e k* K, the Stage; G *g k* K the plain Side, conſiſting of ſeveral lateral Planes projecting beyond each other in their proper Grooves; (*e f g k*) is the Poſtſcene, half plain, and half painted; E F *f e* is the other Side, covered, as

it

it were, with perſpective Drawings and Deſigns; and making but one Piece in Appearance. F *fg* G is the aerial Scene, open on one Part, and properly formed on the other; (i) is the Point of Sight by which the Whole is regulated. This Piece of Perſpective is the proper Scenography of that Part of the Stage in Fig. 4. which is denoted by E F G K *k g f e*.

1645. Hence it is evident, that the Heights of the ſeveral Planes, or Sliders, on the Sides of the Stage, are to be adjuſted by the Inclination of the Line F *i*, or G *i*, to the Line E *i*, or K *i*, on the Stage; ſo that the Height E F or G K, of the firſt Sliders, and (*e f*) or (*g k*) of the laſt, in the perſpective Stage Fig. 5. are the ſame as the Lengths of the Lines denoted by the ſame Letters in Fig. 4. And the ſame is to be obſerved, for the Sliders in any other Scene (*e k n o*) in a remote Part of the Stage.

1646. Another Figure is yet neceſſary to ſhew how the Grooves are to be made on the Sides of the Stage, and their Diſtances aſcertained and proportioned. In order to this let K *k* (Fig. 4.) be the Perſpective of the ground Line for the Length of the Stage divided into equal Parts (as directed in 1554, 1555.) Then let Fig. 6. be the whole Length of the Stage; A B the Width of the Houſe, and E *i* K the Area of the Stage, as in Fig. 4. and draw the Lines A *i*, and B *i*. Then from the Points of equal Diviſion in the perſpective Line K *k* (Fig. 5.) raiſe Perpendiculars, and they will terminate the fore Parts of the lateral Sliders reſpectively, in that Perſpective of the Stage.

1647. Let the middle Line H *i* of Fig. 6. be in a right Line with H *i* in Fig. 5. Then from the ſeveral Points of equal Diviſion in the Line K *k* of Fig. 5. let fall Perpendiculars to the Line K *k* in Fig. 6. and they will interſect it in the Points K, *r*, *s*, *t*, *u*, *w*, *x*; through which, if Lines are drawn parallel to the ground Line A B, they will aſſign the Places and Diſtances of the Grooves (*r* 1, *ſ* 2, *t* 3, &c.) on each Side the Stage K *k* and E *e*. And the Figures 7 and 8 denote the Grooves of the Poſt-ſcene. But there is in different Stages, a different Diſpoſition of the Parts and Machinery; what we have ſaid is ſufficient for a *general Rationale of theatrical Perſpective*, which is all that is here intended.

CHAP.

CHAP. XIII.

The Doctrine of PERSPECTIVE DECEPTION *explained, and exemplified.*

[PLATE IV. *Of* PERSPECTIVE.]

1648. WE have already conſidered the various Illuſions and Deceptions which the *viſual Senſe* is ſubject to from *Catoptrics* and *Dioptrics*; that is, how different the Magnitude, Poſition, Place, Diſtance, *&c.* is of any Object, as it is conceived in the Mind by the *viſual Faculty*, from Rays reflected or refracted to the Eye in regard to the Idea of the ſame Thing excited by direct Rays.

1649. Of theſe Deceptions ſome are very uſeful, witneſs the *Looking glaſs*, *Reading-glaſs*, *&c.* Others afford us a very rational Amuſement; as thoſe Lenſes and Speculums which give an agreeable *Relievo* to perſpective Views, Prints, *&c.* uſed in *optical Machines*. And ſome on the other Hand gives us a great deal of Trouble, as in Caſe of Refractions through the Atmoſphere in Matters of Aſtronomy and Navigation, by which the Celeſtial *Phænomena* are ſhewn in very different Circumſtances from thoſe which are real.

1650. But the Deceptions we here intend to ſpeak of are purely Perſpective, and are deſigned to impoſe on the viſual Faculty in the moſt agreeable and advantageous Manner. Indeed all Perſpective is a Sort of Illuſion, as it repreſents Things not as they really are, but as they appear on an intervening Plane by refracted or reflected Light. And therefore, if in any particular Caſe Things are not juſt as we could wiſh to have them, we can, at leaſt, by this Art, make them appear to be ſuch, and to an inadvertent Obſerver, the Difference from the Truth will not be diſcovered, and that which is only *Perſpective* ſhall be taken for the *real Thing* it repreſents.

1651. We ſhall illuſtrate this Matter by an Example or two. Let A B C D be the ground Plane of the perſpective *Viſta* C *b a* D, (Fig. 7.) at the far End of which, ſuppoſe a high Wall (*b c d e*) at right Angles thereto; and let the Length of the Walk

B C be 40 Yards. Now if it be desired to make this Vista or Walk to appear longer than it really is, it may be thus effected. Suppose it were required to lengthen it by one Half, or to make it appear to be 60 Yards long. Then since on the ground Line C E, the Distance C 40 determines the Length of the Vista C *b*; if from the Point 40 you set off 20 Yards more, it will give the Point 60, from whence drawing a Line to the Point of Distance Z, it will give the Point *(f)* in the perspective Side C *b* of the Vista continued out; by which Means the perspective C f g D may be compleated for a Walk C F on the ground Plane 60 Yards in Length.

1652. Therefore since the additional Part of the perspective *Vista (a b f g)* may be painted on the Wall contiguous to the End *(a b)* and as it will be a perfect Continuation of the said Vista, it will exhibit to the Eye of a Spectator, at a proper Distance, the View of a Walk A F G D just 60 Yards long; nor will he be sensible of the Deception, or that the Part *(a b f g)* is drawn or painted on the lower Part of the Wall. The Truth and Pleasure of such perspective Illusions any one may be convinced of by the notable Instances of such Pieces of Art in *Vauxhall Gardens*, and in many other Places.

1653. After a like Manner you may rectify the Appearance of an irregular Room. Thus for Instance, suppose the Floor of the Room was in Form of the Trapezium A E F K B (Fig. 8.) deficient from a Quadrangle by the Part E I F; then in order to make the Room appear of the usual Form to a Person viewing it at some Distance, it will be necessary to compleat the perspective *Trapezium*, or Floor A E F K B into the Rectangle A I K B, which is done by drawing the Line B Z, intersecting the Side A E continued out in I, and joining I F. Therefore on the Side of the Room E F G H, you draw the Perspective of the several Parts deficient, *viz.* (1.) The perspective Triangle E I F for the Floor. (2.) The corresponding Triangle G H M for the Cieling. (3.) The Part E H M I to compleat the Perspective of the Side. (4.) I M G F to compleat the farther End or Side of the Room. And (5.) on these several perspective Supplements, are to be drawn Windows, Doors, Glasses, Pictures, *&c.* to make the Perspective represent the Room in the Man-

Manner you propoſe to compleat the View of it; and then it will have the agreeable Effect deſired.

1654. We have already taken Notice of the great Uſe, yea, abſolute Neceſſity of perſpective Diſguiſe in all theatrical Purpoſes; the very Eſſence of a Theatre conſiſting in that Sort of Illuſion. Here all the Scenery is a Contrivance to repreſent Things which are not as tho' they really were; and, in ſhort, all the Incidents to a Drama, are one united Syſtem of Deception; and by the Price that is paid for it, one would think there was no Pleaſure ſo great as that of *being deceived*; and which, therefore ought to be conſidered as a high Recommendation of Perſpective, which above all other Arts does moſt agreeably and innocently impoſe upon our Senſes.

CHAP. XIV.

The RULES *of* PERSPECTIVE *applied to* ARCHITECTURE, *in raiſing the* Perſpective ELEVATION *of* BUILDINGS.

[PLATE V. *Of* PERSPECTIVE.]

1655. THERE is certainly no Art in which the Science of PERSPECTIVE is ſo immediately concerned as Architecture, ſince whatever Edifice, or Fabric, is propoſed, the *Ground Plan* thereof muſt firſt be made; then the *Ichnography* or Perſpective of the Plan muſt be drawn; and, laſtly, the *Scenography* or perſpective Elevation of all the Parts muſt be compleated, before a proper Idea can be given of the Deſign; and therefore the Architect, above all Men, is under a Neceſſity of underſtanding the *Rules of Perſpective*, and the Reaſon of them likewiſe, if poſſible.

1656. We ſhall illuſtrate this Matter in regard to the perſpective Elevation of an Edifice by the following Example. Let ABCD be a geometrical Square, on the Ground Plane, in which another Square EFGH is inſcribed; and in that a CROSS, KLMN, which is the Form of the Plan on which

an Edifice is to be erected. Suppoſe it to be viewed obliquely, with an Angle in Front; then, by the Rules delivered (Chap. III.) you find the *perſpective Plan (k l m n)* for the given Point of Sight I, and Diſtance I Z.

1657. Let us now ſuppoſe this perſpective Plan, or Croſs, drawn more at large, and for a leſs Height of the Eye in Proportion to the Diſtance, as *(k l m n)* in Fig. 2. Then it is evident, that if from the ſeveral angular Points you raiſe Perpendiculars, and give to each of them their proper Heights (according to their reſpective Diſtances from the Ground Line A D) by the *Line of* MEASURES, (as directed in Chap. VIII.) You will then, by joining their Apices in a proper Manner by right Lines, form the Scenography of the Building in its linear Extremities; after which its front Sides, gable Ends, *&c.* are to be filled up, and pannelled with Windows, Doors, and architectural Ornaments, with a juſt Diſpoſition of Light and Shadow upon the Whole; in all which there can be no Sort of Difficulty, if the foregoing Precepts be underſtood, as appears by this Example, wherein the ſeveral Parts of the Edifice receive their Form and Proportion immediately by the *Radial* and *Diagonal Lines*, drawn to the Points of Sights and Diſtance, I, Z; it would therefore be ſuperfluous to ſay any Thing more on this Head.

1658. And becauſe the above Rules of Perſpective are in the higheſt Degree neceſſary in truly deſigning and repreſenting the Ruins of Building and Monuments of Antiquity, we thought it would be proper, likewiſe, to give a Specimen of ſuch a perſpective View in all the Variety of Architecture.

1659. In Figure 3. there is a *Side View* of a Colonade of Pillars, all diminiſhing in juſt Perſpective to the Point of Sight at I. There are alſo three Pillars in a *Front View*, but entirely out of Perſpective, being all of an equal Size, (ſee 1587) this is one Reaſon why I made Choice of this Piece (for all the Figures of this Plate except the firſt, are borrowed,) being willing the Reader ſhould ſee in what an imperfect State the Practice of Perſpective is in at this Day, and how ridiculouſly the Laws of this Science are tranſgreſſed, even by the Profeſſors themſelves.

1660. Having

1660. Having ſaid all I think can be neceſſary for perſpective Draughts and Deſigns, either of Superficies or Solids; I have added, for further Illuſtration, two other Figures (*viz.* Fig. 4, and 5.) to ſhew how vaulted Arches, Roofs, the interior Parts of Churches, winding Stairs, diſtant Views thro' Buildings, and many other Particulars, relative to Architecture, appear in perſpective Deſigns. In theſe the Reader will find not a ſingle Article but what is ſtrictly conformable to, and executed by the Methods and Rules of Perſpective above laid down.

CHAP. XV.

Of Sciagraphic PERSPECTIVE, or the ART of SHADOWS.

1661. AS SHADOW is nothing but the *Abſence* of LIGHT, it is a mere *negative*, and therefore can not, properly be the Subject of Art. However, there is what we uſually call the *Art of Shadows*, and as it is a neceſſary Article in Painting and Deſign, and is, for the moſt Part, conducted by the Rules of Perſpective, we ſhall here treat of it in a few Words.

1662. Since whatever intercepts the Rays of Light muſt produce *Shadow*, it is evident, that all Bodies, or Objects which are opake, muſt be attended with Shadow, if placed in the Light which can fall obliquely on them.

1663. Again, it is neceſſary that the Poſition of the Shadow be contrary to that of the luminous Object, whether a *Candle*, the *Sun*, a *Window*, *&c.* becauſe the Rays of all Light are rectilineal.

1664. The Figure of the Shadow depends partly on the Figure of the Object, and partly on the Form of the Rays. For it is evident, that in any Light the Shadow of a Triangle will be different from that of a Circle, or a Square; and the Shadow of a Cone will not be the ſame as that of a Globe or Cube. And, on the other Hand, it is as certain, that the ſame

Body will have its Shadow varied according to the Form of the Rays. For a Cone will caſt a Shadow of a greater Length by intercepting diverging Rays, as thoſe of a CANDLE, than it will do from parallel Rays, as thoſe from the SUN or a *Window*.

1665. But it will be ſufficient here to obſerve, that the *Laws of Shadow*, proceeding from a Privation of diverging Rays, are ſtrictly conformable to, as they reſult immediately from, the *Rules of common Perſpective*; by ſuppoſing a *Candle* inſtead of the *Eye*, and conſidering the Rays of this Light in lieu of the viſual Rays coming from the Object to the Eye.

1666. To demonſtrate this we need only have regard to the foregoing perſpective Figures. Thus, in Plate II. Fig. 1. if a Candle be ſuppoſed to be placed at I, and *(a n t v)* an opake Object at any Height, P *t*, above the Horizon, or Ground Plane, then, ſince all the Rays which fall thereon are intercepted, it is evident, if the Rays which paſs by the extreme Parts thereof be continued to the Ground-plane, as I *a* A, I *n* N, I *t* T, I *v* V, they will include a Space A N T V, which will be wholly *deprived of Light*; and that *dark Space* is therefore called the *Shadow* of that Object, and gives, of Courſe, its true *Limits*, *Figure*, and *Dimenſions*.

1667. Again, (in Fig. 2.) let a *Candle*, or radiant Point be ſuppoſed at I, and let A *b c* D be any opake Object on the Ground or horizontal Plane, then the Rays I *c* C, I *b* B will project its Shadow into the Space A B C D, as is evident by Inſpection.

1668. The Figure and Dimenſions of the Shadow is determined in the ſame Manner, in any Poſition of the Plane whatſoever. Thus ſuppoſe *(a b c d)* (Plate II. Fig. 1.) were an opake Surface; then from a Radiant at I, the Rays, I *a* A, I *b* B, I *c* C, I *d* D, will project its Shadow into the Space A B C D on any Plane ſtanding on the Ground Plane; and if the Plane which receives the Shadow be parallel to that in which the Object is, then the Shadow will be ſimilar to the Object. In ſhort, every Thing, with regard to the Object and its Shadow, will be *inverſely* the very ſame as has been demonſtrated of the various Relations between the Object and its perſpective Repreſentation, and therefore, if theſe be underſtood, the *Doctrine of Shadows* can admit

admit of no Difficulty, unleſs in that Caſe, where the Shadow is not projected wholly on one Plane, but partly on ſeveral.

1669. For Example ſuppoſe *(brtva)* were an irregular, opake Object, and ſo ſituated, with regard to the Radiant at I, that its Shadow falls partly on the Ground Plane, and partly on ſome other Plane elevated above it. Then even in this Caſe, you have nothing more to do, than, by the Rules of Inverſe Perſpective, to determine the Prototype of the given Perſpective *(brtva)* and how much of it is on one Plane, and how much on the other. *See* Chap. VI.

1670. Or thus; ſuppoſe Rays drawn thro' all the angular Points *a*, *b*, *r*, *t*, *v*, to the ſeveral Planes, they will there determine the Extent and Figure of the Shadow; thus I*t* T and I*v* V will determine its Limit T V on the Ground Plane; and the Rays I*r* R, I*b* B will mark out the ſame on the other Plane, as R B; and therefore, all the Space between R B on the elevated Plane, and T V on the Ground Plane, will be occupied by the Shadow.

1671. Or, laſtly, by Calculation, thus; let the Diſtance of the Object Y P = *a*, the Height of the Radiant or Candle Y I = *H*; the Height of the Point *t*, or P *t* = *h*, and P T = *x*, the Diſtance ſought of the Point T, where the Shadow commences. Then, by ſimilar Triangles, I Y T and *t* P T, we have I Y : *t* P :: T Y : T P; that is, $H : h :: a + x : x$; whence this Theorem $\frac{ah}{H-h} = x$. In Words thus, *multiply the Diſtance of the Perpendicular* Y P *by the Height t* P *of the ſhadowing Point; and divide that Product by the Difference between the perpendicular Height of the Radiant and given Point t, and the Quotient will be the Diſtance* P T *of the Shaddow* T, *from the Foot of the Perpendicular* P, *as required.*

1672. In the ſame Manner the Point *N* is determined for the Shadow of the Point *(u)* in the Object; and from thence the Length of the Shadow T N for the Part of the Object *(t n)*. Alſo, if the Shadow falls upon a Wall, you find its Height on the ſaid Wall by the ſame Rule, knowing the Diſtance of the Wall from the Radiant, and the Point to which the Ray tends beyond it, found as above.

1673. It

1673. It is evident, that the Shadows of all perpendicular Lines converge to a Point perpendicularly under the Radiant; and conſequently, if we ſuppoſe the Radiant removed to a very great Diſtance thoſe Shadows will become parallel, which, therefore, is the Caſe of all ſuch Shadows produced by intercepting the *Rays of the Sun.*

1674. Hence the Shadow of a *Parallelogram* will diverge from the Point under the Radiant, if a *Candle*; but if the *Sun*, the Sides of the Shadow will be parallel. The ſame may be obſerved with Regard to the Shadow of a *Cylinder.*

1675. It is alſo evident, that the Shadow of a round Table on the Floor will be *circular*, both from the Candle ſtanding upon it on any Part, and alſo from the Sun-beams (1504.) only in the firſt Caſe, the Shadow will exceed the Dimenſions of the Table; in the latter, it will be juſt equal to it.

1676. The Shadow of a *Globe* lying on a Table muſt neceſſarily be *elliptical*; for the Cone of Rays are intercepted by a Plane perpendicular to its Axis, and paſſing thro' the Globe in ſuch Poſition as to make an Angle with the Horizon, or Table, juſt equal to the Co-Altitude of the Radiant whether *Candle* or *Sun.* In this Caſe the Cone is never *Scalenous*, and can admit of no Subcontrary Sections from the two Planes, therefore the *Elliptic Shadow* can never become *Circular.* (See Chap. IV.)

1677. As we have ſhewn a general Method how the Shadow may be determined for any given Line on given Planes, and as the Shadow of any Superficies is determined by that of each bounding Line, and the Shadow of a Solid reſults from thoſe of its connected Superficies, it is evident that the ſame general Perſpective Rule for Shadows (1671.) of Lines extends to the Determination of the Shadow of any Body whatſoever; the particular Application of which is left to, and is no ſmall Part of the Praxis of the *young* DESIGNER.

1678. The Art of Shadows, in one particular Branch, conſtitutes a Science of itſelf alone, viz. GNOMONICS, or the ART of DIALLING; to which therefore we proceed to apply it; but it will be firſt neceſſary to teach the *Projection of the Sphere* from the Principles of Perſpective above laid down; and then the *Rationale* of making a DIAL will appear in a new Light.

CHAP.

CHAP. XVI.

The RULES *of* PERSPECTIVE *applied to the* Geographical PROJECTION *of the* SPHERE, *for the Conſtruction of* MAPS, CHARTS, *&c.*

1679. THE Method of making *Geographical* and *Aſtronomical Charts* by a PROJECTION of the Sphere *in Plano* is as ancient as any Mathematical Science; but as it is in that Way limited and intricate, we ſhall here ſhew how general and eaſy it is by the Principles and Rules of *Common* PERSPECTIVE, which is not only new, but the natural Source of ſuch a Doctrine.

1680. For if (Fig. 1. Plate II.) the Plane of Projection HF, the *Vertical* Plane QN, and the *Horizontal* Plane L O, were all of a circular Form, and of an equal Radius, they would repreſent the Planes of *three great Circles of the Sphere* whoſe common Interſections would all paſs thro' the Center (*i*).

1681. Let the Planes of three ſuch great Circles of the Sphere be EQTR, QKPR, and EPLT (Plate VI. Fig. 1.) of which the firſt is ſuppoſed perpendicular to the Viſual Ray proceeding from the Center C to the Eye at Y, and is therefore the *Plane of Projection*. The Second is the *Vertical Plane*, and the Third, the *Horizontal Plane*, as all the three are ſuppoſed to interſect each other at right Angles. Conſequently QR is the *Vertical Line*, and E T the *Horizontal Line* of the *Projection*.

1682. Let O be the Place of any Object on the Surface of the *Globe* or *Sphere*, then its Diſtance from the Plane of Projection is the Sine O B of the Arch O G, which meaſures its *Latitude* or Declination from that Plane. Its Diſtance from the Vertical Plane is the Sine O D of the Arch O K of a leſſer Circle K O L, whoſe Plane is parallel to the Plane of Projection, and its Diſtance from the horizontal Plane is the Co-Sine O N of the ſame Arch O K, or the Sine of the Arch O I. All which Sines or Diſtances bear a known Proportion to the Radius of the Globe or Sphere, and therefore are to be eſtimated in that meaſure, as found in the *Trigonometrical* Canon. (708 to 716.)

1683. The Diſtance of the Eye C Y is alſo eſtimated in the ſame Meaſure, viz. in Parts of the Radius: That is, if Radius

= r,

$= r$, then $CY = \frac{1}{2}r$, r, $2r$, $3r$, &c. or nr, which may repreſent any Diſtance of the Eye indefinitely, and thereby render the Theory of Projection, this way conſidered, univerſal. Therefore let us ſuppoſe,

1. *The* EQUATOR, *the* PLANE *of* PROJECTION.

1684. Since the Circle FQTR, in this Caſe, is the Equator, the Vertical Circle QPR, and the Horizontal One EPT will be Meridians; and P the Pole in the Hemiſphere remote from the Eye; and if O be the Place of an Object in that Hemiſphere, then we have $CG : GH :: CB\ (= MO) : BI\ (= OD.)$ that is, as *Radius is to the Sine of the Difference of Right Aſcenſion or Longitude ſo is the Co-Sine of Declination or Latitude to the Diſtance of the Point O from the Verticle Plane.*

1685. Again, we have $CG : GF :: CB\ (= MO) : BS\ (= ON.)$ That is, *as Radius is to the Co-Sine of the Difference of Longitude or right Aſcenſion, ſo is the Co-Sine of Latitude or Declination, to the Diſtance of the Point O from the Horizontal Plane.*

1686. Or thus in Symbols; let $HG = p$, $GF = q$, $OB = x$, $OM = y$; Then by (1684.) $DO = \frac{py}{r}$ = Diſtance of the Point O from the vertical Plane; and by (1685.) $ON = \frac{qy}{r}$ = Diſtance from the horizontal Plane; and x = Diſtance from the Plane of Projection. Having theſe Diſtances from the three Planes, we determine the Diſtances from the vertical and horizontal Lines, in the Plane of Projection, by the *common Rules of Perſpective.*

1687. Thus, ſuppoſe the Equator between the given Point O and the Eye Y, we have $nr + x : nr :: \frac{py}{r} : \frac{npy}{nr + x} = a$ = Diſtance of the Point O from the vertical Line of the Projection; (1470.) whence we have $npy = anr + ax$; and therefore $nr + x : ny :: p : a$; the Analogy for finding a.

1688. Again, we have $nr + x : nr :: \frac{qy}{r} : \frac{nqy}{nr + x} = b$; (1470.) which gives $nr + x : ny :: q : b$, the Analogy for finding b, the Diſtance from the *Horizontal Line.*

1689. If

1689. If the Eye be placed in the Pole oppoſite to P, then $n = 1$; and the two Analogies become (1.) $r + x : y :: p : a$, the Diſtance from the *vertical Line*; and (2.) $r + x : y :: q : b$, the Diſtance from the *horizontal Line*; in this Caſe, the Projections is that of *Ptolomy*; and is repreſented in Fig. 2.

1690. If the Eye be at an infinite Diſtance, then n is infinite, and the two Analogies become (1.) $r : y :: p : a =$ Diſtance from the Vertical; and (2.) $r : y :: q : b =$ Diſtance from the horizontal Line. In both theſe Caſes the *Meridians* are projected into *right Lines* (1464) and the *Parallels* into *concentric Circles* (1504).

1691. If the Eye were placed in the other Part of the Axis at z, then the Point O is between the Eye and Plane of Projection; and in ſuch a Caſe the Analogies will be (1.) $nr - x : ny :: p : a$; and (2.) $nr - x : ny :: q : b$; for the Diſtances from the vertical and horizontal Lines, as is evident from (1470). The firſt of theſe Caſes is called the STEREOGRAPHIC, and the other the ORTHOGRAPHIC PROJECTION *of the* SPHERE.

II. *Of the* PROJECTION *of the* SPHERE *on the Plane of the* MERIDIAN.

1692. If the Meridian be choſe for the Plane of the Projection, as E P T R, (Fig. 3.) then in the Caſe of common Maps of the World, the Equator E Q T is the horizontal Plane; and the Six o'Clock Hour-circle P Q R is the vertical Plane. P and R are the two Poles; and the Eye Y is placed in the common Interſection Q C of the vertical and horizontal Planes continued out.

1693. Let O be the given Point, as before, through which draw the Meridian P O G; then will the Sines and Co-Sines G H, G F, O B, O M, be the ſame as before, and denoted by the ſame Letters. Alſo $O D = \frac{py}{r}$ the Diſtance from the Vertical Plane, $O N = \frac{qy}{r} =$ Diſtance from the Plane of Projection as in the laſt, and $O B = x$ the Diſtance from the horizontal Plane. Then by the Analogies of Perſpective (1687, 1688.) we have $nr + \frac{qy}{r} : nr :: x : \frac{nrrx}{nrr + qy} =$ Diſtance from the

horizontal Line. And as $nr + \frac{qy}{r} : nr :: \frac{py}{r} : \frac{nrpy}{nrr + qy} =$ Diſtance from the vertical Line.

1694. If the Diſtance of the Eye $nr = r$, as in the common *Stereographic Projection*, then $n = 1$; and the two Expreſſions become $\frac{rrx}{rr + qy}$, and $\frac{rpy}{rr + qy}$; which are to each other, as rx to py. And when $rx = py$; then $r : p :: y : x :: r : t$ (712); that is when of the Tangent Latitude O G is equal to the Sine of Difference of Longitude, then *the Point* O *in the Projection will be equally diſtant from the vertical and horizontal Lines.*

1695. When the Point O is in the Equator, $x = o$ and $y = r$; and the Diſtance from the vertical Plane is then $\frac{rp}{r + q}$. But the Ratio of $r + q$ to p, is the ſame as that of p to $r - q$; as will appear if we place the Arch QG in another View; let QTR (Fig. 4.) be the Plane of Projection, or *primitive Circle*, as it is uſually called. Then by the Eye at R the Point G is projected into the Point O in the horizontal Line CT. Put $CR = r$, $GH = p$, $GF = HC = q$, and $QH = r - q$; and draw the Line RG. Then by the Property of the Circle (658) we have $RH : HG :: HG : HQ$; that is, $r + q : p :: p : r - q$, and therefore $p = \sqrt{rr - q^2}$.

1696. Now becauſe $QH = \frac{p^2}{r + q}$ and $CO = \frac{rp}{r + q}$, therefore $QH : CO :: p^2 : rp :: p : r :: GH : QC$; and conſequently QO is the Arch of a Circle. *Therefore all great Circles of the Sphere are projected into Arches of Circles of different Dimenſions.*

1697. Let the Radius of the Circle DOE be $= R$, the Center B, and Diameter AO. Alſo put $CQ = P$, and $BC = Q$, then becauſe $CO = \frac{P^2}{R + Q} = \frac{r^2}{R + Q} = \frac{rp}{r + q}$ (1696) we have $\frac{r}{R + Q} = \frac{p}{r + q}$, and ſo $R + Q : r :: r + q : p :: QC : CO$; therefore $AC : CQ :: RH : GH$; and conſequently the Angle $QAC = QRG$ (657) and the Angle $QBC =$ GCH

GCH (642) therefore making the Angle LCT = QCG, the Line LC will be parallel to BQ, and give the Center B.

1698. Or thus; CO is the Tangent of half the Angle QCG (705); whence it appears that the Tangent of half the Angle or Arch GQ ſet from the Center C, gives the Point O; then through the three Points QOR the Meridian is drawn with Eaſe (695).

1699. The Projection, as hitherto conſidered, is the vulgar Stereographic One, which on every Account, is the worſt that can be for Maps and Charts; becauſe of the great Diſproportion and Diſtortion it occaſions in all the Parts of the Earth's Surface repreſented upon it, as appears from the very unequal Parts of the horizontal Line or Equator intercepted between Meridians equally diſtant on the Globe; therefore this Projection muſt give a very falſe Idea of the geographical Relations of Places, and is conſequently the moſt unfit for ſuch Purpoſes. (See Fig. 5.)

1700. To remedy this Imperfection in a conſiderable Degree, we may ſuppoſe the projected Quadrant of the Equator CT, or CE divided into equal Parts, of any Number $= s$, and let CO be any Number of them denoted by t; that is, let CO be the $\frac{t}{s}$ Part of r, then will $\frac{tr}{s} = \frac{npr}{nr+q}$, or $\frac{t}{s} = \frac{np}{nr+q}$; whence $snp = ntr + tq$, and we get $n = \frac{tq}{sp-tr}$ = Diſtance of the Eye to project that Meridian; for Example, let the Meridian of 10 Degrees from the vertical Plane be projected by the Eye; then $s = 90$, and $t = 10$; $n = \frac{10q}{90p-10r} = 175$ Parts of which $r = 100$.

1701. In Practice, by dividing the Diameter or horizontal Line into equal Parts; you have the Points given thro' which to draw the Meridians (695). And this is uſually called the GLOBULAR PROJECTION; foraſmuch as the Meridians and Parallels are circular, and at equal Diſtances in the Equator and Meridian or vertical Line; and conſequently ſuch a Projection is greatly preferable to the former for general Maps of the World;

a Specimen of which ſee in that Map we have prefixed to our PHILOSOPHICAL GEOGRAPHY.*

1702. There is another very uſeful geographical Projection, *viz.* on the *Plane of the Horizon*, of which we have given a Specimen in Fig. 6. and though any particular Point may be determined on the ſaid Plane by the *Rules of Perſpective*, yet the Whole of it is with more Eaſe and Readineſs laid down from Calculations founded on the Principles of ſpherical Geometry hereafter to be explained.

CHAP. XVII.

The RULES *of* PERSPECTIVE *applied to* Aſtronomical PROJECTIONS *for the Conſtruction of* Celeſtial PLANISPHERES, *the* ANALEMMA, *&c.*

1703. SINCE the Surface of the *Celeſtial Globe* may be projected *in Plano*, in the ſame Manner as that of the *Terreſtrial* One, it follows, that the ſame Theorems which ſerve for the Conſtruction of MAPS, may alſo be adapted to CELESTIAL CHARTS or PLANISPHERES.

1704. For this Purpoſe, inſtead of the *Latitude* and *Longitude* of Places on the Earth, we muſt, in regard to the Sun and Stars, uſe the Words *Declination*, and *Right Aſcenſion*; and then the ſame Symbols will ſerve; that is,

p = Sine of Right Aſcenſion.
q = Co-Sine of that Right Aſcenſion.
x = Sine of Declination.
y = Co-Sine of that Declination.

1705. Alſo the ſame Diagrams (in Fig. 1, and 3.) may here be uſed; if O be the Place of a Star, then its Diſtance from the vertical Plane is $\frac{py}{r}$ = DO; from the horizontal Plane $\frac{qy}{r}$ = NO; and from the Plane of Projection x = OB; the ſame as before (1686).

1706.

* Alſo in a New Map of the World entitled, *The* PRINCIPLES *of* GEOGRAPHY *and* ASTRONOMY *explained*, &c.

Plate VII { PERSPECTIVE — DIALLING.

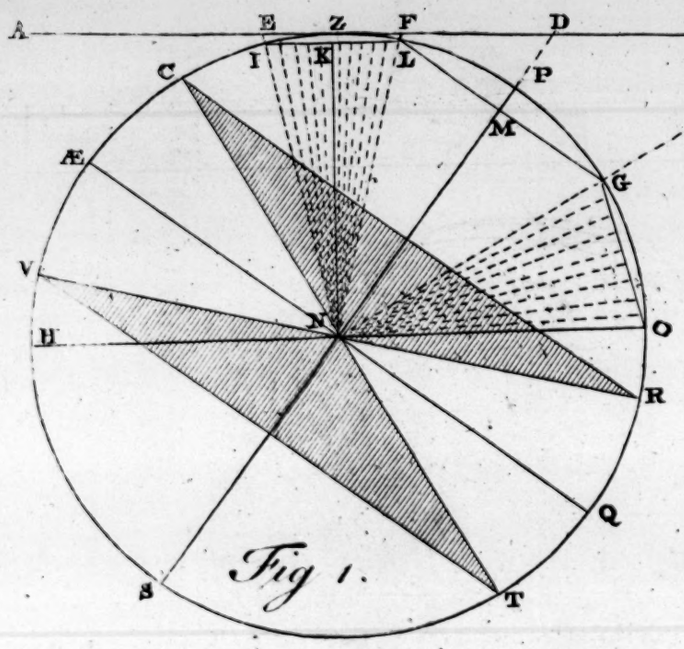


Fig 1.

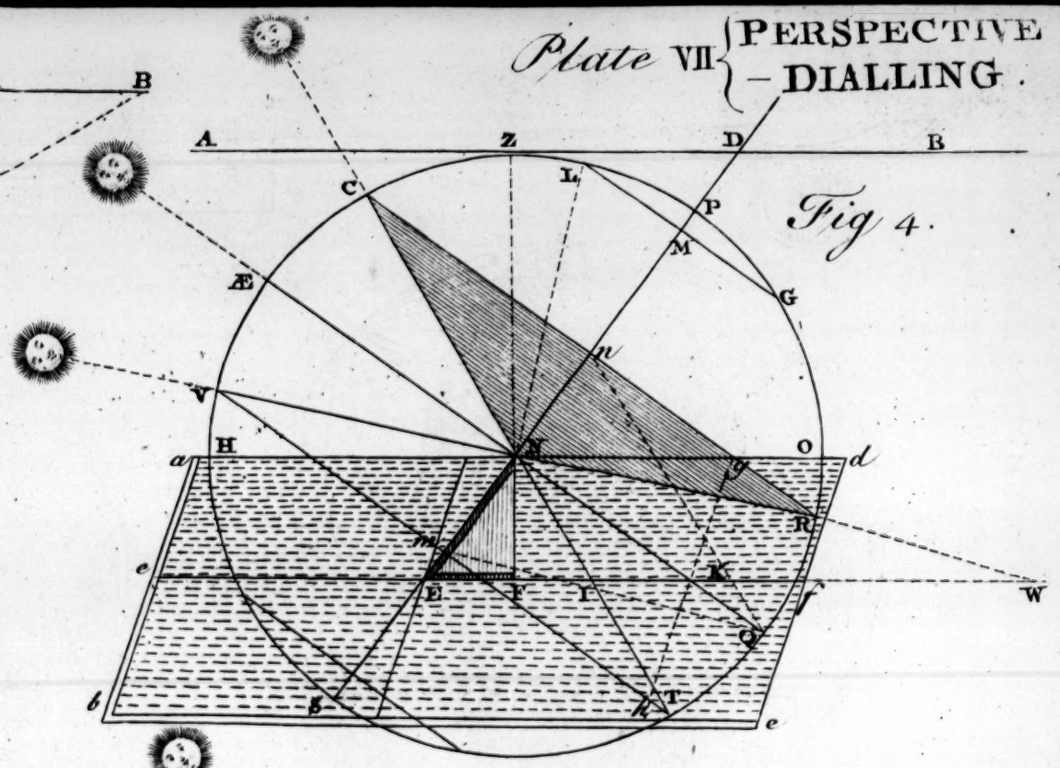


Fig 4.

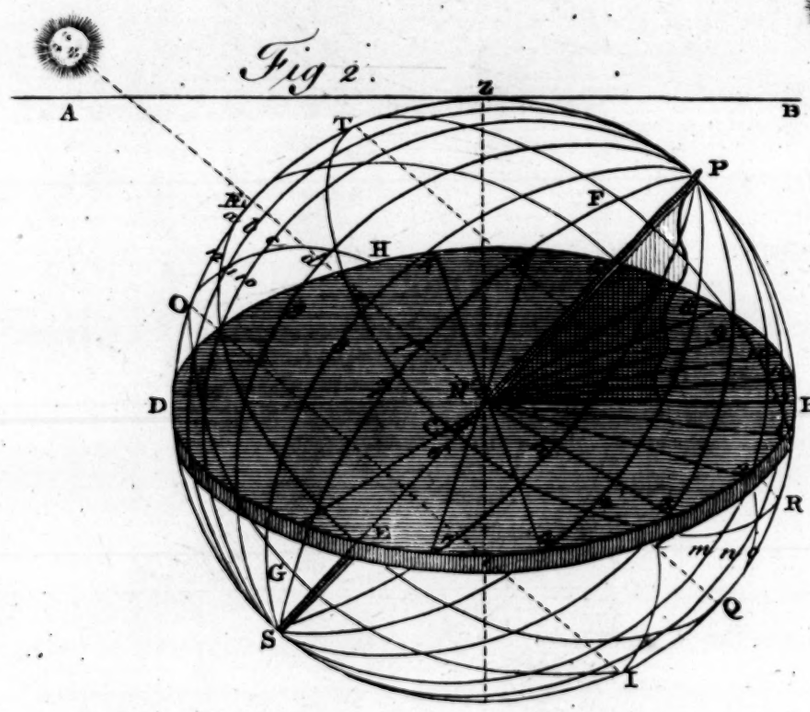


Fig 2.

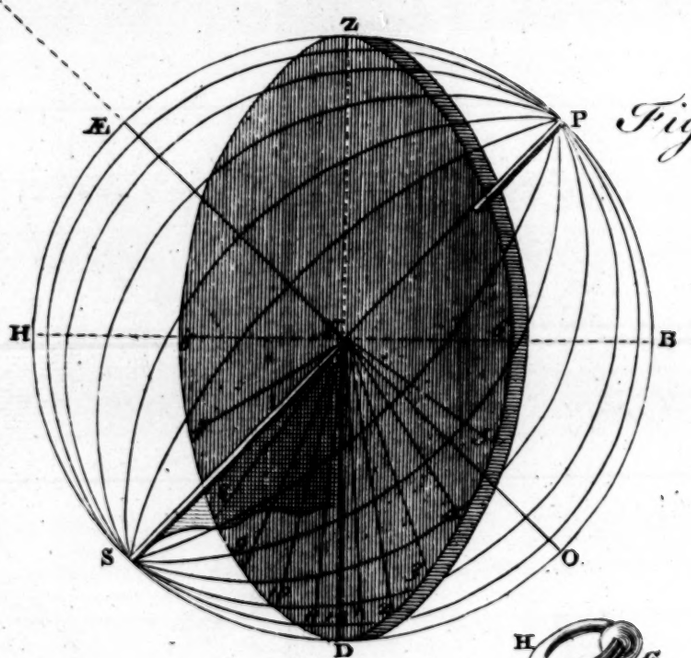


Fig 3.

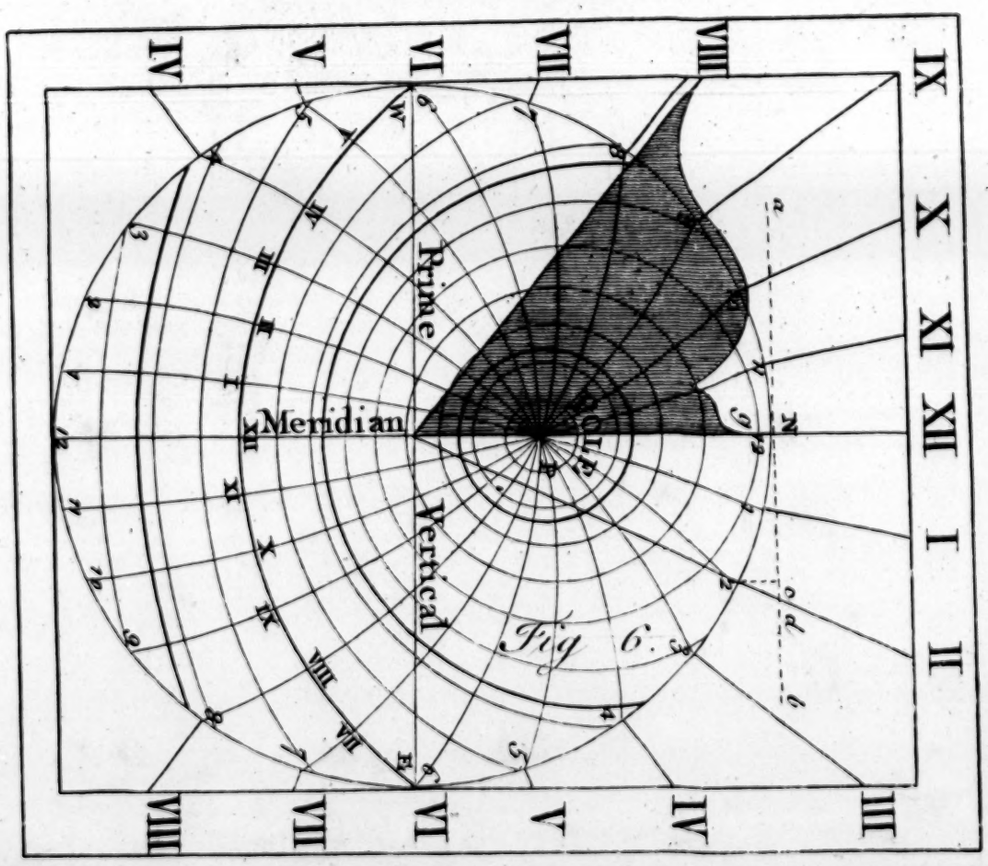


Fig 6.

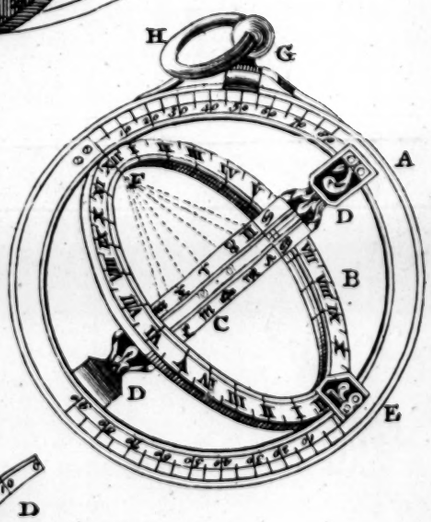


Fig 5.

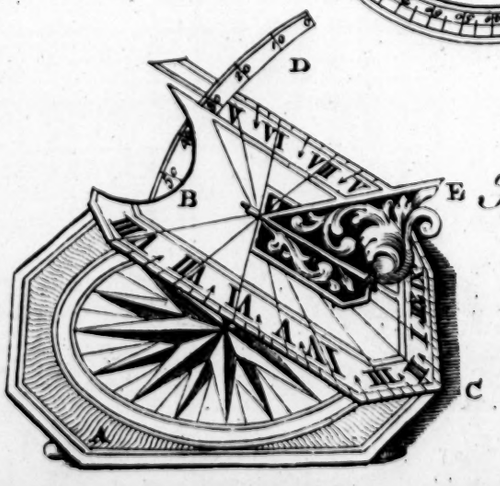


Fig 7.

1706. If the Equator be the Plane of the Projection; then the Pole of the World is the Center, and the Circles of Declination are projected into right Lines, alſo the Parallels of Declination are concentric Circles; all as before (1685.) But in this *Planiſphere* the Pole of the Ecliptic is alſo projected, and all the Circles of Latitude, together with the Ecliptic itſelf, are projected into circular Arches, by the uſual Methods.

1707. In this Projection the *ſolſtitial Colure* is the vertical Plane, and the *equinoctial Colure* the horizontal Plane; the Diſtance of any Star from the former will be $\frac{npy}{nr+x}$; and from the latter, $\frac{nqy}{nr+x}$. But if the Eye be placed in the Pole on the Globe's Surface, the Diſtances of the Star becomes $\frac{py}{r+x}$ and $\frac{qy}{r+x}$; and thus all the Stars may be readily laid down from a Table of their Declinations and right Aſcenſions. Such *celeſtial Planiſpheres* we have by the late Mr. *Senex*, of 24 Inches Diameter.

1708. If the ECLIPTIC be the Plane of Projection, then is its Pole the Center, and all the Circles of Latitude are right Lines; alſo $OB = x$, and $OM = y$, are the Sine and Co-Sine of the Star's *Latitude*; $GH = p$, and $GF = q$, are the Sine and Co-Sine of its *Longitude* (Fig. 1.) and the vertical and horizontal Planes are thoſe of two Circles of Latitude paſſing through the ſolſtitial and equinoctial Points, as before. And therefore by the ſame *Formulæ* $\frac{py}{r+x}$ and $\frac{qy}{r+x}$ the Stars may be laid down in this Projection from *Tables of their Latitude and Longitude*. Such Planiſpheres we have of all the Stars in the *Britiſh* Catalogue by the ſame Hand.

1709. There is alſo a very uſeful Projection of the Heavens on the Plane of the Horizon. In this Caſe the Meridian is the vertical Plane, and the prime Vertical the horizontal Plane. Here the Star's Diſtance from the vertical Line is $\frac{rpy}{rr+qy}$ as before (1695). Then if a, b, t be the Sine, Co-Sine, and Co-Tan-

1706. If the Equator be the Plane of the Projection; then the Pole of the World is the Center, and the Circles of Declination are projected into right Lines, alſo the Parallels of Declination are concentric Circles; all as before (1685.) But in this *Planiſphere* the Pole of the Ecliptic is alſo projected, and all the Circles of Latitude, together with the Ecliptic itſelf, are projected into circular Arches, by the uſual Methods.

1707. In this Projection the *ſolſtitial Colure* is the vertical Plane, and the *equinoctial Colure* the horizontal Plane; the Diſtance of any Star from the former will be $\frac{npy}{nr+x}$; and from the latter, $\frac{nqy}{nr+x}$. But if the Eye be placed in the Pole on the Globe's Surface, the Diſtances of the Star becomes $\frac{py}{r+x}$ and $\frac{qy}{r+x}$; and thus all the Stars may be readily laid down from a Table of their Declinations and right Aſcenſions. Such *celeſtial Planiſpheres* we have by the late Mr. *Senex*, of 24 Inches Diameter.

1708. If the ECLIPTIC be the Plane of Projection, then is its Pole the Center, and all the Circles of Latitude are right Lines; alſo $OB = x$, and $OM = y$, are the Sine and Co-Sine of the Star's *Latitude*; $GH = p$, and $GF = q$, are the Sine and Co-Sine of its *Longitude* (Fig. 1.) and the vertical and horizontal Planes are thoſe of two Circles of Latitude paſſing through the ſolſtitial and equinoctial Points, as before. And therefore by the ſame *Formulæ* $\frac{py}{r+x}$ and $\frac{qy}{r+x}$ the Stars may be laid down in this Projection from *Tables of their Latitude and Longitude*. Such Planiſpheres we have of all the Stars in the *Britiſh* Catalogue by the ſame Hand.

1709. There is alſo a very uſeful Projection of the Heavens on the Plane of the Horizon. In this Caſe the Meridian is the vertical Plane, and the prime Vertical the horizontal Plane. Here the Star's Diſtance from the vertical Line is $\frac{rpy}{rr+qy}$ as before (1695). Then if a, b, t be the Sine, Co-Sine, and Co-Tan-

Tangent of the Star's Azimuth, we have $a : b :: r : t :: \frac{rpy}{rr + qy}$ $: \frac{tpy}{rr + qy}$ = Star's Diſtance from the horizontal Line of the Projection. But when the right Aſcenſion and Declination of a Star is known, together with the Latitude of the Place, or Pole's Height, then the Azimuth of a Star is known alſo; whence its Place on this Projection is eaſily aſſign'd.

1710. It is not common to project the ſtarry Heavens on the Plane of the Meridian, or Circle of Latitude paſſing through the Poles of the World; but if it be required, it may be done by the *Formulæ* in (1694.) by ſubſtituting as directed (1704). However, this Projection is not confined to *Geography*, but is the Ground-work of that celebrated aſtronomical Inſtrument called the ANALEMMA, or ORTHOGRAPHIC PROJECTION of the SPHERE.

1711. For if the Eye be ſuppoſed at an infinite Diſtance, then the Stereographic Forms (1694) $\frac{nrrx}{nrr + qy}$, and $\frac{nrpy}{nrr + qy}$, become x and $\frac{py}{r}$, for the Diſtances of any Point from the horizontal and vertical Lines of the Projection. And in this Caſe, the *Meridians*, *Parallels*, and other Circles, Oblique to the Plane of Projection, are projected into ELLIPSES. A Specimen of this Projection you have in Fig. 7.

1712. That every Oblique Circle PGS is projected into an Ellipſis P*a*FS by parallel Rays is thus demonſtrated (Fig. 8.) Let CP = CG = a, CF = b = Semi-Conjugate of the Curve PFS. Alſo put CM = x, and OM = y, and NM = y. Then by the Property of the Circle (658) we have SM × PM = OM², that is, $a + x \times \overline{a - x} = a^2 - x^2 = y^2$. Hence $\sqrt{a^2 - x^2} = y$ = MO. Draw OB parallel to CM, and B*b* parallel to GF; then OM : NM :: (BC : *b*C ::) GC : FC; that is, $y : y :: a : b$; therefore $y = \frac{b}{a} y = \frac{b}{a} \sqrt{a^2 - x^2}$, which is the Property of an Ellipſis (765) but y = NM = $\frac{py}{r} = \frac{by}{a}$; for $r = a$, $p = b$, (766) therefore the Curve PFS is an *Ellipſis*.

1713.

1713. When the Point O coincides with G, or is in the Equator, then $y = r$, and the Diſtance C F or Semi-Conjugate of the Ellipſis, $\frac{py}{r} = p$ = Sine of right Aſcenſion from the Point Q.

1714. Hence, as $r : p :: p : \frac{p^2}{r}$ = Half the *Latus Rectum* of the Ellipſis (766); but in this Caſe $\frac{py}{r} = \frac{pa}{r}$; therefore $y = p$ = O M, when M is the Focus of the Ellipſis; whoſe Diſtance therefore C M from the Center, is equal to the Co-Sine right Aſcenſion, which is then equal to the Sine (x = O B) of Declination.

1715. Hence if C F, the Semi-Conjugate of the Ellipſis, be made Radius; then, ſince in this Caſe $r = p$, we have the Diſtance from the vertical Line every where $\frac{py}{r} = y$; that is, *all the Semi-Ordinates* N M *of the Ellipſis will be equal to the Co-Sines of Declination, to the Radius* C F.

1716. And whatever has been demonſtrated of the MERIDIAN P O G, and its orthographical Projection, may be applied in the ſame Manner to any PARALLEL A O L when poſited obliquely to the Plane of Projection. For in this Caſe, P C S may be conſidered as the horizontal Line, and A M L as the vertical One; and then, if M L = r = Radius, O D and O N will be the Sine and Co-Sine of right Aſcenſion from K; and conſequently the Diſtances from the horizontal and vertical Planes. Alſo O D = p will be the Diſtance from the horizontal Line, and its Diſtance from the vertical Line will be $\frac{xq}{r}$ (1711).

1717. Hence when Q coincides with R, then $q = r$; and x, the Sine of Declination (or Obliquity) from the vertical Plane is then the Semi-Conjugate of the Ellipſis of the projected Parallel. And therefore, becauſe $r : x :: x : \frac{x^2}{r}$ = *Latus Rectum* (766) = $\frac{xq}{r}$, it is $q = x$, at the Focus of the Ellipſis; where the

the Diſtance of the Focus, from the Center M, is equal to the Sine of right Aſcenſion, or Co-Sine of Declination.

1718. From all which it appears, that ſince the Diameters, and *Foci*, of the Ellipſis of any projected oblique Circle are given by this Method, therefore ſuch an Ellipſis may be drawn, by Points, or mechanically (785) or from a Table of Sines, or laſtly, by the *Sector*, which is the moſt ready Method of all.

1719. The ANALEMMA (Fig. 7.) is that particular Species of the *orthographic Projection*, where the Eye is placed in the common Interſection of the Ecliptic and Equinoctial, at an infinite Diſtance. Theſe Planes will therefore be projected into right Lines; and the Equinoctial will be the horizontal Line; and the equinoctial Colure the vertical Line; as the other Colure is the Plane of Projection.

1720. Here alſo the *Parallels* of *Declination* are all right Lines, as they ſtand at right Angles on the Plane of Projection. The Meridians here drawn, are thoſe of 15 Degrees interval; and conſequently are the *Hour Circles* of the Sphere. As to the Uſes of the *Analemma* in *Aſtronomy*, ſuch as are conſiderable will appear in the Sequel of this Work, it ſuffices, at preſent, that we have ſhewn its Nature and Conſtruction from the Principles of Perſpective. In the mean Time we refer the Curious to *Sutton*'s *Analemma* of 18 Inches Diameter, in which the Meridians and Parallels are drawn thro' every Degree, and many of the principal Stars laid down; but this Inſtrument is yet capable of great Improvements, as we may hereafter ſhew.

CHAP. XVIII.

The PRINCIPLES *of* PERSPECTIVE *applied to the Computation of* SOLAR *and* LUNAR ECLIPSES.

1721. THE PRINCIPLES of PERSPECTIVE are applicable to Projections for *Solar and Lunar* ECLIPSES; and the ſame Rules ſerve here as in all the foregoing Caſes. For let S be the Center of the Sun A C B (Fig. 9.) E the Center of the

the Earth QW; and LEP a Part of its Orbit. Alſo let XIMF be a Circle deſcribed about the Earth's Center in the Plane of its Motion, or of the Ecliptic, at the Diſtance of the Moon. And let HK be the Orbit of the Moon, making the given Angle HKG, with the Ecliptic, at the Time of a *ſolar Eclipſe*; and K the *deſcending Node*.

1722. Now in Projections of this Kind, the moſt commodious Poſition of the Eye is in the Center of the Sun at S, in which Caſe the *horizontal Plane* will be that of the Ecliptic; the *vertical Plane* will be that Circle of Latitude which cuts the Ecliptic in the Points of Conjunction X, and Oppoſition M; and the *Plane* of *Projection* is that which paſſes thro' the Earth's Centre E, and touches its Orbit, when it is in the Line of Ecliptic Conjunction or Oppoſition SM.

1723. It is evident from the Diagram that the Angle AQS is that under which the Sun's Semidiameter appears at the Earth. The Angle ESQ is that under which the Semidiameter of the Earth appears at the Sun, and is call'd, the Sun's *horizontal Parallax*. The Angle EXQ or EMQ is that under which the Earth's Semidiameter appears at the Diſtance of the Moon, and is call'd the Moon's *horizontal Parallax*. And the Angle XQS is the *Difference of the horizontal Parallaxes* of the *Sun and Moon*. For XSQ + XQS = QXE, (632) therefore XQS = QXE — XSQ.

1724. By the Word *Parallax*, is only meant the different Place in which the Sun, or Moon appears, when viewed from the Center of the Earth at E, and from any Point of it's Surface as Q. And, therefore, at the Diſtance of the Phænomenon, it muſt be equal to the Angle under which the Semidiameter of the Earth appears. It muſt, therefore, be greateſt of all when in the Horizon of the Place Q; and vaniſh for Objects at an infinite Diſtance; ſo that the *Stars have no Parallax at all*; and that of the Sun ESQ is almoſt immeaſurably ſmall.

1725. Theſe Things premiſed, we may proceed: Let K be the Place of the Node, and O that of the Moon in her Orbit HK, then it is plain, her Diſtance from the *horizontal Plane* is the Sine x of her Latitude OG; her Diſtance from the *vertical Plane* is $OD = \frac{py}{r}$, and her Diſtance from the *Plane of Projec-*

tion is $ON = \frac{qy}{r}$; for here we retain ſtill the ſame Symbols as before (1686, 1704.) only OG is here the Moon's *Latitude* from the Ecliptic, and GK the *right Aſcenſion*.

1726. Then becauſe the Place of the Moon O is between the Eye at S, and the Plane of Projection at E; the perſpective Analogies (1693.) will be for an ECLIPSE of the SUN.

1727. As $nr - \frac{qy}{r} : nr :: \frac{py}{r} : \frac{nrpy}{nrr - qy} =$ to the Diſtance of the Moon from the *vertical Line*. And as $nr - \frac{qy}{r} : nr :: x : \frac{nrrx}{nrr - qy} =$ her Diſtance from the *horizontal Line*. But in Eclipſes, the Latitude and Diſtance of the Moon, from the vertical Plane, are ſo very ſmall, that we have $y = r$, very nearly; therefore in ſuch Caſes the Analogies become $nr - q : nr :: p : \frac{pnr}{nr - q}$; and $nr - q : nr :: x : \frac{nrx}{nr - q}$.

1728. Now becauſe $nr =$ ES the Diſtance of the Sun, and $q = r =$ EX nearly, the Diſtance of the Moon; and ſo $nr - q =$ SX; therefore ſince SX : SE (= SQ) :: Angle SQX : Angle SXQ or EXQ (718) the firſt Analogy is, in Words; *as the Difference between the horizontal Parallaxes of the Sun and Moon, is to the Parallax of the Moon*; ſo is the Moon's Longitude (at the given Inſtant) *from the vertical Plane, to her Diſtance from the vertical Line.* —— And, *ſo is her Latitude, to her Diſtance from the horizontal Line.* Note, the Longitude and Latitude are here put for their Sines, there being no ſenſible Difference in ſuch very ſmall Arcs.

1729. If therefore from the aſtronomical Tables you take the Moon's horizontal Parallax, and with it, as a Radius, you deſcribe a Circle on E, as QYWZ; then will QW be the horizontal Line, and YZ, the vertical Line, on the Plane of Projection, which if it be about 10 or 12 Inches in Diameter will be ſufficient for the Delineation of all the Phaſes of a ſolar Eclipſe.

1730. For ſince the Diſtance of the Sun SE is vaſtly great in reſpect of the Earth's Semidiameter EQ, therefore the Hour Circles and Parallels will all be *orthographically* projected, and may be ſo drawn on the ſaid Plane (by 1711 — 1718.) and ſince

by

by the above Analogies, the Path of the Moon over the Diſk of the Earth (or Plane of Projection) is determined by finding her Place thereon at two Inſtants of Time, one juſt before the Beginning, the other after the End of the Eclipſe; it will be eaſily ſeen how the Moon, and any particular Place, as LONDON, are ſituated in reſpect of each other, and what are their Diſtances apart at any given Moment, during the Paſſage of the Shadow.

1731. For if from aſtronomical Tables you take the Sun's Semidiameter in your Compaſſes, and with one Foot placed on the given Moment of Time in the Parallel of *London*, you deſcribe a Circle; and then from the ſaid Tables you take the Moon's apparent Semidiameter, and with one Foot of the Compaſſes, placed in the ſame Moment in the Moon's Path, you deſcribe another Circle; theſe two Circles will ſhew how the Diſks of the Sun and Moon are related to each other in the Heaven, in regard to a Spectator at *London*.

1732. Thus if the Circles do not touch, the Eclipſe is not begun, or over; if the Circles juſt touch, the Eclipſe is then juſt beginning or ending. But if the Circle of the Moon lies over that of the Sun, there is an Eclipſe. If the Diameter of the ſolar Diſk or Circle be divided into 12 equal Parts, or *Digits*, then the Number of theſe covered by the lunar Circle, will ſhew the *Phaſe* and Quantity of the Eclipſe. But on theſe Particulars we muſt not enlarge any further at preſent.

Of a LUNAR ECLIPSE.

1733. When the Moon is ſo near the Node at the *Oppoſition* M, that her Latitude is leſs than the Semidiameter V M of the Earth's Shadow Q T W, it will be more or leſs immerſed into, and eclipſed by it. And then by having the Moon's Latitude, and Longitude (or Diſtance reduced to the Ecliptic) from the Plane of the Circle of Latitude paſſing through M, the Point of Oppoſition, you will by the ſame Analogies find her Place for any Inſtants of Time on the Plane of Projection, and thereby be able to delineate her Path. Only here, becauſe the Plane I F is between the Eye and the Object, the *Sum* of the Parallaxes is to be taken; for now it is S M : S Q :: S Q M (or R Q M) :

SMQ; but RQM = QMS + QSM (632) = *Sum of the Parallaxes of the Sun and Moon.*

1734. But to repreſent the lunar Eclipſe, the Diameter of the Earth's Shadow at the Point M muſt be known. In order to that, it is to be obſerved, that the Angle MQR conſiſts of two others, *viz.* RQV and VQM; of which RQV = AQM = apparent Semidiameter of the Sun; and the other VQM = apparent Semidiameter of the Shadow at M. But RQM = QMS + QSM; therefore QSM + QMS — RQV = VQM. That is, from the Sum of the horizontal Parallaxes of the Sun and Moon, ſubtract the apparent Semidiameter of the Sun, the Remainder is the Semidiameter of the Earth's Shadow at M, and is therefore given from the aſtronomical Tables.

1735. With this Semidiameter, as a Radius, deſcribe a Circle on the Plane of Projection about the Point E, and ſhade it to repreſent the Section of the Earth's Shadow. From this Point E, let fall a Perpendicular to the Path of the Moon, and that will ſhew the neareſt Diſtance of the Centers of the Shadow and Moon. And by deſcribing ſeveral lunar Circles at given Inſtants of Time in the Moon's Path, the Phaſe and Quantity of the Eclipſe will appear at each particular Inſtant (1732.) And as the Shadow of the Earth, at the Moon, is in Diameter near three Times as large as the Moon, therefore a lunar Eclipſe may be total to all the World for a conſiderable Time, whereas an Eclipſe of the Sun can be little more than momentarily ſo, as the Diſk of the Moon but very little exceeds that of the Sun, and that too but really happens in Eclipſes.

1736. We have now explained and applied the perſpective Principles to the Doctrine of Eclipſes, and though there may be readier Methods of conſtructing an Eclipſe, there is none ſo punctual and exact as this, when applied by the Method of *Interpolation*; but that will be the Subject of a future Part of this Work.

1737. The ſame perſpective Analogies are applicable to the TRANSIT of VENUS and MERCURY over the Sun's Diſk; and as there is no Difference in the Method of delineating the Path of the Planet, and that of the Moon in a ſolar Eclipſe, but what

what is merely verbal; and that the Diſk of the Sun is to be put for the Diſk of the Earth, it is preſumed what has been ſaid is ſufficient for the Conſtruction of any Tranſit of thoſe Planets.

CHAP. XIX.

GNOMONICS; *or the* PRINCIPLES *of* PERSPECTIVE *applied to the* ART *of* DIALLING.

1738. IN the Application of PERSPECTIVE to DIALLING, the Poſition of the *Eye* and *Plane of Projection* are interchanged; or the Eye is in this Caſe placed in the *Center* of the Sphere, and the *Plane* in *Contact with its Surface*. For in delineating the *Hour-lines* on a given Plane, it muſt be conſidered, that the viſual Ray which connects the Sun, and the Eye deſcribes the Plane of each *Hour-circle*, and of Courſe, in its Motion through any Plane on the Surface of the Sphere, it will trace out the *Hour-line* on that Plane.

1739. And it is quite the ſame Thing whether we conſider the Plane of any *Hour-circle* projected on a given Plane at the Surface of the Sphere by Rays from the Sun, or by viſual Rays from the Eye in the Center; for ſince both the Sun and the Eye are in the projected Hour-plane, the Projection itſelf muſt be a *Right-line* on the given Plane (1464).

1740. And, in ſhort, ſince all great Circles of the Sphere paſs through the Eye in the Center, therefore they will all be projected into Right-lines on a Plane in any Poſition whatſoever. But with regard to leſſer Circles, they will be of one or other of the *Conic-ſections*, as they are poſited relative to the Plane.

1741. Let Æ P Q S (Fig. 1.) be a great Circle of a Sphere, then will it be projected into the Right-line A B, infinitely continued on a Plane touching the Sphere in the Zenith Z. And any Part or Arch Z P is projected into Z D the *Tangent* thereof.

1742. Let I L be the Diameter of any leſſer Circle parallel to the Plane A B, then will its Center K be in the Perpendicular N Z, and is projected into the Point Z; its Semidiameters I K and K L into the Tangents Z E and Z F; which as they are equal to each other, ſhews *the Projection of a leſſer Circle parallel to the Plane, is a* CIRCLE *upon the Plane.*

1743. Let L G be a Diameter of a leſſer Circle in ſuch an oblique Poſition to the given Plane A B, that it may be wholly projected upon it within a finite Diſtance Z B; then it is evident the Center M will be projected into the Point D on the Plane, the Semidiameter I M into the Right-line F D, and the Semidiameter M G into the Right-line D B; and ſince it is evident that D B muſt exceed D F, the Projection cannot be a Circle. But conſidering L N G as a right Cone, its *Section* (if continued out) *by the Plane* A B, muſt be an ELLIPSIS (763).

1744. Again, it is as evident, that if the Diameter G O of any leſſer Circle paſſes through O, the extreme Part of the Diameter H O parallel to the given Plane A B; then, becauſe the Cone G N O (infinitely continued) is cut by a Plane parallel to one of its Sides N O, *the Projection of any Circle* G O *on that Plane will be a* PARABOLA (740).

1745. A ſmall Circle poſited in any other Manner than what is above ſpecified, will be projected into an HYPERBOLA, as is manifeſt from the ſame Principles of the Section of a Cone (765).

1746. We have ſhewn that the Interſection of the Plane of any Hour-circle, with a given Plane on the Surface of the Sphere, is the true Hour-line on that Plane (1738) and the Caſe is the ſame if the given Plane paſſes through the Sphere in any Part, provided it be in a *parallel Poſition*; for the Planes of the ſame Hour-circles will interſect two or more parallel Planes in the ſame Manner; and therefore the *Hour-lines* will be the ſame in all; this is evident from (615 and 631.)

1747. Hence it is common to repreſent the given Dial-plane as paſſing thro' the Center of the Sphere, as in Fig. 2. where it is denoted by D W B E. The Sphere with its Hour-circles, is there orthographically projected (1711) Æ Q is the Equator; Æ P Q S is the Meridian, or Hour-circle of XII; and *a*, *b*, *c*, *d*, *e*, *f*, N, are the Elliptic-projections of the Hour-circles of

12, 11, 10, 9, 8, 7, 6, in the Morning, which are all at the equal Diſtance of 15 Degrees on the Sphere; for 24) 360° (= 15° per Hour in the Sphere's Revolution.

1748. The Planes therefore, of theſe Circles, will interſect the given Plane D W B E in the Lines N 12, N 11, N 10, N 9, *&c.* which will therefore be the Hour-lines on that Plane; and thus they are to be conſidered in all the other Quadrants of the Plane.

1749. But when the Hour-lines are aſſigned or delineated on a Plane, it is neceſſary there ſhould be ſome Contrivance to indicate or point out the Moment of Time when the Sun is ſucceſſively in thoſe horary Planes; now if the Plane of the Dial paſs through the Center of the Sphere, then the Expedient we require is found in the Axis of the Sphere P S. For ſince all the Planes of the Hour-circles interſect each other in the ſaid Axis, the Sun when it comes on any one horary Plane will project that Axis into the Hour-line on the Dial-plane proper to that Hour-circle.

1750. Moreover, as the Axis P S of the Sphere is that Line about which its Motion is performed, it muſt in it's ſelf be conſidered as *abſolutely at Reſt.* For any Line parallel to the Axis, in the Plane of any Hour-circle, has leſs Motion in Proportion as it is nearer to the Axis; and conſequently when it coincides with, or becomes the Axis, it can have no Motion at all.

1751. The Axis therefore of the Sphere retaining always the ſame Poſition, if it be ſuppoſed to conſiſt of a fine inflexible Wire, it will intercept the Sun's Rays in the Planes of the Hour-circles, and therefore its Shadow muſt conſtantly fall on the Hour-lines of the given Dial-plane, correſponding to the reſpective Planes of the ſaid Hour-circles; ſo that *the Axis of the Sphere, by Means of its Shadow, will conſtantly indicate the Moment of Time when the Sun is in any Meridian.*

1752. Now the Plane D W B E is called the horizontal Plane with reſpect to the Place Z, becauſe it is parallel to the horizontal Line A B, touching the Sphere in that Point; and when all the Hour-lines are drawn upon the Plane for all the Meridians or Hour-circles the Sun can be in above the Plane, then

then this Plane with its Hour-lines, and Semiaxis N P compleats the HORIZONTAL DIAL in the Sphere.

1753. The *Semiaxis* N P is called the GNOMON or STILE of the DIAL; and its Elevation or Height B N P above the Plane of the Dial, is always equal to Æ Z the Latitude of the Place. For Æ P = Z B = Quadrant; from each of which take away the common Arch Z P, and there will remain Æ Z = P B. Therefore alſo the Stile's Height is equal to the Elevation of the Pole in the Latitude Z.

1754. If the Plane be in a vertical Poſition, or paſſes thro' the Zenith and Nadir Points Z and D, and is the Plane of the prime Vertical (as Fig. 3.) then ſince the Sun cannot be on or above ſuch a Plane, but from VI to VI; no other Hour-lines need be drawn upon it. And in this Caſe, the Oppoſite or South Pole S is elevated above it in the Angle D N S = Z N P = the Complement of the other Elevation P N B to a Quadrant. Here alſo the other *Semiaxis* N S is the Stile; and this is called an *erect and direct* SOUTH DIAL. And the other Side of the Plane is a *North Dial*, for the Hours before VI, and the Stile is N P.

1755. It is eaſy to obſerve, that this *vertical Dial* is an *horizontal One* in the South Latitude D, which is the Co-Latitude of the Point Z, ſince Z D = 90 Degrees. The Meridian Line N D or Hour-line of XII is directly under the Gnomon or Stile N S, and is therefore called the *Subſtilar-line* or *Subſtile*.

1756. If we ſuppoſe the Plane to continue vertical, but to decline or move towards the Eaſt or Weſt; then it becomes an *Eaſt or Weſt declining* DIAL; and it is evident in ſuch Caſes, the Angle which the Stile N S makes with the Plane is leſſened, and the Subſtile departs from the Hour-line of XII, or rather, this departs from that, towards the Eaſt or Weſt.

1757. When the vertical *Dial* has a Poſition directly *Eaſt* or *Weſt*, then the Axis P S of the Sphere is in the Plane; and conſequently the Stile of an *Eaſt* or *Weſt Dial*, muſt be parallel to the Plane thereof, and the Height of it may be taken at Pleaſure, and will be always equal to the Diſtance of the Hour-lines of III and IX, from the Middle-line of VI. And as the Plane of the Dial

is

is now in the Plane of a Meridian, it is an *horizontal Dial* under the EQUATOR every where, and is therefore called an EQUATORIAL DIAL.

1758. With reſpect to the horizontal Plane D W B E (Fig. 2.) it is evident, that as the Point Z approaches the Pole P, the Plane will approach to that of the Equator Æ Q; in which Caſe the Axis P N becomes perpendicular to the Plane, and marks out the Hour by dividing the Plane into 24 equal Parts; and ſuch an One is called a POLAR DIAL, as being an *horizontal Dial* under the Pole P.

1759. This POLAR DIAL (commonly called an *equinoctial Dial)* with a perpendicular Stile, being drawn on a Plane, and that Plane elevated to the Latitude of a Place, the Hour of the Day will be ſhewn upon it, by the Shadow of the Perpendicular, with the ſame Exactneſs as on a common horizontal Dial for that Latitude; and therefore this Dial is, in its own Nature, an UNIVERSAL DIAL, for *all Latitudes* or Parts of the World.

1760. The Eye being in the Center N of the Sphere projects the *Parallels* of the *Sun's diurnal Motion* on a given horizontal Plane into one of the *Conic Sections*; for let Æ C be the Sun's Declination, then C R is the Diameter of the Parallel of its Motion on that Day; and C N R the Section of the Cone deſcribed in the Sphere by the viſual Ray drawn from the Eye to the Sun. Now ſince C R is parallel to the Equator Æ Q, the Axis P S will be perpendicular thereto; and therefore upon a horizontal Plane at the Pole P, which is parallel to C R, the Projection of the Parallel, which is a Circle, will be a CIRCLE alſo (1504).

1761. Produce the Side of the Cone R N to V; then Æ C = Æ V; and on V R erect the perpendicular L N on the Center N. Then upon an horizontal Plane in the Latitude L, which will be parallel to the Side of the Cone N R or V R, the Projection of the Cone's circular Baſe, or parallel of the diurnal Arch, will be a PARABOLA (1505).

1762. Hence upon every horizontal Plane between the Pole P and the Latitude L, the circular Baſe, or Parallel muſt be projected into an ELLIPSIS, more eccentric in Proportion as the

Plane is nearer the Point L (1507) for every ſuch Plane will cut both the Sides of the Cone C N R produced.

1763. And on the other Hand, upon every horizontal Plane, between the Equator and the Latitude L, the Baſe of the Cone will be projected by the Eye into a HYPERBOLA (1506) becauſe ſuch a Plane muſt cut the Side C N of the Cone C N R, and the Side V N of the oppoſite Cone V N T, when produced.

1764. Becauſe Æ C = L P, therefore Æ L = C P = Complement of the Sun's Declination; therefore in all Latitudes leſs than the Co-Declination, the Projection is an *Hyperbola*; in all Latitudes greater, it is an *Ellipſis*; and in the Latitude equal thereto, it is a *Parabola*. In the Latitude 90°, the *Ellipſis* becomes a *Circle*; and in the Latitude 00, or in the Equator, the *Hyperbola* degenerates into a *Right line*.

1765. Hence we have an eaſy Tranſition to the Shadows caſt by a Gnomon on the Plane of any *horizontal Dial*. For the ſame Ray which projected the Sun's diurnal Path or Parallel on a Plane, projects alſo the Shadow from the Point of the Gnomon on the ſame Plane. Thus ſuppoſe a Plane (*a b c d*) placed parallel to A B, below the Center N of the Sphere, cutting the Axis in E, ſo that the Part of the Axis E N becomes the Gnomon on that Plane, then it is evident, the ſame Ray N C projects the Parallel of C R on the Plane at A B on one Hand, and the Shadow of the extreme Point N of the Gnomon on the Plane (*a b c d*) on the other; and that when the Sun is in the Meridian at C, the Shadow of the Gnomon falls in the Subſtile E *f* on the Point I, at the Diſtance F I from the perpendicular F N.

1766. Now becauſe of the Paralleliſm of the Planes, the Curves deſcribed by the Shadow will be exactly the ſame as thoſe of the projected Parallel of the Sun's diurnal Path; that is to ſay, (1.) Under the Pole P, the Shadow of the Point N will deſcribe a CIRCLE. (2.) At the Latitude L, equal to the Co-Declination of the Sun, it will deſcribe a PARABOLA. (3.) On all other Planes between P and L it will deſcribe ELLIPSES. (4.) In all Latitudes from the Equator to the Point L, the Path of the Shadow will be a HYPERBOLA. And (5.) When the Sun is in the Equator at Æ, the Shadow will fall on K, at Noon,

Noon, and its Path will that Day be the Right-line *g* K *h*. (1760 to 1765.)

1767. Hence it appears, that when the Sun enters *Cancer*, and Æ C = 23° 30′, then L is in Latitude 66° 30′, and conſequently, the Shadow can deſcribe Ellipſes only in the *frigid Zone* L P, when the Sun is altogether above that Horizon; and its Altitude on the South and North Part of the Meridian unequal.

1768. If the Height of the Gnomon N F be made the Radius, then the Angle N I F = C N H is the Sun's Meridian Altitude, and the Angle I N F its Co-Altitude; then ſay, as Radius to N F, ſo is Co-Tantent of the Meridian Altitude, to the Diſtance of the Shadow F I; when the Sun is in C. If C I be the *Ecliptic*, then the Points I, K, and W, will be found by this Analogy for the ſolſtitial and equinoctial Days.

1769. Hence if the Dial-plane be of a ſufficient Length, all the Parallels of the Sun's Declination may be deſcribed thereon at his Entrance into each of the 12 Signs. Thoſe of the firſt Six, or *Summer* half Year, will fall between K and I; and thoſe for the *Winter*, between K and W. In our Latitude of 51° 30′, we have Radius : Tangent of 75° :: F N : F W :: 1 : 3,7 nearly.

1770. Suppoſe Æ Z Q S a graduated Braſs Meridian, ſuſpended from a Ring at Z, which is moveable and ſet to the Latitude of the Place; then if C R be the Tropic of *Cancer*, and V T the Tropic of *Capricorn*; and (*n m*) a Plate of Braſs in the Axis of the Sphere, with a moveable Piece containing a ſmall Hole which may be placed any where between *n* and *m*; the Ray of Light paſſing through that Hole, when properly adjuſted, will, from the Meridian Sun, always fall on the North Point Q of the Equator, for when the Sun is in *Cancer* at C, and the Hole at (*n*) the ſolar Ray will be *n* Q parallel to C N; and when the Sun is in *Capricorn* at V, and Hole at (*m*) the Ray will be in Q parallel to V N, and therefore every Day, at Noon, the Ray will paſs through the Hole to the Point Q, or Hour of XII in the Equator.

1771. Alſo when the Sun is on any other Hour-circle before or after Noon, it will ſtill paſs through the Hole, (adjuſted to the Day,) to the Equator, but juſt as far from Q, as the Hour-cir-

cle is from the Meridian; becauſe the diurnal Motion of the Ray N C, and the Motion of the Ray n Q parallel to it, will neceſſarily be the ſame; therefore the Ray through the Hole in the Axis will ever point out the Hour on the Periphery of the Equator. And this is the *Rationale* of the Conſtruction of the *univerſal* RING-DIAL, the Form of which is exhibited in Fig. 5.

1772. We ſhall finiſh this Theory of *perſpective Dialling*, with obſerving that the Hour-lines of any Dial are laid off by the Tangents of the Angles which the projected Meridians make on the Horizon of the Place. For let W N E S (Fig. 6.) be the Horizon of LONDON, Latitude 51° 30′, on which project the Sphere, and the horary Meridians will interſect the Horizon in the Points 1, 2, 3, 4, &c. The Eye being in the Center C; let the Plane of Projection touch the Sphere in N; then is (*a b*) the *horizontal Line*, and N C the *vertical Line*; and then to ſhew from the Rules of Perſpective, that the Hour-line of II paſſes thro' (2) the Point of Interſection of the 2 o'Clock Meridian with the Horizon; we have C N = r, c 2 = N g = v = verſed Sine, or Diſtance from the horizontal Line; and N c = g 2 = s = Sine of the Arch N 2, or Diſtance of the Point (2) from the vertical Plane. Then becauſe that Point is between the Eye and Plane of Projection, the Analogy is (1691)

as $nr - v : r :: s : \frac{rs}{nr - v} = \frac{rs}{r - v}$ (becauſe here $n = 1$)

$= \frac{rs}{c}$, (putting c = Co-Sine C g) which therefore is the Diſtance N d of the Hour-line of II from the vertical Plane C N.

But $\frac{rs}{c} = t$ = N d = Tangent of the Arch N 2; (for $s : c :: r : t$, or C g : C 2 :: C N : C d (711.) *Therefore the Hour-lines of a horizontal Dial are drawn from the Center* C *thro' the Points of Interſection of the Hour-circles with the Horizon of the Place.**

INSTI-

* *N. B.* Fig. 7. is here inſerted from *Bion*, repreſenting a *general horizontal Dial*; a particular Deſcription of which will hereafter be given.

INSTITUTIONS OF SPHERICAL TRIGONOMETRY.

CHAP. I.

Of the Nature of SPHERICAL TRIANGLES, *with a* SOLUTION *of all their* CASES.

1773. WE have heretofore conſidered the Dimenſions of a SPHERE with regard to its ſuperficial, and ſolid Contents in the *Inſtitutions of plain Geometry*; that which we call *ſpherical Geometry* being converſant chiefly about the *Relation*, *Magnitude*, and *Inclination* of the GREAT CIRCLES of the SPHERE; and more particularly the various TRIANGLES that are form'd by their Interſections on its Surface; whoſe Properties and Affections we now proceed to treat of. And firſt of thoſe which are *Rectangular*.

1774. A *Spherical Triangle*, is that whoſe three *Sides* are ſeverally *the Arches of three great Circles of the Sphere*, interſecting each other in the three Angular Points. And the Angles of every Triangle are meaſured by the Arches of Circles; therefore the Arches of Circles are the Meaſures of every Part of a *Spherical Triangle*.

1775. And as the Quanity of the Arches themſelves is determinable only from the Relations of thoſe right Lines which are called their *Sines*, *Tangents* and *Secants*, as they ſtand conſtructed with the Radius in a plain Triangle, (706 to 708.) Therefore the Solution of a ſpherical Triangle, differs in nothing

thing eſſential to the Operation, from that of a common plain Triangle.

1776. Let A B C (Fig. 1.) be a right-angled ſpherical Triangle on the Surface of the Sphere, and ſuppoſe the Sides continued out indefinitely in the *Baſe* A B to E; the perpendicular A C to H; and the Hypothenuſe B C to I. On the Angle B draw the great Circle G F E thro' D the Pole of the Baſe A B; and on the Angle C, draw the great Circle G H I thro' G, the Pole of Hypothenuſe B C. Then will there be form'd the four Triangles E B F, E D A, I C H and I G F, each having two right Angles, and two quadrantal Sides, as is evident by Inſpection.

1777. There are alſo form'd two other right Angled ſperical Triangles. *viz.* C D F right Angled at F, and G D H, right Angled at H, which may be called *complemental Triangles* with reſpect to the Original, or given Triangle A B C.

1778. For in the firſt C E D the Baſe C F is the Complement of the Hypothenuſe B C, and the Angle at the Perpendicular, D = A E, is the Complement of the Baſe A B. And the Hypothenuſe C D, is the Complement of the Perpendicular A C. And, laſtly, the Perpendicular D F is the Complement of E F = B, the Angle at Baſe; the Angle C at the Perpendicular, being the ſame in both.

1779. And in the other Triangle D G H, the Baſe G H is the Complement of H I = C in the given Triangle A B C. The Perpendicular D H = A C, becauſe A D and C H are Quadrants. The Hypothenuſe D G = E F = B the Angle at Baſe. The Angle at Baſe G = I F = B C the Hypothenuſe of the given Triangle. And the Angle at the Perpendicular, D = A E, the Complement of the Baſe A B.

1780. Now by help of theſe Quadrantal and Complemental Triangles, the Canons or Analogies for the Solution of all the Caſes of a right Angle, ſpherical Triangle A B C, may be eaſily derived from the two following Theorems.

THEOREM I.

In a right Angled ſpherical Triangle, the Radius (or Sine of 90 Degrees) to the Sine of the Baſe, as the Tangent of the Angle at Baſe is to the Tangent of the Perpendicular. For let Æ B P (Fig. 2.) be a fourth Part of the *Orthographic Projection* of the Sphere on the Plane

Plane of the *Meridian* (1710.) And let A B C be a right-angled ſpherical Triangle form'd on the Globe, by the three great Circles Æ B, B F, and A P. Then is B C the Sine of the Hypothenuſe, and A B the Sine of the Baſe (1690.) C H is the Sine of the Perpendicular A C, and F E the Sine of the Angle at Baſe B; alſo A D, and Æ G, are the Tangents of the ſame Parts. And becauſe of ſimilar Triangles A B D and Æ B G, (657) we have *as Æ B Radius is to A B the Sine of the Baſe, ſo is Æ G the Tangent of the Angle at Baſe, to A D the Tangent of the Perpendicular A C.* Q. E. D.

1781. THEOREM II.

The Radius (or Sine of 90 *Degrees) is to the Sine of the Hypothenuſe, as the Sine of the Angle at Baſe is to the Sine of the Perpendicular, in every right-angled ſpherical Triangle.* For by Reaſon of the ſimilar Triangles C B H, and F B E, we have F B : C B :: F E : C H. *Q. E. D.*

1782. By theſe two Theorems, all the Caſes of a right Angled ſpherical Triangle are ſolved. That is, of the three Sides and three Angles, if any two are given, beſides the right Angle, the other three Parts may be found; and here it muſt be obſerved, that either of the two Sides A B or A C is reckon'd the Baſe, as the Angle B or C is given. And therefore there can be but a ſixfold Variety, or different Caſes in regard to any right-angled Triangle, which here follows.

1783. CASE I.

Given the Baſe A B, *and Angle* B; (Fig. 1.) *to find the Perpendicular* A C. In the quadrantal Triangle E B F; it is *s* B E : *s* B A :: *t* F E : *t* A C; that is R : *s* B A :: *t* B : *t* A C (1780).

1784. *To find the Hypothenuſe* B C; in the quadrantal Triangle A D E, we have *s* D E : *s* D F :: *t* A E : *t* C F; that is, R : *cs* B :: *ct* A B : *ct* B C.

1785. *To find the Angle* C. In the quadrantal Triangle F G I, we have, as *s* F G : *s* D G :: *s* D : *s* G H; that is, R : *s* B :: *cs* A B : *cs* C (1781).

1786. CASE

1786. CASE II.

Given the perpendicular AC, *and Angle at Baſe* B; *to find the Baſe* AB. In the quadrantal Triangle FBE we have *t* EF : *t* AC :: *s* BE : *s* BA; ∴ *t* B : *t* AC :: R : *s* BA.

1787. *To find the Hypothenuſe* BC; we have *s* EF : *s* AC :: *s* BF : *s* BC; ∴ *s* B : *s* AC :: R : *s* BC.

1788. *To find the Angle* C. In the quadrantal Triangle HCI, it is, *s* DC : *s* DH :: *s* DF : *s* HI; ∴ *cs* AC : R :: *cs* B : *s* C.

1789. CASE III.

Given the Hypothenuſe BC, *and the Angle at Baſe* B; *to find the Baſe* AB. In the quadrantal Triangle ADE, we have *s* DF : *s* DE :: *t* CF : *t* AE; ∴ *cs* B : R :: *ct* BC : *ct* AB, or R : *cs* B :: *t* BC : *t* AB.*

1790. *To find the Perpendicular* AC. In the quadrantal Triangle EBF, there is *s* BF : *s* BC :: *s* EF : *s* AC; ∴ R : *s* BC :: *s* B : *s* AC.

1791. *To find the Angle* C. In the quadrantal Triangle HCI, it is *s* CF : *s* CI :: *t* DF : *t* HI; that is, *cs* BC : R :: *ct* B : *t* C.

1792. CASE IV.

Given the Baſe AB, *and Perpendicular* AC; *to find the Hypothenuſe* BC. In the quadrantal ADE, there is *s* AD : *s* DC :: *s* AE : *s* CF; that is, R : *cs* CA :: *cs* AB : *cs* BC.

1793. *To find the Angle* B; we have *s* AB : R :: *t* AC : *t* B (1786).

1794. *To find the Angle* C; *s* AC : R :: *t* AB : *t* C; by the ſame Reaſon as in the laſt.

1795. CASE V.

Given the Hypothenuſe BC, *and the Perpendicular* AC. *To find the Baſe* AB. *As cs* AC : R :: *cs* BC : *cs* AB (1792).

1796. *To find the Angle* B; *s* BC : R :: *s* AC : *s* B (1787).

1797. *To find the Angle* C; in the quadrantal IGF, we have *t* IF : *t* DH :: *s* GI : *s* GH; that is, *t* BC : R :: *t* AC : *cs* C.

1798. CASE

* Becauſe the *Co-Tangents* of any two Arches are inverſely as their Tangents, as is ſhewn hereafter at (1831).

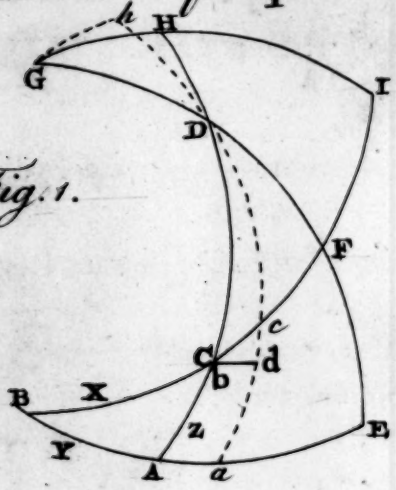


Fig. 1.

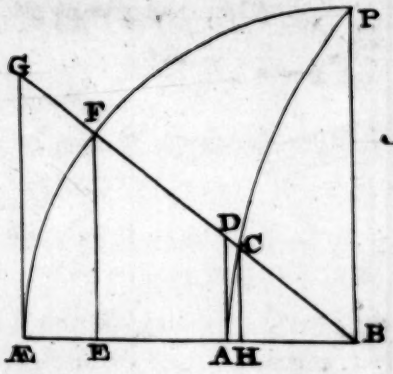


Fig.

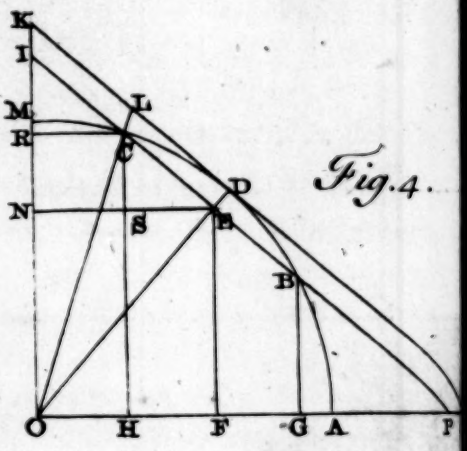


Fig. 4.

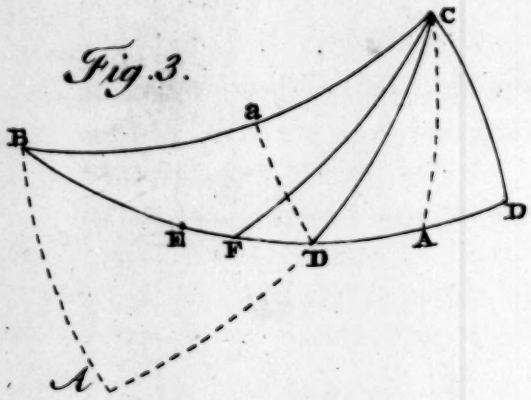


Fig. 3.

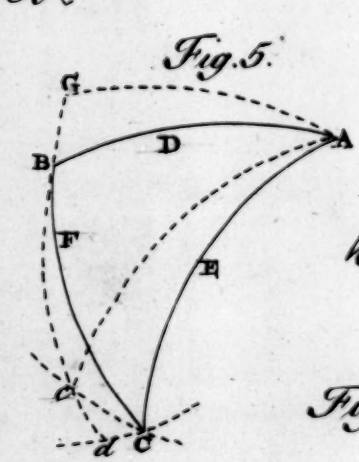


Fig. 5.

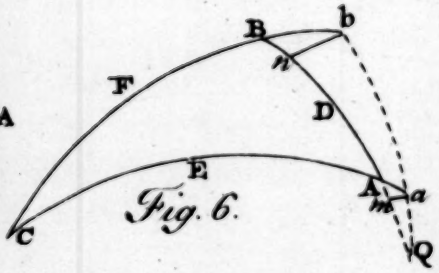


Fig. 6.

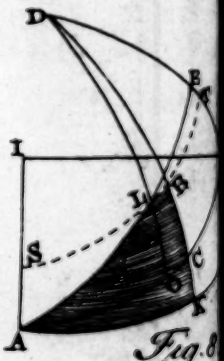


Fig. 8

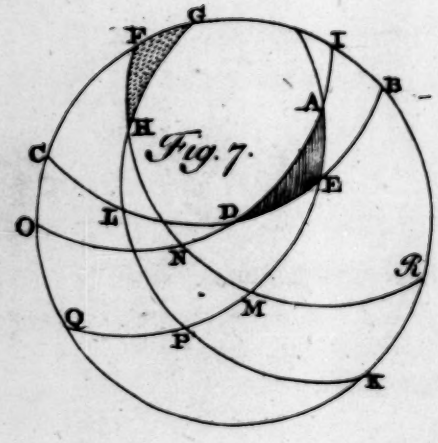
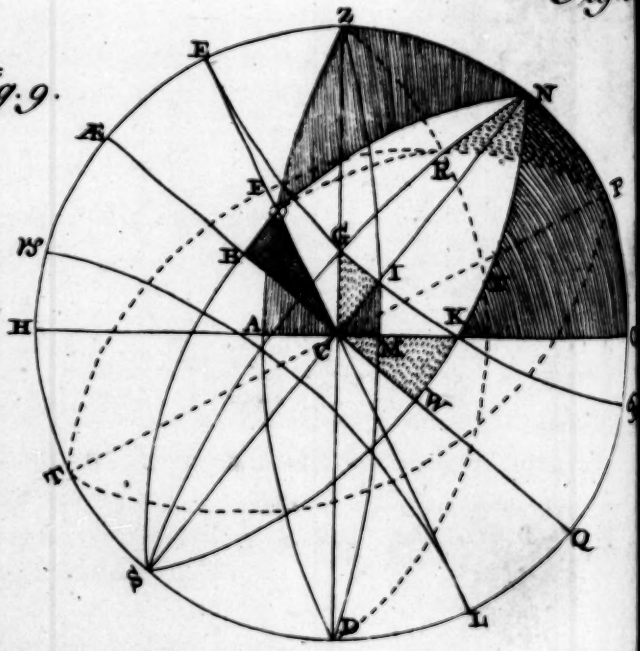


Fig. 7.

Fig. 9.



1798. CASE VI.

Give the Angles B and C; *to find the Baſe* A B. As *s*B : *cs*C :: R : *cs*AB (1785).

1799. *To find the Perpendicular* A C. As *s*C : *cs*B :: R : *cs*AC (1785).

1800. *To find the Hypothenuſe* BC; in the quadrantal ICH, we have *t*HI : *t*DF :: *s*CI : *s*CF; which gives *t*C : *ct*B :: R : *cs*BC.

1801. The two Angles B and C of every right-angled ſpherical Triangle, are together greater than the right Angle A; for the Angle B = EF = GD; and the Angle C = IH; but IH + DG, is greater than IH + HG = 90 Degrees, or a right Angle, whence the Theorem is evident.

CHAP. II.

The Method of determining the FLUXIONS *of the* SIDES *and* ANGLES *of a right-angled* SPHERICAL TRIANGLE.

[PLATE I. *Of* SPHERICAL TRIGONOMETRY.]

1802. IF the Semi-diameter of the Earth bore no ſenſible Proportion to the Diſtance of the heavenly Bodies; if we could view them not through a *refracting Medium*; if the Poſition of the Earth's Axis were *immutable*, and laſtly, if *there were no Obliquity of the Ecliptic*, then the Subject of this Chapter would be unneceſſary.

1803. But asthe Caſe now ſtands, we find a ſenſible Difference between the *true and apparent Place* of the *Sun*, *Moon*, and *Planets*, on Account of their *Parallax* (1724.) *Refraction* (1321.) *Motion* of the *Earth's Axis*, *&c.* by Means of all which the *Latitude*, *Longitude*, *Right-Aſcenſion*, *Declination*, *Altitude*, *Azimuth*, *Amplitude*, *Hour-Angle*, *&c.* will all be affected in the ſame Manner as if the great Circles by which they are aſcertained were continually moveable through a very ſmall Space, one Way and the other, about their Axis in the Sphere.

1804. Therefore it becomes neceſſary, where Exactneſs is required, (as in *Aſtronomy*, *Navigation*, *&c.*) to make a proper Correction of the Computations made by ſpherical Triangles, by ſuppoſing ſuch a ſmall Motion in any one of the Circles of which that Triangle conſiſts, and from thence to find the *Fluxionary Increaſe or Decreaſe* of its other Sides and Angles reſpectively, occaſioned thereby; which will expreſs the Quantities ſought from the *Parallax*, *Refraction*, *&c.* given.

1805. Therefore (in Fig. 1.) if we ſuppoſe the Circle H D A to be moveable upon the Pole D; and to change its Situation from H D C A to b D c a; then it is evident, that in the three variable Triangles A B C, D C F, and D H G, there will (beſides the Right-angle) be one Part *conſtant* in each, and the other Parts all variable; thus the Angle B = E F, is conſtant in A B C; the Side D F in D C F; and the Hypothenuſe G D in D G H.

1806. For Diſtinction Sake, let X, Y, Z, denote the *Hypothenuſe*, *Baſe*, and *Perpendicular* of the Triangle A B C, and let C d be drawn at Right-angles to the Circle b D a. Then is A a = $\dot{Y}$, the *Fluxion* of A B; but — $\dot{Y}$ is the Fluxion of its Complement A E; C c = $\dot{X}$, and dc = $\dot{Z}$. Alſo the Fluxion of the Angle C = I H, is H e = $\dot{C}$. Which Fluxionary Parts are thus determined.

1807. In the fluxionary quadrantal Triangle A D a, we have (1780.) s D A : s D C :: A a : C d, or R : s D C :: A a : C d = $\frac{s\,DC \times \dot{Y}}{R} = \frac{cs\,AC \times \dot{Y}}{R}$.

1808. The ſmall fluxionary Triangle c C d may be eſteemed *Rectilineal*, which gives this Analogy. As t C (= c) : R :: C d : cd; that is, t C : R :: $\frac{cs\,AC \times \dot{Y}}{R} : \frac{cs\,AC}{tC} \times \dot{Y} = cd = \dot{Z}$ (1807).

1809. Alſo, in the ſame Triangle, as s C : R :: C d : C c :: $\frac{cs\,AC}{R} \times \dot{Y} : \frac{cs\,AC}{sC} \times \dot{Y} = Cc = \dot{X}$.

1810.

1810. Laſtly, $s\,DA : s\,DH :: Aa : He$ (1781); but $DH = AC$; therefore $R : s\,AC :: \dot{Y} : \frac{s\,AC}{R} \times \dot{Y} = He = \dot{C}$, or the Fluxion of the Arch IH which meaſures the Angle C.

1811. In the Triangle CDF, having one Side DF conſtant, the Fluxions of the other Parts are derived from the above Equations by Subſtitution of equal Parts; for becauſe $c\,s\,AC = s\,DC$, and $+\dot{X} = -\dot{X}$ in Quantity; therefore $\frac{s\,DC}{s\,C} \times \dot{Y} = -\dot{X}$ (1809), the Fluxion of the decreaſing Side or Baſe CF.

1812. For the ſame Reaſon $\frac{s\,DC}{t\,C} \times \dot{Y} = -\dot{Z} = cd$, the Fluxion of the decreaſing Hypothenuſe DC (1808).

1813. Laſtly, we have $\frac{c\,s\,CD}{R} \times \dot{Y} = \dot{C} = He$, the Fluxions of the increaſing Angle C, or Arch IH. The Fluxion of the Angle D, or Arch AE is negative, or $Aa = -\dot{Y} = -\frac{R}{c\,s\,CD} \times \dot{C}$.

1814. In the Triangle DGH, the Hypothenuſe DG being conſtant; the Fluxions of the variable Parts are found by Subſtitution of Equals as before. Thus $\frac{c\,s\,DH}{c\,t\,GH} \times \dot{Y} = \dot{Z}$, or Fluxion of the Side DH, which is Poſitive (1808).

1815. Alſo, it is $\frac{c\,s\,DH}{c\,s\,GH} \times \dot{Y} = \dot{X}$, the Fluxion of the Angle G, which is alſo poſitive (1808).

1816. And then for the Fluxion of the Side GH, (which will be, in this Caſe, negative) we have $\frac{s\,DH}{R} \times \dot{Y} = \dot{C}$.

1817. In the Demonſtrations hitherto, the Letters X, Y, Z, are uſed in Reference to three Triangles, and ſometimes they ſeparately denote an Angle, and ſometimes a Side. But if we have regard to one Triangle only, we muſt uſe ſuch Symbols as will denote the ſeveral Parts abſolutely. Therefore let H be the Hypothenuſe; A and a, the two Sides; and B, b, the two

Angles adjacent to them reſpectively. Then will the Equations in each Caſe or Triangle be transformed, in the following Manner.

1818. Case I.

When *one Angle* B is invariable, we have $c\,s\,AC \times \dot{Y} = t\,C \times \dot{Z}$ (1808.) become $c\,s\,a \times \dot{A} = t\,b \times \dot{a}$, and ſo of the Reſt, whence we derive the following Equations and Analogies.

1. $c\,s\,a\,\dot{A} = t\,b\,\dot{a}$; $\therefore \dot{A} : \dot{a} :: t\,b : c\,s\,a$ (1808).
2. $c\,s\,a\,\dot{A} = s\,b\,\dot{H}$; $\therefore \dot{A} : \dot{H} :: s\,b : c\,s\,a$ (1809).
3. $s\,a\,\dot{A} = R\,\dot{b}$; $\therefore \dot{A} : \dot{b} :: R : s\,a$ (1810).
4. $t\,b\,\dot{a} = s\,b\,\dot{H}$; $\therefore s\,b : t\,b :: c\,s\,b : R :: \dot{a} : \dot{H}$.

1819. Case II.

When *one Side* A is invariable.

1. $s\,H\,\dot{B} = -s\,b\,\dot{a}$; $\therefore \dot{B} : -\dot{a} :: s\,b : s\,H$ (1811).
2. $s\,H\,\dot{B} = -t\,b\,\dot{H}$; $\therefore \dot{B} : -\dot{H} :: t\,b : s\,H$ (1812).
3. $c\,s\,H\,\dot{B} = R\,\dot{b}$; $\therefore \dot{B} : \dot{b} :: R : c\,s\,H$ (1813).

1820. Case III.

When the *Hypothenuſe* H is conſtant.

1. $c\,s\,A\,\dot{B} = c\,t\,a\,\dot{A}$; $\therefore \dot{A} : \dot{B} :: c\,s\,A : c\,t\,a$ (1814).
2. $c\,s\,A\,\dot{B} = c\,s\,a\,\dot{b}$; $\therefore \dot{B} : \dot{b} :: c\,s\,a : c\,s\,A$ (1815).
3. $s\,A\,\dot{B} = -R\,\dot{a}$; $\therefore \dot{B} : -\dot{a} :: R : s\,A$ (1816).
4. $c\,t\,a\,\dot{A} = c\,s\,a\,\dot{b}$; $\therefore c\,s\,a : c\,t\,a :: s\,a : R :: \dot{A} : \dot{b}$.

1821. From theſe Equations, it is evident, there is a threefold Value or Expreſſion for the Fluxion of every variable Part; and many Equations to expreſs the Ratio of any two Fluxions, beſides thoſe here ſpecified. The great Uſe of this Doctrine of *Fluxionary Trigonometry* we ſhall hereafter particularly illuſtrate and exemplify in all the above-mentioned Caſes.

CHAP.

CHAP. III.

THEOREMS *for the* SOLUTION *of the* CASES *of* Oblique Spherical TRIANGLES.

1822. WHEN each of these three Angles of a spherical Triangle is greater or less than a *Right-angle*, it is called an *Oblique Triangle*, as BCD (Fig. 3.) In such an one all the Angles are uncertain, and therefore three Parts out of the Six must here be given, to find the Rest; and consequently there will also be *Six Cases* in the Solution of an *Oblique Triangle*. In order to which, the *Ten* following Theorems must be premised.

1823. THEOREM I.

In every oblique spherical Triangle, the Sines of the Sides are proportional to Sines of the opposite Angles. To demonstrate this; let fall the Perpendicular CA on the Side BD (continued out Fig. 3.) and it will be as $R : sBC :: sB : sAC$; and again, $R : sCD :: sD : sAC$ (1781). Therefore $R \times sAC = sBC \times sB = sCD \times sD$; whence $sBC : sDC :: sD : sB$. *Q. E. D.*

1824. THEOREM II.

In an oblique spherical Triangle BDC, *drawing the Perpendicular* CA, *the Tangents of the Sides* BC *and* CD *are reciprocally proportional to the Co-Sines of the vertical Angles.* For in the right-angled Triangle ACB, it is, $R : csACB :: tBC : tAC$ (1789.) and in the Triangle ACD, it is $R : csDCA :: tDC : tAC$. Therefore $csACB \times tBC = R \times tAC = csDCA \times tDC$; consequently $tBC : tDC :: csACD : csACB$. *Q. E. D.*

1825. THEOREM III.

In the oblique Triangle BDC, *with the Perpendicular* AC, *the Co-Sines of the Sides* BC *and* CD *are directly proportioned to the Co-Sines of the Parts of the Base* AB *and* AD. For in the right-angled Triangle ACB, it is, $R : csAC :: csAB : csBC$,

(1792) and for the ſame Reaſon R : *cs* A C :: *cs* A D : *cs* D C. Therefore (as above) *cs* A B : *cs* B C :: *cs* A D : *cs* D C. *Q. E. D.*

1826. THEOREM IV.

In the ſame oblique Triangle B D C, *the Co-Sines of the Angles at Baſe are directly as the Sines of the vertical Angles.* For as R. : *s* A C B :: *cs* A C : *cs* B (1785) and as R : *s* A C D :: *cs* A C : *cs* D. Therefore *cs* B : *cs* D :: *s* A C B : *s* A C D. *Q. E. D.*

1827. THEOREM V.

In the ſame oblique Triangle B C D ; *the Sines of the Parts of the Baſe* A B *and* A D, *are inverſely as the Tangents of the Angles at the Baſe.* For R : *s* A B :: *t* B : *t* A C (1783.) and R : *s* A D :: *t* D : *t* A C. Therefore *s* A B × *t* B (= R × *t* A C) = *s* A D × *t* D, whence *s* A B : *s* A D :: *t* D : *t* B. *Q. E. D.*

1828. THEOREM VI.

As the Sum of the Sines of two unequal Arches is to their Difference ; ſo is the Tangent of half their Sum to the Tanget of half their Difference. For let A C M be a Quarter of a Circle (Fig. 4.) in which take the two unequal Arches A B, A C ; their Difference is B C, which is biſected by the Radius O D. Then A D is half their Sum, and D C = D B half their Difference. The Sines of the Arches are B G and C H ; and the Co-Sines O G and O H. Through the Points B and C draw the right Line I P, and parallel to it draw K Q touching the Circle in the Point D ; alſo draw N E parallel to A O, and E F perpendicular thereto. And laſtly, through C draw O L, then is D Q the Tangent of half the Sum of the Arches A D, and D L the Tangent of half their Difference C D ; and D K is the Tangent of D M the Complement of A D. Then becauſe C D = B D, or C E = E B, we have alſo H F = F G, and therefore C H + B G = 2 E F, and C H — B G = 2 C S (221). But 2 E F : 2 C S :: E F : C S (656.) :: E P : E C :: D Q : D L (657.) therefore C H + B G : C H — B G :: D Q (Tangent of A D) : D L (Tangent of D C). *Q. E. D.*

1829. THE-

1829. THEOREM VII.

The Sum of the Co-Sines of two unequal Arches is to their Difference, as the Co-Tangent of half the Sum of the Arches to the Tangent of half their Difference. For OG + OH = 2OF, and OG — OH = 2HF (221.) Therefore 2OF : 2HF :: OF : HF (:: EN : SE) :: EI : EC :: DK : DL; therefore the Sum of the Co-Sines OG + OH : OG — OH (their Difference) :: DK (the Co-Tangent of AD) : DL, the Tangent of DC. *Q. E. D.*

1830. THEOREM VIII.

In any oblique Triangle BDC, *(having drawn the Perpendicular* AC, *and biſected the Baſe in* E,*) it will be, as the Co-Tangent of half the Sum of the two Sides is to the Tangent of half their Difference, ſo is the Co-Tangent of half the Baſe, to the Tangent of the Diſtance* (AE) *of the Perpendicular from the middle Point* (E) *of the Baſe.* For we have *cs*AB : *cs*AD :: *cs*BC : *cs*DC (1824). Whence *cs*BC + *cs*DC : *cs*BC — *cs*DC :: *cs*AB + *cs*AD : *cs*AB — *cs*AD, (648, 649.) But *cs*BC + *cs*DC : *cs*BC — *cs*DC :: $ct\frac{BC+DC}{2} : t\frac{BC-DC}{2}$ (1829.) Alſo *cs*AB + *cs*AD : *cs*AB — *cs*AD :: $ct\frac{AB+AD}{2} : t\frac{AB-AD}{2}$ :: *ct*BE : *t*AE; and hence by Equality (652) $ct\frac{BC+DC}{2} : t\frac{BC-DC}{2}$ :: *ct*BE : *t*AE. *Q. E. D.*

1831. THEOREM IX.

In any oblique ſpherical Triangle, the Tangent of half the Baſe is to the Tangent of half the Sum of the Sides, as the Tangent of half the Difference of the Sides is to the Tangent of the Diſtance of the Perpendicular from the middle Point of the Baſe. For ſince KOQ (Fig. 4.) is a Right-angle, and OD perpendicular to the Baſe KQ; it is DQ : DO :: DO : DK (660.) Therefore $DK = \frac{DO^2}{DQ}$, or becauſe DO^2 is conſtant, it is $DK : \frac{1}{DQ}$, or the Co-Tangent of Arches are as the Tangents reciprocally. There-

Therefore $ct\frac{BC+DC}{2} : ctBE :: tBE : t\frac{BC+DC}{2} :: t\frac{BC-DC}{2} : tAE$ (1829.) *Q. E. D.*

1832. THEOREM X.

As the Co-Tangent of half the Sum of the Angles at Base, is to the Tangent of half their Difference, so is the Tangent of half the vertical Angle to the Tangent of the Angle which the Perpendicular makes the Line bisecting the vertical Angle. Let the Circle CF bisect the vertical Angle BCD (Fig. 3.) Then it is, $R : sACB :: csAC : csB$ (1798) and $R : sACD :: csAC : csD$. Therefore by Equality (652) and Permutation, we have $csB : csD :: sACB : sACD$; and therefore it is $csB + csD : csB - csD :: sACB + sACD : sACB - sACD$ $(:: ct\frac{B+D}{2} : t\frac{B-D}{2}$ (1829) $:: tBCF : tACF$ (1828.) *Q. E. D.*

CHAP. IV.

The SOLUTION *of the* SIX CASES *of* Oblique Spherical TRIANGLES.

1833. CASE I.

GIVEN *the two Angles* B *and* D; *and the Side* BC *opposite to one of them; to find the other Angle and Sides* (Fig. 3.) The Analogy for the Side CD is, as $sD : sBC :: sB : sCD$, (1823.)

1834. *To find the Angle* BCD; we have $csBC : R :: ctB : tACB$ (by 1791.) Then, it is, $csB : csD :: sACB : sACD$ (1826); whence the whole Angle BCD is known.

1835. *To find the Side or Base* BD; we have $R : csB :: tBC : tAB$ (1784, 1831.) Then, as $tD : tB :: sAB : sAD$ (1827.) Whence $AB + AD = BD$ required.

1836. CASE

1836. CASE II.

Given two Angles B and B C D, *and a Side* B C *between them; to find the other Angle* D. First, R : *cs* B C :: *t* B : *ct* B C A (1791, 1831); whence A C D is also known. Then *s* A C B : *s* A C D :: *cs* B : *cs* D (1826).

1837. *To find the Side* D C. Say as, R : *cs* B C :: *t* B : *ct* A C B (1791, 1831) whence A C D is known. Then *cs* D C A : *cs* B C A :: *t* B C : *t* D C (1824).

1838. *To find the Side* B D. The Proceſs is here the ſame, as for the Side C D, if from the End B of the given Side B C, you let fall the Perpendicular B A oppoſite to the given Angle B C D.

1839. CASE III.

Given two Sides B C *and* C D, *and an Angle* B *oppoſite to one of them; to find the other Side* B D. Say, as R : *cs* B :: *t* B C : *t* A B (1789, 1831). Then *cs* B C : *cs* D C :: *cs* A B : *cs* A D (1825). Whence A B ± A D = B D ſought.

1840. *To find the included Angle* B C D. Firſt ſay, as R : *cs* B C :: *t* B : *ct* A C B (1791, 1831). Then *t* D C : *t* B C :: *cs* A C B : *cs* A C D (1824). Whence A C B ± A C D = B C D.

1841. *To find the oppoſite Angle* D. Say, as *s* D C : *s* B :: *s* B C : *s* D (1823).

N. B. It muſt here be obſerved, that when the Side D C is leſs than B C, this Caſe will be ambiguous in all its Parts; ſince it will be uncertain from the *Data*, whether the Angle D is Obtuſe or Acute; ſince the Sine of an Angle, and of its Complement to 180 Degrees, is the ſame.

1842. CASE IV.

Given two Sides B D *and* B C, *and the included Angle* B; *to find the other Side* C D. Say, as R : *cs* B :: *t* B C : *t* A B (1789, 1831.) whence A D is alſo known; then *cs* A B : *cs* A D :: *cs* B C : *cs* C D (1825).

1843. *To find the Angle* D. Say, as R : *cs* B :: *t* B C : *t* A B (1791, 1831.) whence A D is known; then *s* A D : *s* B D :: *t* B : *t* D (1827).

1844. *To find the Angle* C. Let fall the Perpendicular D*a* on the given Side BC, and oppoſite to the given Angle B; then R : *cs* B :: *t* BD : *t* B*a*, and then *a*C is known; conſequently *sa* C : *sa* B :: *t* B : *t* C as above (1843).

1845. CASE V.

Given all the three Sides BD, BC, *and* CD; *to find an Angle, ſuppoſe* B. Say $t\frac{1}{2}$BD : $\overline{t\frac{1}{2}BC + \frac{1}{2}CD}$ :: $\overline{t\frac{1}{2}BC - \frac{1}{2}CD}$: *t* AE, the Diſtance of the Perpendicular from the middle Point E of the Baſe BD (1831.) whence AB is known. Then *t* BC : *t* AB :: R : *cs* B (1797).

1846. CASE VI.

Given all the three Angles, B, C, *and* BCD; *to find a Side, ſuppoſe* BC. Then ſay, as *c t* $\frac{1}{2}\overline{B + D}$: *t* $\frac{1}{2}\overline{B - D}$ :: *t* BCF : *t* ACF = Angle included between the Perpendicular AC, and the Line CF, which biſects the vertical Angle C; from whence the Angle BCA is known. Then ſay, as *t* B : *c t* BCA :: R : *cs* BC (1800, 1831).

N. B. *There are other Methods of ſolving the Caſes of oblique ſpherical Triangles, but they are more complex and difficult both in Theory and Operation, and therefore are not to be expected in an* Elementary Inſtitution. *Nor ſhall we here inſiſt on thoſe Solutions of Triangles which depend on the given* Sums *and* Differences *of* Sides *and* Angles, *or of their* Sines, Tangents, Verſed-ſines, *&c. as they are very intricate, and rarely neceſſary in Practice.*

CHAP.

CHAP. V.

The Method of determining the FLUXIONS *of the* SIDES *and* ANGLES *of Oblique Spherical* TRIANGLES.

1847. CASE I.

LET ABC be an *oblique ſpherical Triangle*, (Fig. 5.) and let the Side AC move on the Point or Angle A, and by its Motion deſcribe a ſmall Portion of a Parallel C*c*; at the ſame Time let C*d* be Part of a Parallel deſcribed by BC on the Pole B; and draw the great Circle B*c* interſecting C*d* in *d*. Then let D, E, and F, denote the three Sides AB, AC, and BC reſpectively, of which AB and AC are *conſtant*, and all the other Parts variable.

1848. Then we ſhall have $R : sE :: sCAc : sCc$ (1781) for the fluxionary Triangle C*c*A is right-angled at (*c*); and becauſe the Sine of a very ſmall Arch is nearly equal to the Arch itſelf, therefore $R : sE :: CAc\ (= \dot{A}) : Cc = \frac{sE}{R} \times \dot{A}$.

1849. Alſo, $R : sF :: CBd\ (= \dot{B}) : Cd = \frac{sF}{R} \times \dot{B}$. Again becauſe $ACc = dCB =$ Right-angle (1847); therefore $ACB = dCc$, and becauſe of the Right-angle at *d*, and the fluxionary Triangle *d*C*c* Rectilineal, we have $R : sdCc\ (= ACB) :: Cc : cd :: \frac{sE}{R} \times \dot{A} : \dot{F} = \frac{sE \times sC}{R^2} \times \dot{A}$ (1848).

1850. But $sC : sD :: sB : sE$ (1823); therefore $sE \times sC = sD \times sB$; conſequently it is alſo $\dot{F} = \frac{sD \times sB}{R^2} \times \dot{A}$.

1851. Again; $R : cs.dCc\ (= C) :: Cc : Cd :: \frac{sE}{R} \times \dot{A} : \frac{sE \times csC}{R^2} \times \dot{A} = \frac{sF}{R} \times \dot{B}$ (1849). Whence we have $\dot{B} = \frac{sE \times csC}{R \times sF} \times \dot{A}$.

1852. And laſtly, $ct.cCd(=C) : R :: Cd : cd :: \frac{sF}{R} \times \dot{B} : \dot{F} = \frac{sF}{ctC} \times \dot{B}$.

1853. In the ſame Manner it is ſhewn (by making the like Conſtruction at B) that $C = \frac{sD \times csB}{R \times sF} \times \dot{A}$; and $\dot{F} = \frac{sF}{ctB} \times \dot{C}$

1854. Let f = Secant, and cf = Co-Secant of an Arch EB (Fig. to Inſt. 705) then it is, CH (= ED) : CE :: CG (= CE) : CF; that is, $sEB : R :: R : fGE$, or $cfBE$; therefore $R^2 = sBE \times cfBE$.

1855. Hence becauſe $R^2 = \frac{sD \times sB}{\dot{F}} \times \dot{A}$ (1850) $= sD \times cfD$ (1854.) Therefore we have, from the above Equations, the following Analogies for the Fluxions of the Sides and Angles of this Triangle, *viz.*

1. $\dot{A} : \dot{F} :: cfD : sB$.
2. $\dot{A} : \dot{B} :: R \times sF : sE \times csC$ (1851).
3. $\dot{A} : \dot{C} :: R \times sF : sD \times csB$ (1853).
4. $\dot{B} : \dot{F} :: ctC : sF$ (1852).
5. $\dot{C} : \dot{F} :: ctB : sF$ (1853).
6. $\dot{B} : \dot{C} :: ctC : ctB$ (1851, 1853).
   $:: tB : tC$ (1831).

Case II.

1856. *If one Side* D, *and an adjacent Angle* B *be invariable*, then this Caſe will be reduced to that of a right-angled Triangle, by letting fall the Perpendicular AG (from the End of the given Side, and oppoſite to the given Angle B) on the Side BC continued out. For then there will be found the right-angled Triangle ACG, in which the Side AG (and the Segment GB) is conſtant; and ſo this Caſe becomes the ſame with Caſe II. (1811, 1819).

1857. Case III.

If one Side D, *and an oppoſite Angle* C, *be invariable* (Fig. 6.); then are the other Parts variable by the Motion of the given Side D or AB. Let the Side AB move into the Situation (*ab*) and be-

becauſe it is ever $ab = AB$, it is impoſſible that (ab) ſhould be parallel to AB (631); therefore ab will meet AB, continued out, in ſome Point Q. On the Point Q, as a Center, deſcribe the ſmall Arches bn, am; then becauſe $AB = ab = nm$, if from the firſt and laſt, you take away the common Part An, there will remain $Bn = Am$.

1858. Then ſince the fluxionary Triangles $Amа$ and Bnb may be conſidered in their *Naſcent* State as *Rectilineal*, and right-angled at m and n; therefore we ſhall have $Am : Aa :: csA : R$; and $Bn : Bb :: csB : R$; whence $Am \times R = Aa \times csA = Bn \times R = Bb \times csB$; therefore $Aa : Bb :: csB : csA :: \dot{E} : \dot{F}$.

1859. Again, we have $sC : sD :: sB : sE$ (1823) $:: \dot{;}B : \dot{;}E$. For becauſe the flowing Quantities or Sines of B and E are in the conſtant Ratio of sC to sD (1856). The Fluxions of the Sines, *viz.* $\dot{;}B$ and $\dot{;}E$ will be in the ſame conſtant Ratio (788). But the Fluxion of the Sine is to the Fluxion of its Arch as Co-Sine to Radius (874). Therefore $\dot{;}B = \frac{csB}{R} \times \dot{B}$, and $\dot{;}E = \frac{csE}{R} \times \dot{E}$; Whence $sC : sD :: csB \times \dot{B} : csE \times \dot{E}$, which gives $\dot{B} : \dot{E} :: sC \times csE : sD \times csB$.

1860. In the ſame Manner it is proved that $\dot{A} : \dot{F} :: sC \times csF : sD \times csA$.

1861. By comparing the Analogies in (1858, and 1859,) we get $\dot{B} : \dot{F} :: sC \times csE : sD \times csA$.

1862. And by comparing this with the Analogy in (1860) we have $\dot{A} : \dot{B} :: csF : csE$.

1863. The Analogies in (1859, 1860) may be more ſimply expreſs'd by Tangents; thus, it is $\dot{B} : \dot{E} :: \frac{sC}{csB} : \frac{sD}{csE}$; but $sC : sD :: sB : sE$; therefore $\dot{B} : \dot{E} :: \frac{sB}{csB} : \frac{sE}{csE} :: tB : tE$, becauſe the Sine has the ſame Ratio to the Co-Sine as the Tangent has to Radius $= 1$. (ſee *Fig.* to 705.) Thus alſo, it is $\dot{A} : \dot{F} :: tA : tF$.

1864 CASE IV.

When two Angles A, *and* D, *in the Triangle* A E D (Fig. 7.) *are invariable.* But this is reduced to the first CASE (1847) by considering, that the Angles of the Triangle A E D, are equal to the Sides of another Triangle F G H, formed by Circles connecting the Poles G, F, H, of the Sides of the Triangle A E D. For let the Sides D E, D A be produced to Quadrants in O and C, and on the Point D describe a Circle B K O G. Let G be the Pole of the Circle C D B, F the Pole of the Circle O D I, and H the Pole of the Circle A E Q. Then is G C = F O = 90°, from which take the common Part F C, and there remains F G = C O = Angle D. In the like Manner F N = H M = 90°; Subduct H N, and there remains F H = N M = Angle A. Lastly, G L = H P = 90°, take away H L, and we have G H = L P = Angle P E L, the Complement of the Angle A E D.

1865. Hence also it appears, that the *Sixth Case* of oblique Triangles (1846) is reducible to the *Fifth*, by changing the given Angles A, E, D into Sides of another Triangle F G H.

1866. I have now premised all that I judge does properly belong to the *Elementary Part* of the *Doctrine of spherical Triangles.* As to the *Five circular Parts* of Lord *Neper*, I think it an Artifice of more Ingenuity in the Invention, than of Use in Practice, and have here omitted it. Also those Methods of solving oblique spherical Triangles by *given Sums, Differences, Products, &c.* of their several Parts, are not to be considered as *first Principles*, but rather the Inventions of Art resulting from those Principles, and may be explicated by them when ever they occur in Practice. However we cannot think these Elements compleat without such as determine the *Area* of a spherical Triangle, which therefore we shall add in the next Chapter.

CHAP.

CHAP. VI.

The Method of determining the AREA *of a* SPHERICAL TRIANGLE.

1867. IN *Fig.* 8. Let A C F be a Quadrant of the Equator, the Pole D; S B T a Parallel to the Equator; A B E an oblique Circle croſſing the Equator in A; A K V an oblique Circle below the Equator; D B K a Meridian, and D L O another drawn indefinitely near to it. By this Means there will be form'd an oblique ſpherical Triangle A B K which is divided into two right angled Ones A B C, and A C K, by the Equator A F.

1868. Let p = Periphery of a great Circle, and $x = s\,BC$, or *Sine* of B C, the Latitude of the Zone A H E F, whoſe Surface is $x \times AF = x \times \frac{1}{4}p$; becauſe xp = Surface of the whole Zone continued round the Globe, (838.) therefore alſo the Surface of the indefinitely ſmall Part L O C B is $= x \times OC = s\,BC \times OC$. But this is alſo the Fluxion of the Triangle A B C; for that of the Zone, and of the Triangle at the Point B will be the ſame (792) as is evident from the Reaſoning there uſed.

1869. The Fluxion of the Angle A B C is $= \frac{s\,BC}{R} \times OC$

(1810.) But $s\,BC \times OC : \frac{s\,BC}{R} \times OC :: 1 : \frac{1}{R} :: R : 1$.

That is, the Fluxion of the Triangle A B C is to the Fluxion of the Angle B, *is in the conſtant Ratio of Radius to Unity.* And therefore the contemporaneous Fluents will be in the ſame Ratio.

1870. In the *Naſcent State* of the Triangle, when the Sides may be conſidered as *Rectilineal*, the Angle at B is equal to the Complement of the Angle A to a Right-angle (633). But as the Triangle flows, or encreaſes, this Angle B alſo flows and encreaſes to a larger Quantity, till at laſt the Triangle A B C becomes the quadrantal Triangle A E F, and the Angle B becomes a Right-angle at E, the whole Increaſe therefore of the

Angle

Angle B is equal to the Angle A. And it is this increaſing Part of the Angle B, which is the Fluent of the Fluxion $\frac{s\,BC}{R} \times OC$. Therefore *the Area of the Triangle* A B C *is to the Increaſe of the Angle* B, *or the Exceſs of* A + B *above a Right-angle, in the conſtant Ratio of* R *to* 1.

1871. The ſame Thing is ſhewn with regard to the Triangle A C K; therefore it is evident, *that the Area of any oblique ſpherical Triangle* A B K *is conſtantly proportional to the Encreaſe of the three Angles above two Right-angles.*

1872. Hence in regard to the Triangles A B C and A E F, it will be as A : A E F :: A + B — 90 : A B C; and in the ſame Manner in the Triangle C A K and F A V, it is A : F A V :: A + C — 90 : A C K. Therefore with regard to the whole oblique Triangle A B K, we have the Angle B A K to the whole quadrantal Area E A V, as A + B + K — 180° to the Area of the Triangle A B K.

1873. Let S = Surface of the Globe, T = Area of the Triangle A B K, Q = Area of the quadrantal Area E A V, N = Angle B A K = E V, and M = Sum of the three Angles of the Triangle = N + B + K, then it is ſelf-evident, *that the quadrantal Space* E A V *is ſuch a Part of a Hemiſphere as the Arch* E V *is of a great Circle*; that is, Q : $\frac{r}{2}$ S :: N : p; but N : Q :: M — 2p : T (1870); therefore p : $\frac{1}{2}$ S :: M — $\frac{1}{2}p$: T, or 2p : M — 2p :: S : T; that is, *as* 720 : N + B + K — 180 :: *the Surface of the Globe* S : *Surface of the Triangle* A B K.

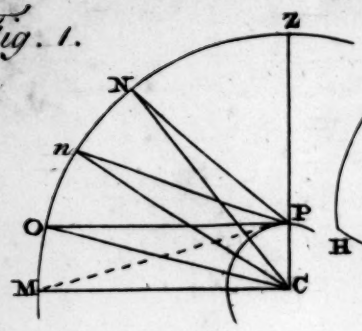
1874. But the Surface of a Globe or Sphere is equal to four Times the Area of its great Circle, or S = 4 A (839); therefore $\frac{1}{2}p$: $\frac{1}{4}$ S :: 180 : A :: M — 180 : T. But A = $\frac{1}{2}pr$, or half the Radius (r) multiplied into the Periphery (830); therefore (180 : 180r ::) 1 : r :: M — 180 : T.

1875. The Radius (r) expreſſed in Degrees is = 57°,2957795 (884); therefore if the three Angles of a Triangle leſſened by 180°, be multiplied by 57,2957795, the *Product will be the Area of that Triangle in Square Degrees.* And becauſe in one *Square Degree,* there are 3600 *Square geographical Miles,* or 4830,25 *Engliſh Miles,* therefore the Area of the Triangle may be expreſſed in Miles of either Sort.

CHAP.

Plate II. *Of* Spherical Trigonometry.

Fig. 1.



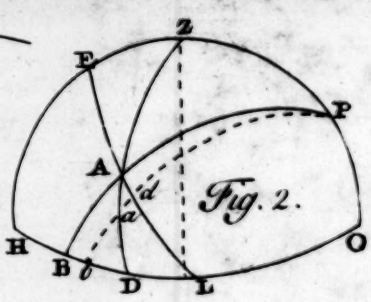
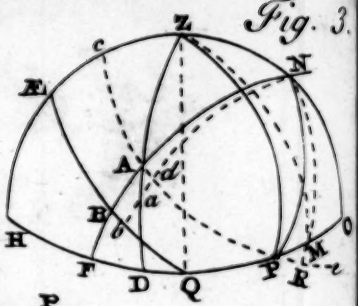


Fig. 2.

Fig. 3.



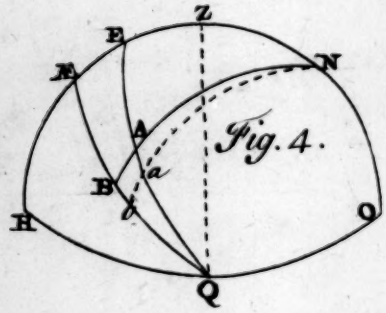
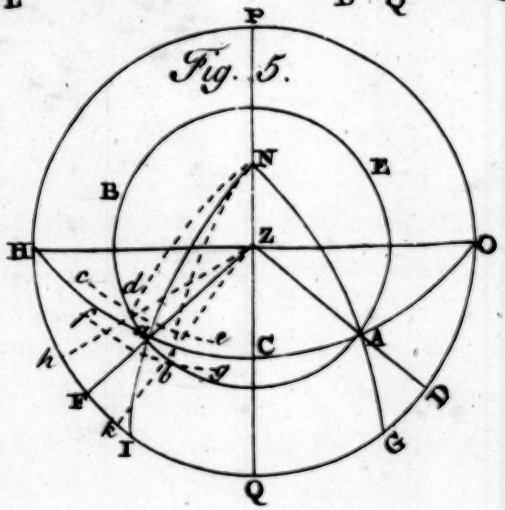


Fig. 4.

Fig. 5.



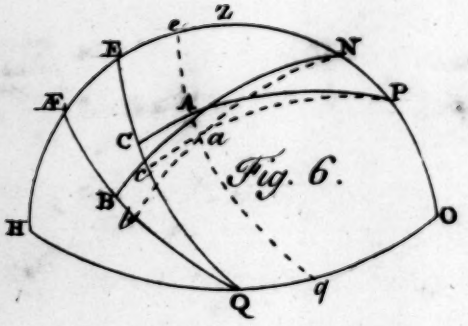
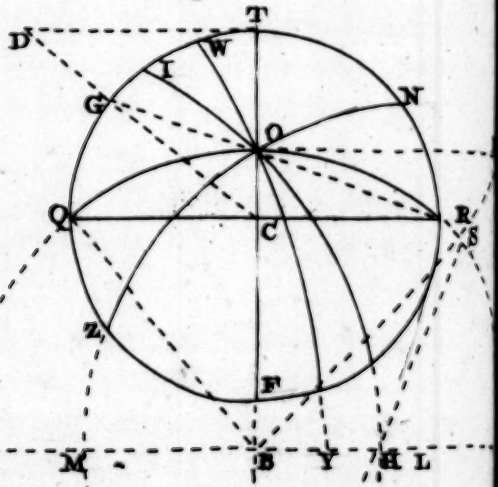


Fig. 6.



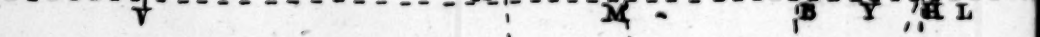
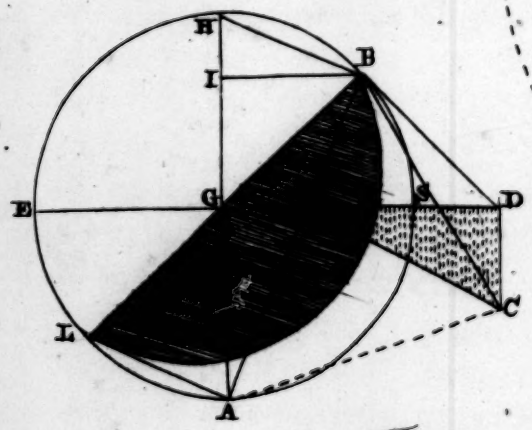


Fig. 7.

Fig. 8.



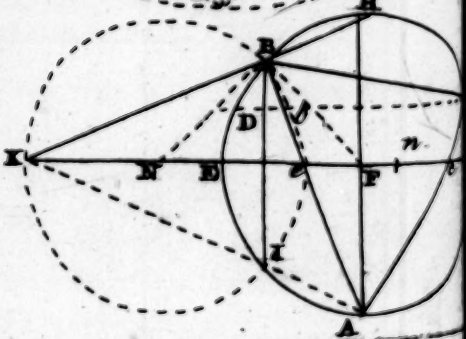


Fig. 9.

CHAP. VII.

The foregoing PRINCIPLES *applied to the* SOLUTION *of* PROBLEMS *in* ASTRONOMY, GEOGRAPHY, NAVIGATION, DIALLING, *&c.*

1876. THE Conſtruction of a Diagram for preſenting at one View the greateſt Numbers of *ſpherical Triangles*, in order to the Solution of *aſtronomical Problems*, is that of Fig. 9. which is a *ſtereographical Projection of the Sphere* on the Plane of that Meridian ÆNQS, which is called the *ſolſtitial Colure* (1707).*

1877. In this Scheme, ÆQ is the EQUINOCTIAL, and its two Poles N, S. EL is the ECLIPTIC, and its Poles P, T. HO is the HORIZON, its Poles Z, the *Zenith*, and D the *Nadir*. ZD, the *prime* VERTICAL, and its Poles H, O. E♋ is the Tropic of *Cancer*; ♑L, the Tropic of *Capricorn*. ⊙ is the *Sun's Place* in the Ecliptic; N⊙S a *Hour-circle*, and Z⊙D a *vertical Circle* paſſing through the Sun ⊙. PRT a *Circle of Longitude*, and NRS a *Circle of Declination* paſſing through a *Star* at R.

1878. Now theſe various Circles of the Sphere by their Interſections, form many *ſpherical Triangles*, both *Rectangular* and *Oblique*. In a right-angled Triangle, if *any two Parts are given*, it is a *Problem* to find the reſt. And in oblique Triangles, the Problem requires three different Parts to be given, as we have ſhewn (717—721). But *one Angle* in moſt Triangles, Right or Oblique, is of no Conſequence in aſtronomical Problems, and therefore does not enter the *Data*.

1879. In a right-angled Triangle, there are, therefore, four ſignificant Parts, *viz. three Sides*, which call *a*, *b*, *c*; and *one Angle*, which let us denote by (*n*). Then ſince there are ſix Combinations of two Quantities in four, *viz. an*, *bn*, *cn*, *ab*, *bc*, *ac*, there will be at leaſt *ſix Problems* reſulting from every right-angle Triangle.

* It is here ſuppoſed, the Reader is acquainted with the Names and Uſes of the CIRCLES of the SPHERE, as they have been at large explained in the GENTLEMAN and LADIES PHILOSOPHY.

1880. Again, in an oblique Triangle there will be five ſignificant Parts, *viz.* *three Sides*, *a*, *b*, *c*; and *two Angles*, *m*, *n*; and there will be *ten Combinations of three Quantities in five*, viz. *abm*, *bcm*, *acm*, *abn*, *bcn*, *acn*, *amn*, *bmn*, *cmn*, *abc*; therefore *ten aſtronomical Problems* will ariſe from every oblique Triangle.

1881. We ſhall now ſpecify thoſe right-angled Triangles that are of moſt Uſe in *Aſtronomy*, which are as follow, *firſt*, the Triangle B C ☉, right-angled at B; in which B C is the Sun's *right Aſcenſion*, or its Complement to the Point C. C ☉ is the Sun's *Longitude*, or Complement to C. B ☉ is the Sun's *Declination*; and the Angle B C ☉ the *Obliquity of the Ecliptic*.

1882. *Secondly*; the Triangle A ☉ C, right-angled at A; in which the Side C ☉ is the Sun's *Longitude* in the Ecliptic from C; the Side A ☉, its *Altitude* above the Horizon. A C is the *Azimuth* from the Point C. And the Angle A C ☉ is the Sum of the *Co-Latitude* and *Obliquity of the Ecliptic*, *viz.* Æ H + Æ E.

1883. *Thirdly*; Let the Sun be in the Tropic of *Cancer*, and in the Point G in the prime Vertical, when due *Eaſt* or *Weſt*. Then in the Triangle V C G, right-angled at V, the Side V C is the *Hour from Six*. The Side V G = Æ E, the Sun's *Declination*. The Side C G the Sun's *Altitude when Eaſt or Weſt*. And the Angle V C G = Æ Z, the *Latitude of the Place*.

1884. *Fourthly*; On the ſame Day the Sun is at I, at the Hour of Six preciſely; then in the Triangle I C M, right-angled at M; the Hypothenuſe C I is the Sun's *Declination*; I M, the Sun's *Altitude at Six*; and C M the *Azimuth* from C. And the Angle I C M = N O = Æ Z the *Latitude* of the Place.

1885. *Fifthly*; On the ſame Day the Sun is in the Horizon at K, and there is formed the Triangle W C K right-angled at W; in which W K is the Sun's *Declination*. C W is the *aſcenſional Difference*, or the *Hour* from *Six* of its *Riſing and Setting*; C K, the *Amplitude* of Riſing and Setting from the *Eaſt or Weſt*. And the Angle W C K = O Q, is the *Co-Latitude* of the Place.

1886. *Sixthly*; In the Triangle N K O, right-angled at O, the Side N O is the *Latitude*; the Side N K the *Co-Declination*; and

and the Side K O the *Co-Amplitude* or *Azimuth* from O. And the Angle K N O = W Q the *Hour* from Midnight.

1887. What we have ſaid with Reſpect to the Sun in *Cancer*, is the ſame for its Place in any other Part of the Ecliptic; and the ſame Triangles are to be drawn for the ſame Purpoſes in the Winter Signs, as we have here done for the Summer. And further, it is to be obſerved, that ſince any Star may be ſuppoſed at ☉, it is evident all the ſame Triangles ſerve for a Star as for the Sun. Therefore in the *ſix Triangles* already enumerated, no leſs than *ſeventy-two Problems* may be ſtated in Reference to the Sun and Stars; beſides many other Triangles that might be added to theſe, to encreaſe the Number of aſtronomical Problems.

1888. In the oblique Triangle ☉ Z N, the Side ☉ Z is the *Co-Altitude* of the Sun; ☉ N, the *Co-Declination*; Z N, the *Co-Latitude* of the Place. The Angle Z N ☉ is the *Hour* from Noon, and the Angle ☉ Z N or ☉ Z Æ is the *Azimuth* from the North or South Part of the Horizon.

1889. This Triangle, therefore, affords *ten Problems* more in reſpect of the SUN (1878) and if we conſider the oblique Triangle Z R N, we ſhall find the Sides and Angles the ſame in regard to a STAR at R, and conſequently other *ten Problems* will thence be produced.

1890. Again; in the oblique Triangle R N P, the Side R N is the Star's *Co-Declination*; the Side R P, its *Co-Latitude*; and the Side N P = Æ E the *Obliquity of the Ecliptic*. Alſo the Angle N P R is the Star's *Longitude* from the Point E; and the Angle R N P its *Azimuth*. From hence we have ten Problems more; and therefore this and the laſt Triangle afford *twenty principal Problems about the Stars only*.

1891. In the laſt Place, let ☉ be the Moon or a Planet, then if we draw a great Circle through ☉ and R, the Place of a Star; we ſhall have an oblique Triangle ☉ Z R, in which the Side ☉ Z is the Moon's *Co-Altitude*; Z R, that of the Star, and ☉ R the *Diſtance* of the Moon from the Star. Alſo the Angle ☉ Z R the *Difference of the Azimuths* of the Moon and Star. This Triangle affords *four* very uſeful Problems relative to the Moon and Stars. Upon the Whole, in theſe few Triangles in this one Projection only, more than *one hundred aſtronomical Problems* are contained.

1892. In regard to GEOGRAPHY, if we ſuppoſe Z and R to repreſent two Places on the Surface of the *Terreſtrial Globe*; then in the Triangle Z R N, the Side Z N is the *Co-Latitude* of the Place Z; the Side R N is the *Co-Latitude* of the Place R; and the Side Z R is the *Diſtance of the two Places* on the Surface of the Globe. The Angle Z N R = Æ V, is the *Difference of Longitude* of the two Places, and the Angle R Z N is the *Bearing* of the Place R from Z. This one Triangle therefore furniſhes *ten principal geographical Problems*, which depend on the Doctrine of the Sphere.

1893. In NAVIGATION, particularly that Part of it called ORTHODROMICS, which treats of the *Art of Sailing* on a *great Circle* of the Sphere, a ſpherical Triangle is concerned. Thus let Z be the Port from whence a Ship is to ſail to another Port X; then in the Triangle Z N X; the Side Z N is the *Co-Latitude* of the Port Z; the Side N X is the *Co-Latitude* of the Port X; the Side Z X is the *neareſt Diſtance* between them or the *Ship's Way*. Alſo the Angle N Z X is the *Courſe* to be ſteered at Z; and the Angle Z N X is the *Difference of Longitude* between the Ports.

1894. But in ſailing from Z to X, the Courſe is perpetually altering, or the Angle made by the Ship's Way on the great Circle Z X with every Meridian is different, or the Latitudes always changing, therefore it is neceſſary to calculate this for every 5 or 6 Degrees of Longitude, that the Ship may be kept upon, or near the circular Arch Z X, every where; and this is the whole Art of great *Circle-ſailing*, which is entirely converſant in the *Solution* of *ſpherical Triangles*. A great Variety, therefore, of *nautical Problems* here offer themſelves from this one View of a ſpherical Triangle.

1895. As to LOXODROMICS, or ſailing upon a *Spiral*, or *Rhumb-line*, as the Theory of that Curve has not yet been conſidered, nor the *Rationale* of that Sort of Sailing been fully explained by Writers on Navigation, we have determined to treat of that Subject more particularly in another Chapter.

1896. In DIALLING, the Triangle N O K, right-angled at O, is uſed in finding all the Angles N K O, which the ſeveral Hour-circles N K S make with the Horizon. For N O being the Latitude of the Place, and the Angle O N K being given,

the

the Arch of the Horizon O K is found by Caſe I. of right-angled Triangles (1783). But this is only touched upon here, to ſhew how very uſeful and extenſive the Doctrine of *ſpherical Trigonometry* is in moſt Arts and Sciences. What relates to the Conſtruction of the *Scale of Latitudes* and *Hours* in practical Dialling, as alſo of a new *Dialling Sector*, and a *new univerſal* HORIRONTAL DIAL, ſhewing the Hour by the ſame Gnomon in all Latitudes, will be treated of more fully hereafter.

CHAP. VIII.

The Application of FLUXIONARY SPHERICS *in* aſtronomical COMPUTATIONS, *relative to* PARALLAXES, REFRACTIONS, equal ALTITUDES, PRECESSION *of the* EQUINOXES, *&c.*

1897. WE have ſhewn how the *Places* of the heavenly Bodies, the *Times* of their *Riſing*, *Setting*, *&c.* and their various *Phænomena* in regard to their *Right-aſcenſion*, *Declination*, *Amplitude*, *Azimuth*, *Longitude*, *Latitude*, &c. are to be calculated by the common Proceſſes of *ſpherical Trigonometry*. But the minute Variations and Alterations which happen in thoſe Quantities by Means of a *Parallax*, *Refraction*, *Receſſion* of the *Equinoxes*, *ſpiral Motion of the Sun*, &c. are of too much Moment in Aſtronomy not to be moſt ſcrupulouſly attended to; and as they are beſt of all computed by *Fluxionary Spherics*, the Principles of which have been explained, we therefore now illuſtrate that Method by ſome Examples.

1898. As the moſt conſiderable of theſe Variations are of that Sort which ariſe from a *Parallax*, and greatly affect the moſt intereſting Subjects of this Science, *viz.* the MOON and the PLANETS, it will be neceſſary to premiſe the following Theorem, *viz.* The *Parallax of a Planet is always proportioned to the Sine of its apparent Diſtance from the Zenith*, which is thus ſhewn. Let C

be

be the Center of the Earth, P the Place of a Spectator on its Surface; Z the Zenith, and let N and *n* be two Places of a Planet in an Azimuth-circle Z O; draw P N and C N, alſo P *n* and C *n*, then are the Angles P N C, P *n* C, the *Parallaxes* of the Planet at the Altitudes N O, *n* O, (1724). To compare which, we have P C : C N :: *s* P N C : *s* N P C = *s* N P Z. Alſo P C : C *n* (= C N) :: *s* P *n* C : *s* C P *n* = *s* *n* P Z; therefore it is *s* P N C : *s* P *n* C : : *s* Z P N : *s* Z P *n*. But theſe parallatic Angles at N and *n*, being very ſmall, *are as their Sines*; whence the Propoſition is evident.

1899. Hence it follows, *that any Parallax at* N *is to the horizontal Parallax at* O (*viz.* that of the Planet ſeen from P in the Horizon at O) *as the Sign of its Zenith Diſtance (nearly)* to the Radius. Alſo it appears, that the parallatic Angle vaniſhes at the Zenith Z; and alſo when the Diſtance P N or C N becomes immeaſurably great. So that the *fixed Stars can have no Parallax* of this Kind. Laſtly, we have, *the Diſtance of a Planet* N C *from the Sun, to the Semidiameter of the Earth* C P, *as the Sine of the Parallax* C N P, *to the Sine of the Zenith Diſtance* Z P N.

1900. As the Angle Z P N is greater than Z C N by the Quantity of the Parallax P N C, it is evident the *apparent Zenith Diſtance of a Planet exceeds the true*, by juſt that Quantity. Therefore let A be the true Place of a Planet; Z A its Zenith Diſtance, P the Pole of the Ecliptic; and P A a Circle of Latitude paſſing thro' the Planet. Alſo let (*a*) be the apparent Place, and draw the Circle of Latitude P *a b* thro' the Planet A. Let E A L be drawn parallel to the *Ecliptic*; and let H L O be the Horizon. (Fig. 2.)

1901. Then in the oblique Triangle A Z P, the Side Z P, and the Angle Z adjacent, are conſtant (1855). And the fluxionary Parts are (1.) A *a* = $\dot{a}$ (1817, 1818) the Parallax in Altitude. (2.) The Parallax in Longitude = A P *d* = $\dot{B}$. And (3.) The Parallax in Latitude *a d* = $\dot{H}$. Then becauſe the Parallax in Altitude is known from the *horizontal Parallax* (1898) in aſtronomical Tables; therefore we have *s* H : *s* b : : $\dot{a}$: $\dot{B}$ = A P *d* (1819). That is, in Words, *as the Co-Sine of the Planet's Latitude, is to the Sine of the Angle* Z A P, *ſo is the Parallax in Altitude to the Parallax in Longitude.*

1902.

1902. Again, to *find the Parallax in Latitude*; it appears from the fluxionary Triangle A *a d*, right-angled at *d*, that A *a* : *a d* :: R : *s a* A *d* = *s* E A Z, that is, R : *cs* Z A P :: A *a* : *a d* :: $\dot{a}$: $\dot{H}$; which is in Words, *Radius is to the Co-Sine of the Angle* Z A P, *as the Parallax in Altitude, is to the Parallax in Latitude.*

1903. If E L be the Ecliptic itſelf, and the Planet be in, or very near it, then *s* H = *s* A P = R; and the *s* b = *s* Z A P = *cs* Z A E; therefore *s* H : *s* b :: R : *cs* Z A E (1899) :: *t* Z A : *t* E A (1797) : $\dot{a}$: $\dot{B}$; that is, *as the Tangent of the Planet's Zenith Diſtance is to the Tangent of its Diſtance from the* Nonageſima *Degree* E, *ſo is the Parallax in Altitude, to the Parallax in Longitude.*

1904. Alſo, in this Caſe, we have R : *cs* b :: R : *s* Z A E :: *s* Z A : *s* Z E (1795) :: $\dot{a}$: $\dot{H}$; that is, *the Sine of the Zenith Diſtance, is to the Co-Sine of the Altitude of the* Nonageſima *Degree as the Parallax in Altitude to the Parallax in Latitude.*

1905. If we put M = horizontal Parallax; then, becauſe (1898) $\frac{s\,Z\,A}{R} \times M = \dot{a} = \frac{t\,Z\,A}{t\,A\,E} \times \dot{B}$, (1903) we have $\dot{B} = \frac{s\,Z\,A \times t\,A\,E}{R \times t\,Z\,A} \times M = \frac{s\,Z\,A \times s\,Z\,A\,P}{R^2} \times M =$ (1823) $\frac{s\,Z\,P \times s\,Z\,P\,A}{R^2} \times M$. Therefore $R^2 : s\,Z\,P \times s\,Z\,P\,A ::$ M : $\dot{B}$. That is, *the Square of Radius is to the Rectangle of the Sines of Altitude of the* Nonageſima *Degree, and the Planet's Longitude from thence, as the horizontal Parallax is to the Parallax of Longitude.*

1906. Again, $\frac{s\,Z\,A}{R} \times M = \dot{a} = \frac{s\,Z\,A}{s\,Z\,E} \times \dot{H}$ (1904) whence $\dot{H} = \frac{s\,Z\,A \times s\,Z\,E}{R \times s\,Z\,A} \times M = \frac{s\,Z\,E}{R} \times M$. Which gives R : *s* Z E :: M : $\dot{H}$; that is, *Radius is to the Co-Sine of the Altitude of the* Nonageſima *Degree, as the horizontal Parallax to the Parallax in Latitude.* Hence becauſe R and M are conſtant Quantities, the Parallax in Latitude will ever be as the Co-Sine of the Altitude of the Nonageſima Degree.

1907. As to the *Parallax* in *Right-aſcenſion* and *Declination*, the Analogies are the ſame as before; for H Z O being the Meridian, Æ Q, the Equinoctial, its Pole N, H O the Horizon, A

the

the Place of the Planet, Z D a vertical Circle, and N F a Hour-circle paſſing through it (Fig. 3.) it is evident the Triangle Z A N, and its fluxionary Parts A *a*, *a d*, A *d*, are all the ſame as in the foregoing Figure, and therefore the ſame Analogy as was uſed (1903) for finding the Parallax of *Longitude*, finds here the Parallax of Right-aſcenſion, *viz.* *s* H (= *s* A N) : *s* b (= *s* Z A N) :: *ȧ* (= A *a*) : *ḃ* (= B N *b*). Alſo for the *Parallax of Declination* (*a d*) it is (1904) as R : *cs* b :: *ȧ* : *ḣ* = *a d*.

1908. The Effects of *Refraction of Light thro' the Atmoſphere*, are next to be conſidered; the general Nature of refracted Light has been ſhewn at large (1321, *&c.*) and that Effect is to elevate Objects in Appearance, or to make them appear higher than their true Places, which is juſt contrary to the Effect of the Parallax which depreſſes them (1899).

1909. It appears alſo from the *Theory*, that the more oblique the Rays are, the more they will be refracted; or the Rays will be more refracted as they are nearer to the Horizon; therefore the horizontal Refractions are greateſt of all, and in the Zenith there is no Refraction at all, which is the Caſe alſo in regard to the *Parallax*.

1910. Therefore if (*a*) be the true Place of a Planet (Fig. 2.) then by Refraction it will be elevated to A in the vertical Z D, which Refraction *a* A in Altitude being found by Obſervation, you will from thence find the Diminution *d* P A in *Longitude*, and the Alteration (*a d*) of *Latitude*, correſponding to the ſame by the Analogies for the *Parallax* (1903, 1904). And alſo thoſe of *Right-aſcenſion* and *Declination* (Fig. 3.)

1911. The horizontal Refraction makes a Difference in the *Time of Riſing and Setting* of the heavenly Bodies, and alſo of their *Amplitude* from the Eaſt or Weſt Points of the Horizon. Thus let (*c c*) be a Parallel of the Sun or Star's Declination, then without the Refraction P would be the Point in which it would aſcend the Horizon, Q P the *true Amplitude*; and P N O the *Hour* from *Midnight*. But if the horizontal Refraction be equal to R M, then will the Sun or Star be thereby elevated to the Horizon in the Point M, and its *apparent Amplitude* will be Q M, and the *apparent Time of Riſing* will be M N O.

1912.

1912. In the Triangle Z N P, the two Sides Z N and N P are conſtant (1847). The Side Z N = D, the Side N P = E, the Side Z P = F; the Angle P Z N = B, the Angle P N Z, or P N O = A, and the Angle Z P N = C. And we have $sB : cfD :: \dot{F} : \dot{A}$ (1855) that is, *the Co-Sine of the true Amplitude* Q P *is to the Secant of the Latitude* Æ Z, *as the horizontal Refraction* R M = $\dot{F}$ *is to the Angle* P N M, *which is the Difference in Time of the Riſing or Setting of the Sun or Star thereby occaſioned.* Or otherwiſe thus; becauſe $sB : sE :: sA : sF = R$ (1828) therefore $sB = \frac{sE \times sA}{R}$. Again, $cfD = \frac{R^2}{sD}$ (1854) therefore $sB : \frac{R^2}{sD} :: \dot{F} : \dot{A}$, hence $sB = \frac{R^2 \times \dot{F}}{sD \times \dot{A}} = \frac{sE \times sA}{R}$ which Equation gives this Analogy $sD \times sE \times sA : R^3 :: \dot{F} : \dot{A}$, as before.

1913. Again, for the fluxionary Variation of the *Amplitude* P M, we have $sF : ct\,C :: \dot{F} : \dot{B}$, (1855) but in this Caſe, $sF = R$, and $ct\,C = tNPO$, therefore $R : tNPO :: sB (= sPO) : tNO$ (1780). Therefore $sB : tNO :: \dot{F} : \dot{B}$; that is, *the Co-Sine of the true Amplitude is to the Tangent of the Latitude, as the horizontal Refraction is to the Difference* P M *between the true and apparent Amplitude.*

1914. This apparent Amplitude is neceſſary to be known, eſpecially by Navigators, becauſe the *Variation of the Needle* depends upon it, ſince that important Article is nothing more than the Difference between the obſerved Amplitude on the real Horizon, and that of the magnetical Card.

1915. It appears from what we have formerly ſhewn, that the Orbit of a Planet about the Sun is elliptical; and ſince that is the Caſe of the Earth, its Motion will be variable, ſometimes quicker, and ſometimes ſlower, ſuch as is expreſſed in *Seconds* of a Degree in the following Table *per Hour*, for the ſeveral Months of the Year.

January	*December*	153″
February	*November*	152
March	*October*	150
April	*September*	148
May	*Auguſt*	145
June	*July*	143

1916. In Fig. 4. let Æ Q be a Quadrant of the Equinoctial, E C of the Ecliptic; B A N an Hour-circle passing through the Sun at A, and in one Hour after let its Place be at (*a*). In the right-angled Triangle A B Q, the Angle at Q is constant; the Declination is A B; whose Fluxion is found from the Equation $t\,b\,\dot{a} = s\,b\dot{H}$ (1818) which gives $t\,b : s\,b :: R : c\,s\,b :: \dot{H} (= A\dot{a}) : \dot{a} = AB - \dot{a}b$. In Words, *Radius is to the Co-Sine of the Angle* Q A B, *as the horary Motion in Longitude* A *a to the horary Variation in Declination.*

1917. Since $\dot{a}$ is always as $c\,s\,b$, it is evident that the hourly Increase or Decrease of Declination will be least of all when the Sun enters *Cancer* and *Capricorn*, and greatest of all at the Equinoxes Q, where it amounts to a *whole Minute of a Degree.*

1918. Therefore suppose H P O Q (Fig. 5.) be the Horizon, Z the Zenith, N the Pole of the *Equinoctial* H C O; also let A B E be an *Almicanter* or Circle of Altitude intersecting the Equinoctial in the Points A and *a*; and draw Z A D and Z *a* F; also the Hour-circles N A G and N *a* I at equal Distances from the Meridian P N Q. Then if the Sun or Star had no Motion in Longitude, its Altitude above the Horizon at equal Times before and after Noon, would be the same, that is A D = *a* F; also the Declination would continue the same, or N A = N *a*. But, as we have shewn, the Declination alters hourly; and when the Circle of Declination N A is thereby *lessened*, as in all the *ascending Signs*, it is evident the Sun which crossed the Almicanter at A in the Morning, will not cross it at (*a*) in the Afternoon, but at a Point more westerly, because N *d* is by Supposition less than N *a*; therefore the Time or Hour-angle C N *d* will exceed that of the *Forenoon* C N A. On the other Hand, when the Declination N *a* from the North *increases* to N *b* (as in all the *descending Signs*) then the Sun comes sooner to the Almicanter at (*b*) and since the *Decrease* or *Increase* of Declination is known for any interval of Time in describing the Arch A C *a* by (1916) therefore the fluxionary Angle *a* N *d*, or *a* N *b* is known from the Triangle Z *a* N, where the two Sides Z *a* and Z N are constant.

1919. For by Case I. (1855) we have this Analogy for finding the Fluxion of the Angle N (there called B) *viz.* $s\,F : c\,t\,C :: \dot{F} (= a\dot{d}) : \dot{B} (= a\,N\,d)$ that is, *as the Co-Sine of the Sun's*

Decli-

Declination (= *s a* N = F) *is to the Co-Tangent of the Angle* Z *a* N (= C) *ſo is the horary Difference of Declination* (a d) *in the Time between the two equal Altitudes, to the Seconds contained in the fluxionary Angle a* N *d*. On the equinoctial Days N A = 90°, or *s* F becomes Radius.

1920. If the Moments of equal Altitudes at A and *d* before and after Noon, be obſerved by a Clock or Watch, then the ſmall Equation of Time *a* N *d*, juſt found, is to be ſubducted from that interval of Time, and the Remainder will be the Time of moving through A *a*; and the Half thereof will give the Moment of the Sun or Star's Appulſe to the Meridian at C, or the *equated Time at Noon*. And this muſt be done in the ſix Signs from *Capricorn* to *Cancer*, and in the other Six, the Equation will be *a* N *b* to be added to the obſerved Time of equal Altitudes at A and *b*.

1921. To find the Variation of the Azimuth *a* Z *d* or *a* Z *b*, we have the firſt Analogy (1855) *viz.* *s* B (= *a* N Z) : *c ſ* D (= Z N) :: $\dot{F}$ (= *a* N — *d* N) : $\dot{A}$ = *b* Z F. In Words, *as the Sine of the Angle* A N C, *or Time before Noon, is to the Secant of the Latitude, ſo is the Variation of Declination in the Interval of equal Altitudes to the Variation of the Azimuth* F *h*, *or* F *k*.

1922. The Poles of the World, or Equinoctial, are found to have a Motion about the Poles of the Ecliptic, contrary to the Order of the Signs, and therefore the Interſections of theſe two Circles or equinoctial Points muſt have a real retrograde Motion; and conſequently the fixed Stars will have their Diſtances from the equinoctial Coloure continually increaſing. Now this Motion is at the Rate of 50″ *per Annum*, at a Mean; and ſo much therefore will the Longitude of the Stars be annully augmented. The *phyſical Cauſe* of this Motion will be hereafter explained.

1923. Let H Z Q be the Meridian N, the Pole of the equinoctial Æ Q, and P the Pole of the Ecliptic E Q. Alſo let A be the Place of a Star, P A C a Circle of Latitude, and N A B a Circle of Declination paſſing thro' it, make C P *c* = 50″, and let *e q* be parallel to the Ecliptic E Q, cutting *c* P in (*a*). Then is A *a* the Space through which the Star advances in one Year in its Parallel, and C *c* its Difference of Longitude in the Ecliptic. And the Difference of Declination, *viz.* A B — *a b* is

found by the first Analogy (1855) *viz.* $c\int D\ (= NP) : sB\ (= ANP) :: \dot{A}\ (= Cc) : \dot{F} = AB - ab$, the Difference of Declination required.

1924. And to find the *Difference of Right-aſcenſion*, we make uſe of the fourth Analogy of (1855) *viz.* $sF\ (= AN) : ctC\ (= A) :: \dot{F} : \dot{B} ::$ *Difference of Declination : Difference or Increaſe of Right-aſcenſion.*

N. B. Moſt of thoſe fluxionary Quantities, which we have given Rules for calculating here, are exemplified in Numbers, and applied to Uſe, in my NEW PRINCIPLES of GEOGRAPHY and NAVIGATION.

CHAP. IX.

Theorems for the Stereographic PROJECTION *of the* SPHERE *in* Plano.

1925. THE Doctrine of *Spherical Projections*, whether *Stereographic*, *Globular*, *Orthographic*, or *Gnomonical*, is wholly derived from the Principles of Perſpective as we have largely ſhewn, and from thence demonſtrated the Properties of each particular Species of Projection in a general Manner; but before we can proceed to an Application of this Art, to the Projection of *ſpherical Triangles*, *Dialling*, *Aſtronomy*, *&c.* we muſt premiſe a few more Principles, eſpecially with regard to the *ſtereographic Projection*, in this Place.

1926. If we look back to Fig. 4. Plate VI. of *perſpective Projections*, we ſhall obſerve, that if any great Circle of the Sphere be oblique to the Plane of Projection, or of the *primitive Circle* QTR; then if the Arch GT meaſures its Obliquity or Elevation above the Plane of the *Primitive*, that Circle will be ſo projected by an Eye at R, that its higheſt Point at G will be in the Diameter at O, diſtant from the Center C, by the Tangent CO of half the Complement QG of its Elevation (1698).

1927.

1927. Again, becauſe the Arch QL = GT (1697) and QB parallel to CL, the Angle BQC = QCL = Elevation of the Circle; therefore the Diſtance of the Center B, from the Center C of the Primitive, is CB = *Tangent* TD *of the Circle's Elevation* TCD.

1928. Let the Arch QOR (Fig. 7.) be compleated into a Circle AOEP, and draw the Diameter ABE; then this may be conſidered as another *primitive Circle*, on which any oblique Circle may be projected by ſetting off the Tangent of its Elevation from the Center B either Way in the Diameter AE (1927). Thus becauſe the Tangent EK (= BO) of an Arch ES of 45° is equal to Radius BE or AB, therefore the Point A is the Center of the oblique Circle OHP elevated above the Primitive in an Angle of 45° = ES = EOH. If the Elevation be of a leſs Number of Degrees than 45, the Center will fall in the Diameter within the Circle; if greater, without. Thus let BL be the Tangent of 30°, then L is the Center of the Circle OMP, whoſe Elevation is the Angle AOM = 30°. If BV be = Tangent of 75 Degrees, then V is the Center of the Circle OYP, making the Angle YOE = 75° = its Elevation above the Primitive; and ſo for any other oblique Circle.

1929. Hence appears the Method of drawing two oblique Circles through any given Point O in the Primitive, to contain a given Angle with each other, by Means of the Tangents of their Elevations. But the Centers of thoſe Circles are alſo found by the *Secants of their Elevation*, ſet off from the Point O, the extreme Point of the projected Diameter OP; for ſince the Angle QBO = GCQ (1697) the Angle CQB = QBA = DCT; and therefore the Triangles CTD and CBQ are equiangular and equilateral; for CT = CQ = Radius; and TD = CB = Tangent of GT (1927); therefore CD = BQ = OB = Secant of GT the Elevation; whence the Center B is thereby given.

1930. Hence the Angle OQT made by the Interſection of two Circles QT and OQ is equal to the Angle made by their *Radii*; for CQ = Radius of QT, and BQ = Radius of QO, and the Angle CQB = TCD = OQT the Elevation.

1931. Since, therefore, the Angle contained by the Radii of the two Circles in the Projection is equal to the Inclination of

the

the Planes of thoſe Circles, it is evident, the Planes themſelves in the Projection muſt contain the ſame Angle as they do in the Sphere; that is, any Angle O Q T is of the ſame Quantity in the Projection as it is on the Globe itſelf.

1932. But as this is a principal Point in the Doctrine of Projections, we ſhall give another Demonſtration of it, thus. Let the Eye at A (Fig. 8.) project the Angle S B R upon the Plane E S placed right before it; and ſuppoſe B D a Tangent to the Circle S B, and B C to the Circle R B, in the Point of the Interſection B. Therefore the Plane in which the Tangents B D and B C are contained, as alſo the Plane of the Circle E S are both perpendicular to the Plane of the Circle B E A S; and ſo their common Interſection C D will be perpendicular to the Line E S produced. Now the Eye at A projects the Tangent B D into F D, and the Tangent B C into F C, becauſe the Point B is projected into F by the viſual Ray A B. Through the Center G draw the Diameters A H and B L, and B I parallel to E S, and join A L, B H. Then is the Angle D B A = B L A (665) = A H B (645) = A B I (659) = B F D (631); that is, D B A = B F D, and therefore B D = D F. Then in the Triangles C D F, and C D B, ſince the Angles D B C and D F C are ſubtended at the ſame Diſtance by the ſame Line D C, they muſt be equal. But the Angle D B C = S B R on the Globe; which therefore is equal to its Projection D F C on the Plane E S continued.

1933. Any Circle B C (Fig. 9.) placed oblique to the Plane of the Circle E G is projected into a *Circle* upon that Plane by an Eye placed at A in the Pole of the Primitive E G. For draw D C parallel to E G, then becauſe the Arch D A = C A, the Angle A B C = A C D (643) and the Angle A is common; therefore the Triangles A B C, and A C *b* or A *c e*, are ſimilar (621). Therefore the viſual Cone A C B is cut by the Planes C D and E G *ſubcontrarily*, and conſequently the Projection of its Baſe B C will be a Circle at *b* C, or *e c* (1510).

N. B. B C is a *ſmall Circle*, but had it paſſed through the Center F it would have been a *great Circle*, and the Demonſtration the ſame.

1934. Hence as *e* F is the Tangent of half the Arch B H, and F *c* = Tangent of half H C; the Points *e* and *c* are given; there-

therefore the Line *e c* biſected, gives (*n*) the Center of the Circle *e c* in the Projection.

1935. A ſmall Circle B I, perpendicular to the Plane of Projection E G, will be projected by the Eye at A into the Circle K *e*, whoſe Center N is in the Line E G, produced, Diſtant from the Center F of the Primitive, by the *Secant of the Arch* B E, (or the Diſtance of the Circle B I from its Pole E) and the Radius thereof will be equal to the *Tangent* of the ſame Arch. For draw A B cutting the Line G E in *e*; and through I draw A K interſecting the Line G E (produced) in K; then the Line K *e* biſected, gives the Center N, on which let the Circle K B I be deſcribed. Draw N B, K B H, and F B; the two Triangles A B H and K H F being right-angled at B and F, and having one Angle at H common, have alſo the Angle B A H = F K H. But B A H = $\frac{1}{2}$ B F H (642) and F K H = $\frac{1}{2}$ F N B; therefore B F H = F N B; and becauſe F N B + B F N = B N F + B F N = to a right Angle; therefore N B F is a right Angle; and conſequently N B is perpendicular to F B, and is the *Tangent* of the Arch E B; and N F is the *Secant* of the ſame. *Q. E. D.*

1936. From theſe Theorems, it is evident, we have certain Rules for deſcribing any Circles great or ſmall, on a given Plane; and therefore the Sphere may be projected on the Plane of any one of its Circles at Pleaſure. We have already given a Specimen of three of theſe Projections in Plate VI. of our *Inſtitutions of Perſpective*, in Fig. 2, 5, and 6. with their Rationale, as derived from the Principles of that Science. But the *Praxis* or Method of drawing the Circles in each, are more immediately deduced from the Theorems we have juſt now premiſed.

1937. Thus the PROJECTION on the Plane of the EQUATOR (in Fig. 2. of that Plate) conſiſts wholly of *right Lines*, and *concentric Circles*. The firſt of which are the great Circles, or Meridians, at right Angles to the Plane of Projection, and whoſe Planes all paſs through the Eye; they are therefore all projected into right Lines. The Circles are the *Parallels* of Latitude, whoſe common Center is the Pole C, or Center of the Primitive, and their Diſtance from the ſaid Center is equal to the Tangent of half their Diſtance from the Pole, and are therefore eaſily drawn by (1934).

1938. The PROJECTION of the SPHERE on the Plane of a MERIDIAN, is that of Fig. 5. where all the Meridians are *great Circles* oblique to the Plane of Projection, and drawn by finding their Centers as directed in (1927) in the ſame Manner as the oblique Circles O H P, O Y P, and O M P, were drawn (1928); and all the Parallels of Latitude, being *ſmall Circles*, perpendicular to the Plane of Projection, are drawn by finding their Centers as directed (in 1934 or 1935).

1939. A PROJECTION on the Plane of the HORIZON, is that of Fig. 6. of Plate VI. and the *great Circles* or Meridians are drawn by the ſame Rules in this as in the former Projection; for let T Q F R (Fig. 7.) repreſent the Plane of the Horizon where the Elevation of the Pole O is T G = T Q O, then it is evident, the Center of the *Six o'Clock* Hour-circle Q O R is at B, the Center of the Primitive A O E P, on which, if the Meridians or Hour-circles are drawn, as above directed (1938) they will, if continued beyond the Pole O, be the proper Meridians or Hour-circles for the *horizontal Projection* T Q F R. Thus for Inſtance, the Hour-circle of XII is O P in one, and T F in the other; that of *Six o'Clock* is A O E P in one, and Q O R in the other; the Hour-circle of II. is O Z P in one, and Z O N in the other; and ſo of the Reſt; therefore they are the ſame in both Projections, and are drawn for both at the ſame Time, all which is evident by Inſpection.

1940. The *Parallels* of *Latitude* are all oblique to the Horizon or Primitive, but are all *Circles* in the Projection (1933) and are drawn by ſetting off the Tangent of half their leaſt and greateſt Diſtances from the Pole of the Primitive, (1934, or 1935,) or Zenith of the Sphere; and biſecting that Interval, you will have the Center of every Parallel in the Line F T continued out beyond T. The *Equator* is drawn by the ſame Rule; and the prime Vertical is a right Line Q C R, as its Plane paſſes through the Eye in the Nadir or lower Pole of the Horizon.

1941. Hence it appears how great the Affinity is between Projections of the Sphere of different Denominations, or rather *that they are all but the ſame Thing* in different Views, and all performed by the ſame Rules. It is hoped the Reader will, from the Method we have here taken, have a clear Idea of the *Rationale* and *Praxis* of this moſt uſeful Branch of Science.

INSTI-

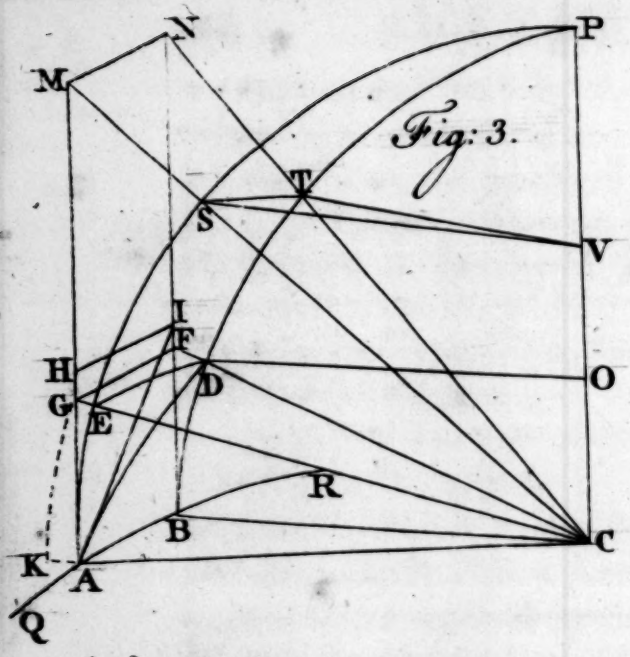


Fig: 3.

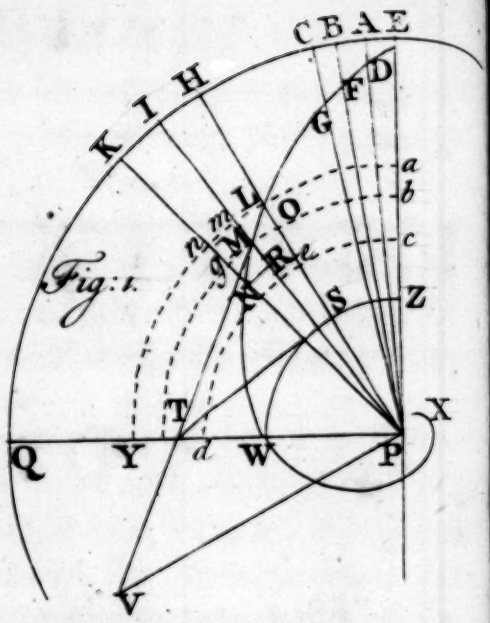


Fig: 1.

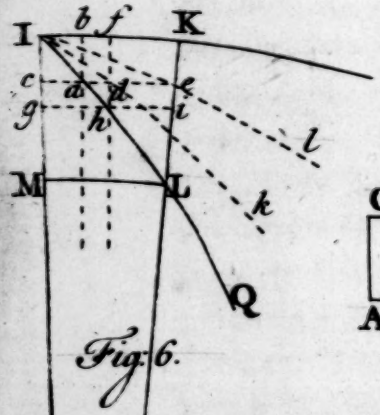


Fig: 6.

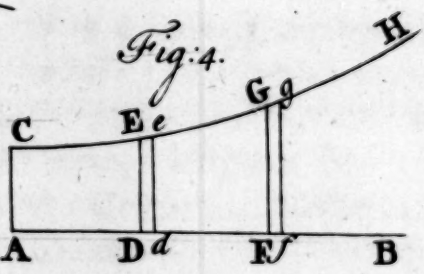


Fig: 4.

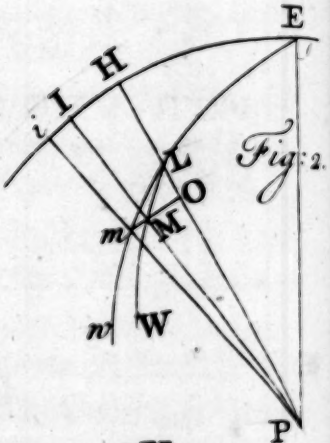


Fig: 2.

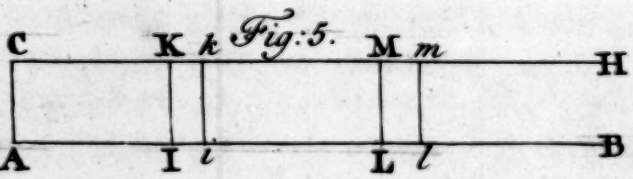


Fig: 5.







Fig: 12.

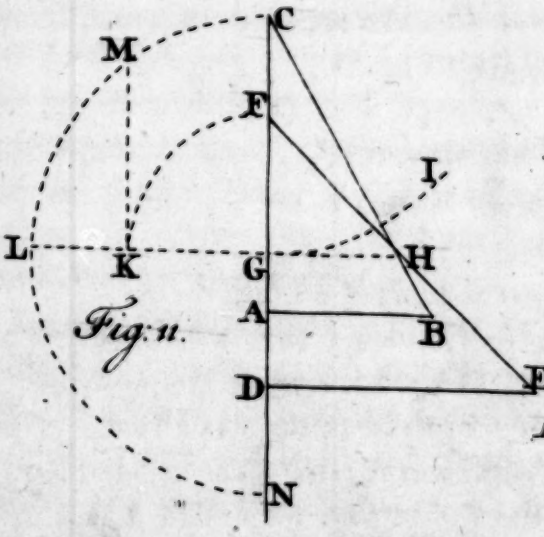


Fig: 11.



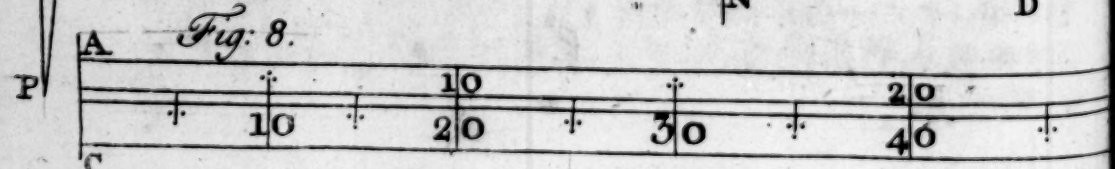


Fig: 8.

INSTITUTIONS OF LOXODROMICS; OR, The THEORY of NAVIGATION on a RHUMB LINE demonſtrated. And from thence each particular SPECIES of SAILING is deduced.

CHAP. I.

The THEORY *of* NAVIGATION *on a* RHUMB LINE *demonſtrated, from its* Genuine PRINCIPLES; *with the Nature and Conſtruction of* MERCATOR's Chart.

1942. THE WAY or COURSE of a Ship, is on the Surface of a GLOBE; and therefore the *ſhorteſt Way* from one Place to another, is in the *Arch of a great Circle*; and when a Ship is conducted on ſuch an Arch, it is called ORTHODROMICS, or GREAT CIRCLE SAILING.

1943. This Method of Sailing, were it *eaſy* and *convenient*, would certainly be the beſt of all others; but ſince a *great Circle* makes different Angles with the Meridians it paſſes over, therefore the *Line* of the *Ship's Courſe* will be perpetually varying; and there will be no *conſtant Rule* for her Conduct. But for eve-

ry ſmall Diſtance ſhe ſails, a new Calculation for the Angle of the Courſe muſt be made, and this would render Navigation extremely difficult and perplexed.

1944. The eaſieſt Method, therefore, of Sailing, is to ſteer her Courſe in ſuch a Manner, that it ſhall every where make the ſame Angle with the Meridians ſhe paſſes over; and then there will be a *conſtant Guide* to direct her Progreſs, *viz.* the COMPASS BOX and NEEDLE.

1945. For ſince by the Property of *Magnetiſm*, the Needle does in every Place make a *given* Angle with the Meridian, if a Ship be ſteered upon a *given Point* of the Compaſs, the Courſe will always make a given Angle with the Meridian alſo; that is, it will croſs every Meridian under equal Angles, while the Needle continues in the ſame Poſition with regard to the Meridian.

1946. But if in different Places, the Needle makes different Angles with the Meridian, that is eaſily diſcovered by the Manner, and the Quantity of its *Variation* obſerved, and then the *Navigator* can readily keep the Ship's Courſe ſtill upon the ſame Point of the Compaſs. And this is the moſt *eſſential Part* of the Art of Sailing.

1947. Therefore it will be neceſſary to enquire into the Nature of that *Line* which a Ship deſcribes in her Courſe, or which interſects all Meridians under equal Angles, and is uſually called the RHUMB, or *Rhumb-Line*. It cannot be a *Right-line*, as being on the Surface of a Globe; it cannot be a *Circle*, as we have ſhewn (2.) it muſt therefore be a *particular Curve* which conſtantly approaches the Pole, and by cutting each Meridian under the ſame Angle, it muſt make perpetual Gyrations about the Pole, but can never terminate in it. The RHUMB, therefore, or Ship's Courſe, is of that *Species of Curves* called SPIRALS.

1948. Of SPIRALS, there are many different Sorts, one of which is called the *Equiangular* or *Logarithmic* SPIRAL; and this is the *Nautical* SPIRAL, or RHUMB-LINE, as we ſhall here demonſtrate. Let E Q be the Quadrant of a Circle, P the Center, and P E, P A, P B, P C, ſeveral Radii drawn very near to each other, and equidiſtant; that is, let E A = A B = B C. Then let the *equiangular Spiral* E W X be drawn, making

making equal Angles with the Radii every where; *viz.* A D E = B F D = C G F, *&c.* Then ſince the Triangles D P E, F P D, G P F are equiangular, and therefore ſimilar; we ſhall have E P : D P :: D P : F P :: F P : G P, *&c.* therefore E P, D P, F P, G P, are in *geometrical Proportion*, while their correſponding Arches E A, E B, E C, are in *arithmetical Proportion*; wherefore the latter are *Logarithms* of the former, (140) that is, A E is the Logarithm of the Ratio of D P to E P; and B E is the Logarithm of the Ratio of F P to E P; and C E, the Logarithm of the Ratio of G P to E P, and ſo on. Therefore the Curve E W X is the *Logarithmic* SPIRAL.

1949. Let a Tangent L V be drawn to the Curve at L, and draw the Radii P L H, P M I, P N K very near to each other, and from the Points M, N, let fall the Perpendiculars M O, N R; and upon P H erect the Perpendicular P V to interſect the Tangent in V. Then becauſe the ſmall Triangles L M O, M N R, are ſimilar, we have L O : L M :: M R : M N :: L O + M R : L M + M N :: L P : L W X :: L P : L V. Therefore the whole Length of the infinite Spiral L W X is equal to the finite Line L V.

1950. From P take any Diſtance P W, and deſcribe the Arch of a Circle W S Z, and at S erect the Perpendicular S T, cutting the Tangent in T, then in the ſame Manner it is ſhewn that the intercepted Part of the Spiral L W is equal to the right Line L T, and of Courſe the remaining Part W X is equal to T V.

1951. Now let the ſame Figure repreſent the common *ſtereographic Projection* of the Surface of the *terraqueous* Globe on the Plane of the *Equator*, of which E Q is a Quadrant; P the *Pole* of the World; H P, I P, K P, ſeveral *Meridians*; and L W X, the *Rhumb-Line*, or Ship's Courſe, making equal Angles with them all. Then let H L be a given Latitude, and L P will be equal to the *Tangent of half the Complement* of that Latitude (1698, 1926); and a Ship paſſing from L to M makes her *Difference of Longitude* I H, *Departure* M O; and *Difference of Latitude* L O.

1952. Then it is evident, *that the Differences of Longitude* A E, B E, C E, *are Logarithms of* D P, F P, G P, *the Tangents of half the Complements of the Latitudes* A D, B F, C G; and H E, I E, K E, *are Logarithms of* L P, M P, N P, *the Tangents of*

half the Co-Latitudes of the Places L, M, and N, *in reſpect to the Radius* P E.

1953. Then, ſuppoſe a Ship ſails from the Latitude L on the Rhumb E L M W to the Latitude M in the Parallel M O (Fig. 2.) the Logarithm of the Ratio of M P to E P is the Arch I E; but if ſhe ſails on any other Rhumb L *w*, ſhe will come to the ſame Parallel in the Point (*m*,) and then the Logarithm of the ſame Ratio *m* P (= M P) to E P, will be the Arch *i* E. Conſequently as there may be an Infinity of Rhumbs ſuppoſed to be drawn through the Point L, there will be an *infinite Variety of Scales of Logarithms* to be found in the Arches of the Equator for expreſſing the ſame Ratios.

1954. In ſailing on the ſame Rhumb L W, (Fig. 1.) it is, *as Radius to the Secant of the Angle of the Courſe* O L M, *ſo is* L O : L M :: M R : M N :: L O + M R : L M + M N :: L P : L W X; *ſo is the Arch of the Meridian equal to the Co-Latitude, to the Length of the whole Spiral or Rhumb from the Latitude or Point* L. For the Triangles L M O, M N R, are equiangular and ſimilar.

1955. Hence if L O = M R, then L M = M N; therefore the *Length of the Rhumb between any two equidiſtant Parallels is the ſame*; or the Length of the Rhumb L M is always proportioned to the Difference of Latitude L O. For L O : L S :: L M : L W. (1950.)

1956. Again, as Radius to the Tangent of the Courſe O L M, ſo L O : M O :: M R : N R :: L O + M R : M O + N R :: L P or Co-Latitude, to the Sum of all the indefinitely ſmall Arches of the Parallels, or whole *Departure* made in ſailing obliquely from one Meridian to another.

1957. Hence in very ſhort Courſes, where the Diſtance ſailed L M, and the Difference of Latitude L O may be eſteemed rectilineal; we have *Radius to the Tangent of the Courſe, ſo is Difference of Latitude* L O *to the Departure* M O.

1958. In ſailing on different Rhumbs L M and L *m* to the ſame Parallel *m* O, (Fig. 2.) it will be *as the Tangent of the Courſe* M L O *to the Tangent of the Courſe m* L O, ſo is the Departure M O to the Departure *m* O, and ſo is the Difference of Longitude H I to the Difference of Longitude H *i*.

1959.

1959. And hence it appears, *that the Differences of Longitude made in Sailing upon two different Rhumbs* L W, and L *w*, *are as the Tangents of the Angles, which they make with the Meridians*; and therefore the Differences of Logarithms (I H, *i* H), of the ſame Ratio (*viz.* of M P or *m* P to E P) in the two different Scales appropriate to the two Rhumbs, will be likewiſe as thoſe Tangents reſpectively.

1960. In very ſhort Voyages, the ſmall Triangle L M O may be eſteemed *Right-lined*, and uſed as ſuch without ſenſible Error, becauſe a ſmall Portion of the Surface of a Globe, near 8000 Miles in Diameter, will differ inſenſibly from a Plane. Hence *all Caſes of Sailing* (except that for the *Longitude*) are ſolved by a *ſimple plain Triangle*, this is therefore called PLAIN SAILING, whoſe Principles we have now premiſed.

1961. But when the Length of Voyages makes it neceſſary to conſider the ſaid Triangle really as it is, *viz.* a *curvilineal One* throughout; and that every Part conſiſts of a Curve of differing Species, *viz.* L O the Arch of a *great Circle*; M O, the Arch of a *ſmall Circle*; and L M the Arch of a *Spiral*, it will eaſily appear, that under ſuch Circumſtances it will be no eaſy Matter to reſolve *Caſes of Sailing* on the *Surface of the* GLOBE *itſelf*, or by the *Globular Chart*.

1962. Therefore to facilitate the Praxis of ſo excellent and neceſſary an Art, Methods have been contrived to delineate the Surface of the Globe on a Plane, in ſuch Manner, that all the *Calculations* relative to NAVIGATION, ſhould ſtill be ſubject to *plain Trigonometry*, and yet productive of Accuracy and Truth beyond all neceſſary Degrees.

1963. To illuſtrate this Affair, let C (Fig. 3.) be the Center of the Globe; P, the Pole; A P, B P, two Meridians ſtanding on the Arch of the Equator Q R. Let E D, S T, be Arches of Parallels whoſe Radii are D O, T V. At the Points A, B, are erected the two Perpendiculars or Tangents A M, B N; and through the Points E and D, are drawn the Secants G C, F C. Then is the Arch of the Parallel E D thereby projected into the Right-line G F on the Plane between the two Tangents. Therefore G F = A B, in the ſame Manner the Arch S T is projected into M N = A B, all which is evident from the Doctrine of *Gnomonical Projection* (1738, *&c.*)

1964.

half the Co-Latitudes of the Places L, M, and N, *in reſpect to the Radius* P E.

1953. Then, ſuppoſe a Ship ſails from the Latitude L on the Rhumb E L M W to the Latitude M in the Parallel M O (Fig. 2.) the Logarithm of the Ratio of M P to E P is the Arch I E; but if ſhe ſails on any other Rhumb L *w*, ſhe will come to the ſame Parallel in the Point (*m*,) and then the Logarithm of the ſame Ratio *m* P (= M P) to E P, will be the Arch *i* E. Conſequently as there may be an Infinity of Rhumbs ſuppoſed to be drawn through the Point L, there will be an *infinite Variety of Scales of Logarithms* to be found in the Arches of the Equator for expreſſing the ſame Ratios.

1954. In ſailing on the ſame Rhumb L W, (Fig. 1.) it is, *as Radius to the Secant of the Angle of the Courſe* O L M, *ſo is* L O : L M :: M R : M N :: L O + M R : L M + M N :: L P : L W X; *ſo is the Arch of the Meridian equal to the Co-Latitude, to the Length of the whole Spiral or Rhumb from the Latitude or Point* L. For the Triangles L M O, M N R, are equiangular and ſimilar.

1955. Hence if L O = M R, then L M = M N; therefore the *Length of the Rhumb between any two equidiſtant Parallels is the ſame*; or the Length of the Rhumb L M is always proportioned to the Difference of Latitude L O. For L O : L S :: L M : L W. (1950.)

1956. Again, as Radius to the Tangent of the Courſe O L M, ſo L O : M O :: M R : N R :: L O + M R : M O + N R :: L P or Co-Latitude, to the Sum of all the indefinitely ſmall Arches of the Parallels, or whole *Departure* made in ſailing obliquely from one Meridian to another.

1957. Hence in very ſhort Courſes, where the Diſtance ſailed L M, and the Difference of Latitude L O may be eſteemed rectilineal; we have *Radius to the Tangent of the Courſe, ſo is Difference of Latitude* L O *to the Departure* M O.

1958. In ſailing on different Rhumbs L M and L *m* to the ſame Parallel *m* O, (Fig. 2.) it will be *as the Tangent of the Courſe* M L O *to the Tangent of the Courſe m* L O, ſo is the Departure M O to the Departure *m* O, and ſo is the Difference of Longitude H I to the Difference of Longitude H *i*.

1959.

1959. And hence it appears, *that the Differences of Longitude made in Sailing upon two different Rhumbs* L W, and L *w*, *are as the Tangents of the Angles, which they make with the Meridians*; and therefore the Differences of Logarithms (I H, *i* H), of the ſame Ratio (*viz.* of M P or *m* P to E P) in the two different Scales appropriate to the two Rhumbs, will be likewiſe as thoſe Tangents reſpectively.

1960. In very ſhort Voyages, the ſmall Triangle L M O may be eſteemed *Right-lined*, and uſed as ſuch without ſenſible Error, becauſe a ſmall Portion of the Surface of a Globe, near 8000 Miles in Diameter, will differ inſenſibly from a Plane. Hence *all Caſes of Sailing* (except that for the *Longitude*) are ſolved by a *ſimple plain Triangle*, this is therefore called PLAIN SAILING, whoſe Principles we have now premiſed.

1961. But when the Length of Voyages makes it neceſſary to conſider the ſaid Triangle really as it is, *viz.* a *curvilineal One* throughout; and that every Part conſiſts of a Curve of differing Species, *viz.* L O the Arch of a *great Circle*; M O, the Arch of a *ſmall Circle*; and L M the Arch of a *Spiral*, it will eaſily appear, that under ſuch Circumſtances it will be no eaſy Matter to reſolve *Caſes of Sailing* on the *Surface of the* GLOBE *itſelf*, or by the *Globular Chart*.

1962. Therefore to facilitate the Praxis of ſo excellent and neceſſary an Art, Methods have been contrived to delineate the Surface of the Globe on a Plane, in ſuch Manner, that all the *Calculations* relative to NAVIGATION, ſhould ſtill be ſubject to *plain Trigonometry*, and yet productive of Accuracy and Truth beyond all neceſſary Degrees.

1963. To illuſtrate this Affair, let C (Fig. 3.) be the Center of the Globe; P, the Pole; A P, B P, two Meridians ſtanding on the Arch of the Equator Q R. Let E D, S T, be Arches of Parallels whoſe Radii are D O, T V. At the Points A, B, are erected the two Perpendiculars or Tangents A M, B N; and through the Points E and D, are drawn the Secants G C, F C. Then is the Arch of the Parallel E D thereby projected into the Right-line G F on the Plane between the two Tangents. Therefore G F = A B, in the ſame Manner the Arch S T is projected into M N = A B, all which is evident from the Doctrine of *Gnomonical Projection* (1738, *&c.*)

1964.

1964. But these Arches E D, S T, are when compared with the corresponding Arch A B of the Equator, as their respective Radii D O, T V to the Radius A C of the Globe. Or A B is to E D, *as Radius to Co-Sine of the Latitude*; for D O is the Sine of D P the Complement of the Latitude B D or A E.

1965. Again, the Arches E D, S T, are to their Projections G F or M N, as E C or S C, to O C and M C; or *they are enlarged in Proportion of the Secant of the Latitude to the Radius.*

1966. Suppose a Ship sails on the Rhumb A D L from A to D; then is A E the Difference of the Latitude, A B the Difference of Longitude; the Angle E A D the Course; A D the Distance sailed; and D E the Arch of the Parallel at D. Now it is required to delineate this mixtilineal Triangle A E D on a Plane in such Sort, that all the Parts *may have the same Ratio of Magnitude and Position as they now have on the Surface of the Globe.*

1967. In order to this, it must be considered, that in whatever Proportion any one Part is altered by projecting the Triangle E A D on the Plane A M N B, the other Parts must be altered in the same Ratio, to produce *Similarity*, and thereby to keep the given Positions of all the Parts to each other. But by the Projection, the Part or Perpendicular E D is enlarged to G F, in the Ratio of E C to G C (1965.) and therefore the Side or Base A E must be enlarged in the same Proportion on the Line A M.

1968. Now it is evident, the Arch A E described with the Radius A C is to the Arch G K described with the Radius G C, as Radius to the Secant of the Arch A E, or Angle A C E; if therefore in the Right-line A M, we take A H equal to the Arch G K, and draw H I parallel to A B (or Chord of the Arch A B) and draw the diagonal A I; we shall have the right-lined Triangle on the *Plane*, every Way similar and proportional to the Triangle E A D on the Globe: For it is H I (= G F) : E D :: G C : E C :: A H (= G K) : A E :: A I : A D.

1969. Hence the Angle of the Course H A I is the very same as the Angle E A D on the Globe; and therefore the Position of the Point I with regard to the Point A in the Plane, is the very same as that of the Port D in respect of the same Point A on the Globe; so that the *Easting* or *Westing* on a Rhumb A D making

ing

ing any Angle E A D with the Meridians, is truly preſerved and repreſented in the Triangle H A I *in Plano*; and conſequently the *whole Affair of Navigation* is by ſuch a Projection reduced to *the Solution of a plain right-angled Triangle*, and thereby becomes a very eaſy Taſk as far as Calculations are concerned. And this Projection of the Surface of the Globe on a Plane, is called MERCATOR's PROJECTION, or CHART.

CHAP. II.

The THEORY *of* NAVIGATION *by a* TABLE *of* Logarithmic TANGENTS *demonſtrated.*

1970. LET Radius of the Equator P E $= r$ (Fig 1.); the Latitude of the Place H L $= z$; the Radius of the Parallel at L $= x$; and the Longitude or Arch of the Equator E H $= y$; then will H I $= \dot{y}$; and L O $= \dot{z}$; and we have P H : P L :: $r : x :: \dot{y} : \frac{x\dot{y}}{r} =$ L m, the Fluxion of the Parallel at L. And putting $t =$ Tangent of the Angle L M O or L M m, we have L m : m M :: $\frac{x\dot{y}}{r} : \dot{z} :: t : r$. Whence $\dot{z} = \frac{x\dot{y}}{t}$; and ſo $\frac{t\dot{z}}{x} = \dot{y}$; and $\frac{rt\dot{z}}{x} = r\dot{y}$.

1971. To conſtruct this Equation, let the quadrantal Arch of the Meridian be extended into a Right-line A B (Fig. 4.); at A erect a Perpendicular A C $= t$; and make A D $= z$, the Latitude of the Place; upon D erect the Perpendicular D E, which is to A C $= t$, as r to x, ſo that it may be every where D E $= \frac{tr}{x}$; and thus by conceiving Perpendiculars to be raiſed on every Point of the Meridian A B, we ſhall have the Curve C E H produced, and of ſuch a Nature that the curvilineal Space A D E C, divided by Radius, will be equal to the Arch of the Equator expreſſing the Longitude E H made in ſailing from the Equator E to the Latitude Z on the given Rhumb E L W (Fig. 1.)

1972.

1972. For let *de* be drawn infinitely near, and parallel to DE; then AD $= z$, and D*d* $= \dot{z}$, and DE $\times$ D*d* $= \frac{tr\dot{z}}{x}$ $= r\dot{y}$ (1970.) But the Fluent of $\frac{tr\dot{z}}{x}$ is the Space ADEC, and that of $r\dot{y}$ is ry; therefore $\frac{\text{ADEC}}{r} = y =$ the Longitude upon the Course EL.

1973. After the same Manner we have $\frac{\text{AFGC}}{r} =$ the Longitude answering to the Latitude AF upon the same Course. Consequently $\frac{\text{DFGE}}{r} =$ the Difference of Longitude corresponding to the Difference of Latitude DF upon the same Rhumb.

1974. It is easy to conceive the Space ADEC, and its Fluxion DE*ed* may be transformed into equal rectangular Spaces or Parallelograms, as AIKC, and IK*ki*, and because in this Case IK $=$ AC $= t$, therefore I*i* $= \frac{r\dot{z}}{x}$, which is to D*d* $= \dot{z}$, as r to x, that is, *as Radius to the Co-Sine of the Latitude, or as the Secant to the Radius.* But this is the Proportion of the protracted Arch to the Length of the natural Arch of the Meridian (1967, 1968.) And since AD is the Length of the latter, AI will represent the former, *viz.* the *enlarged meridional Arch*, used in *Mercator's Chart.* (1969.)

1975. Hence since the Rectangle AIKC $=$ AI $\times$ AC $=$ ry, or, (putting AI $= Z$,) $Zt = ry$, therefore $r : t :: Z : y$, or *Radius is to the Tangent of the Course, as the protracted Arch or Degrees of Latitude is to the Arch of the Equator expressing the Longitude.*

1976. Hence when the Rhumb or Spiral ELW intersects the Meridians at an Angle of 45 Degrees, we have $r = t$; and then $z = y$; *the Degrees of Latitude, therefore, in the protracted Meridian of Mercator's Chart, are a* SCALE *of* LONGITUDES *in sailing upon a Course or Rhumb of* 45°.

1977. Because $r : x :: S : r$ (putting S for *Secant* of the Latitude) therefore $\frac{r}{x} = \frac{S}{r}$, and so $\frac{tr\dot{z}}{x} = \frac{tS\dot{z}}{r} = r\dot{y}$ (1970); and

when

$t = r$, as in Caſe of the Rhumb of 45°, then $S\dot{z} = r\dot{y}$; and if $r = 1$, as in the Canon of natural Sines, Tangents, and Secants; then $S\dot{z} = \dot{y}$; whence it appears, *that the Fluxion of the Longitude* $(\dot{y})$ *is equal to the Fluxion of a Space or Area conſiſting of Secants ſucceſſively erected on the fluxionary Points* $\dot{z}$ *of the Meridian.*

1978. Becauſe $\dot{z}$ is conſtant in the Equation $S\dot{z} = \dot{y}$, we have $S = \dot{y}$, or the Fluxion of the Longitude $(\dot{y})$ will be as the variable Secant S; and ſo the Sum of all the Fluxions or the *Longitude itſelf, will be as the Sum of all the Secants* to any given Latitude. And hence the Reaſon why Mr. *Wright* took this Method to conſtruct his Line of meridional Parts, *viz. by the conſtant Addition of the Secants*, as they are found in the Canon. See his *vulgar Errors in Navigation* corrected; where the *nautical meridian Line* firſt made its Appearance.

1979. It has been ſhewn, that the *Fluxion of the Logarithm of any Number is equal to the Fluxion of the Number divided by the Number itſelf* (849). But the Longitudes are Logarithms of the Tangents LP, MP, &c. to the Radius EP (1952) ſuch a Tangent therefore is a Number, which put $= n$; then, when $r = 1$, we have $t\,S\dot{z} = \dot{y} = \frac{\dot{n}}{n}$ (1977) which gives $n\dot{y} = \dot{n}$, and therefore $1 : n :: \dot{y} : \dot{n}$; or PH (= PE) : PL :: IH : L*m* (Fig. 1.)

1980. And alſo, when $t = 1$, as upon a Rhumb of 45°, we have $S\dot{z} = \frac{\dot{n}}{n}$, whence it appears, that the *Fluent* of $S\dot{z}$, which is an Arch of the *nautical Meridian*, (1978) *is equal to the Fluent of* $\frac{\dot{n}}{n}$, *or the Logarithm of the Tangent of half the Co-Latitude of that Arch.* Therefore the *whole nautical meridian Line is a Scale of Logarithmic Tangents of the half Complements of the Latitudes, upon that Rhumb which makes an Angle of* 45° *with the Meridian.* Therefore this we ſhall call the *nautical Scale of Logarithmic Tangents.*

1981. But it is evident from the original Equation $\frac{ADEC}{r}$ $= y =$ Longitude EH, or Logarithm of the Ratio of PL to PE (1972.) that the Logarithm of the ſame Ratio will be vari-

able according to the different Values of r, or the Radius of the Globe PE. For HI is the Logarithm of the Ratio of the Tangent PL to Radius PH or PE, when $r = PE = 1$, and the Angle $MLO = 45°$. But if r or PE be expounded by any Number leſs or greater than Unity, it follows, that the Length of $HI = y$ will be encreaſed or diminiſhed in the ſame Proportion; for the Area ADEC being conſidered as conſtant, the Logarithm (y) will be always as $\frac{1}{r}$ or *inverſely as the Radius.*

1982. From the fluxionary Equation $\frac{t r \dot{z}}{x} = r \dot{y}$ (1970), it appears that near the Beginning of the Rhumb Line in the Equator, where $x = r$, that Equation becomes $t \dot{z} = r \dot{y}$, and then $r : t :: \dot{z} : \dot{y}$. And therefore upon the Rhumb of 45°, where $t = r$, we have $\dot{z} = \dot{y}$. Therefore *the Fluxion $(\dot{y})$ of the Logarithm of the Ratio of the firſt mean proportional Tangent to the Radius is equal to the infinitely ſmall Difference $(\dot{z})$ between that proportional and Radius.*

1983. Hence if Radius $PE = 1$ (Fig. 1.) and the firſt Proportional be $PD = 1 - \dot{z}$, the ſecond $PF = \overline{1 - \dot{z}^2}$, the third $PG = \overline{1 - \dot{z}^3}$, &c. then will the Logarithms be $AE = \dot{z}$, $BE = 2\dot{z}$, $CE = 3\dot{z}$, &c. therefore $\overline{1 - \dot{z}^n}$ is any Proportional $1 - z$, and its Logarithm is $n\dot{z}$. But ſince $\overline{1 - \dot{z}^n} = 1 - z$, it is $\sqrt[n]{1 - z} = 1 - \dot{z} = \overline{1 - \dot{z}^n}^{\frac{1}{n}}$; whence $1 - \overline{1 - z}^{\frac{1}{n}} = \dot{z}$. But by the *Newtonian Theorem* for extracting the Roots of Binomials (306) we get $\overline{1 - z^n}^{\frac{1}{n}} = 1 - \frac{1}{n} z - \frac{1}{2n} z^2 - \frac{1}{3n} z^3 - \frac{1}{4n} z^4$, &c. or $\frac{1}{n} \times z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \frac{1}{4} z^4 + \frac{1}{5} z^5$, &c. $= 1 - \overline{1 - z^n} = \dot{z}$; whence $z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \frac{1}{4} z^4 + \frac{1}{5} z^5$, &c. $= n\dot{z} =$ Logarithm of the Number $1 - z$.

1984. This Series is the ſame with that derived from the *Hyperbola* (829), and when computed in Numbers gives 2302585 for the Logarithm of the Ratio of 10 to 1, or of 1 to 0,1. Conſequently

ſequently $2302585 = n\dot{z}$; or ſo many mean Proportionals there are in the Ratio of 10 to 1 in this Scale of Logarithms which are thoſe firſt invented by *L. Neper*.

1985. But in Practice it was found much more convenient to put the Logarithm of the Ratio of 10 to $1 = 1000000 = n\dot{z}$, as in the common logarithmic Canon. Hence we have $n : n :: 2302585 : 1000000 :: 1 :: 0{,}43429448$. Theſe Numbers, therefore expreis the Ratio of the Logarithms in each Syſtem reſpectively.

1986. From what foregoes, it appears, that the infinite Series juſt now found, anſwers to the Space ADEC $= y$ (1972) when $r = 1$; and $n\dot{z} = y =$ the Length of the *nautical Meridian* for the Latitude z. Then putting R for the Logarithm of the Tangent of 45° or Radius, and T for the tabular Tangent of any other Arch, as that which denotes the half Co-Latitude of HL; we have, as $EP : LP :: 1 : 1 - z = t =$ natural Tangent of $\frac{1}{2}$ LP. Therefore the Ratio of $\frac{EP}{LP} = \frac{1}{1-z} = \frac{1}{t}$, but the Logarithm of $\frac{1}{t}$ is equal to the Logarithm of Radius 1 leſs the Logarithm of t. That is, the tabular Logarithm of $\frac{1}{t} = R - T$.

1987. But the tabular Logarithm of any Ratio is to *Neper*'s Logarithm of the ſame, as 1 to 2,302585 (1985); therefore $\overline{R - T} \times 2{,}302585 =$ the Length of the protracted meridional Arch HL $= z$, in the Scale of *Neper*'s Logarithms. Put $N = 2{,}302585$, and $\mathcal{T} =$ Logarithm Tangent of any other Arch, as that of $\frac{1}{2}$ NP. Then becauſe $NR - NT$ is the Length of the Arch HL, and $NR - N\mathcal{T}$ is the Length of the Arch KN, we have $NT - N\mathcal{T} = \overline{T - \mathcal{T}} \times N$, the Difference of the Latitude L and N expreſſed in Parts, of which the Radius EP is 1.

1988. If the Radius be expreſſed in Minutes of a Degree, then the *protracted meridian Arch will be had in Minutes likewiſe.* Now the Circumference of a Circle is to Radius as 6,28318 to 1. But in that Circumference there are 360° or 21600′; therefore ſay, as $6{,}28318 : 1 :: 21600' : 3437',747$, which is the

Number of Minutes in the Radius. Put this Number 3437′,747 $= m$; then $m\mathrm{N} \times \overline{\mathrm{T}-t}$ will expreſs the Length of the Difference of Latitude in Minutes of the *protracted Meridian*; and thus a Table of *meridional Parts* or *Minutes*, may be computed from the *common Table of logarithmic Tangents.*

1989. For Example, *let it be required to find the meridional Parts in the Arch of the Meridian contained between the Tropic of Cancer and the arctic Circle.* Then the Latitude of the Tropic is 23° 30′, its Complement is 66° 30′, and the Half thereof is 33° : 15′, whoſe logarithmic Tangent is 9,816658 $= \mathrm{T}$; the Latitude of the arctic Circle is 66° : 30′, its Complement 23° : 30′, and the Half of that 11° : 45′, whoſe logarithmic Tangent is 9,318064; then $\overline{\mathrm{T}-t} = 0{,}498594$; and $m\mathrm{N} = 7915{,}704$; therefore $m\mathrm{N} \times \overline{\mathrm{T}-t} = 0{,}498594 \times 7915{,}704 = 3946{,}722$, the *meridional Parts or Minutes* in that protracted Difference of Latitude.

1990. As in all Circles, the *Radii* P A, P E, P I, (Fig. 6.) are proportioned to ſimilar Arches in their Peripheries A B, E F, I K; and as thoſe Arches are Logarithms of the Ratios of D P to A P, G P to E P, L P to I P, in ſailing on the Rhumbs A N, E O, I Q, making the ſame Angle with the Meridians in all; therefore it follows that the *Radii* are the *Modules* or conſtant Expreſſions of the Ratio of the Logarithms pertaining to their ſeveral Peripheries or Syſtems reſpectively. Thus if I P be *Radius* or the Module for the Logarithms of *Neper*'s Sort, and be expreſſed by *Unity*, then the *Module* or *Radius* for the Logarithms of the common Tables will be 0,43429448 $=$ E P (1985) and then if the Arch I K be *Neper*'s Logarithm of that Ratio of 10 to 1, E F will be the Logarithm of the ſame Ratio in the tabular Syſtem. So that if I K $= 2{,}302585$, we have E F $= 1$.

1991. Again, if A N be the Rhumb which makes an Angle of 45° with the Meridians, and A B an Arch of *one Minute*; then making A C $=$ A B, we have A B D C a Square, and D C $=$ A C $=$ A B; and ſince the Radius A P is in this Caſe to be conſidered as divided into *Minutes* and equal to 3437′,747, (1988) which is but a Part of P I; the ſaid Radius P I, if it be divided throughout into the ſame equal Parts or Minutes, will contain juſt 10000. For I P : A P 1 : 1 : 0,3437747, which therefore

is

is the *Module* for the Syſtem of Logarithms pertaining to the Rhumb of 45°, or the *nautical meridian Line.*

1992. The *Modules*, therefore, or *Ratios* of the three Syſtems of Logarithms are theſe, *viz.* for *Neper*'s Logarithms PI = 1,000000; for the *Tabular* or *Brigg*'s Logarithms PE = 0,434294; and for the nautical Logarithms we have PA = 0,343775. Therefore if EO and IQ be the ſame Rhumb of 45°, then by making EH = EF; and IM = IK; the Triangles ACD, EHG, and IML will be ſimilar; and we have AC : DC :: EH : GH :: IM : LM.

1993. Hence becauſe AC = AB = 1 Minute, we have as 0,343775 : 1 (:: PA : PI :: AB : IK) :: 1 : 2,90888 = IK. And this is the Meaſure of the *natural Sine* and *Tangent* of 1 Minute in the Tables. Again, AP : EP :: 0,343775 : 0,434294 :: (AB : EF ::) 1 : 1,2633114 = EF, the Logarithm of the ſame Ratio in *Brigg*'s *Syſtem.*

1994. Since then the Logarithms of the ſame Ratio pertaining to the *ſame Rhumb*, are expreſſed by the different Numbers 1, 1,263, 2,909, in the three different Syſtems above-mentioned. It is eaſy to conceive, that upon different Rhumbs in the ſame Syſtem, the ſame Numbers will expreſs or be the Logarithms of the ſame Ratio. For let PI be Radius = 1; and take Ic = AC, and Ig = EH, and draw the Parallels *ce* and *gi*; and the Squares I*cab* = ACBD, and IG*hf* = EHGF; then it is I*c* : *ca* :: I*g* : *gh* :: IM : ML = *ce*; whence I*c* : IM :: *ca* : *ce* :: 1 : 2,909 :: Tangent of *a*I*c* = 45° : Tangent of *e*I*c* = 71° : 1′ : 42″. And therefore if a Ship ſails on the Rhumb I*el* making that Angle with the Meridians, *Neper*'s *Logarithms will be a Scale of logarithmic Tangents of the half Co-Latitudes every where.*

1995. In the ſame Manner we have I*c* : I*g* :: *ca* : *gh* = *cd*, that is 1 : 1,2633 :: Tangent of *a*I*c* = 45° : Tangent of *d*I*g* = 51° 38′ 9″. Conſequently, if a Ship ſails on a Rhumb I*dk* making that Angle with the Meridians, *then* (*becauſe* cd = EF) *the logarithmic Tangents*, in the common Canon, are *thoſe which every where expreſs the Ratio of the Tangents of half the Co-Latitudes to Radius*; and therefore the Difference *of the logarithmic Tangents of the half Co-Latitudes will be as the Difference of Longitude*

cor-

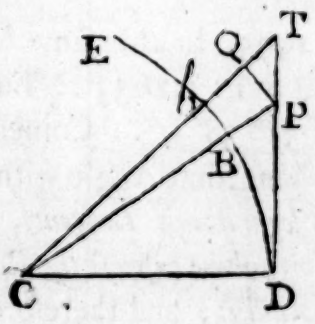
corresponding to the given Difference of Latitudes on the Rhumb of 51° 38′ 9″.

1996. But this Difference of logarithmic Tangents or Longitudes, is an Arch of the Equator, whose Radius, or natural Tangent in the Tables, is 1. Therefore, putting $\overline{T-\mathcal{T}}$ for this Difference of logarithmic Tangents (1987) we must say, as 0,434294 : 1 : 4342′,94 : 10000′ = Minutes of a Degree in the Radius or Arch = 1 (1990). Then say, as the Arch 1 : 10000 :: $\overline{T-\mathcal{T}}$: $\overline{T-\mathcal{T}} \times 10000$ = Minutes of a Degree in an Arch of the Equator expressed by $\overline{T-\mathcal{T}}$.

1997. Hence the principal Cases of *Mercator's Sailing* are solvable by the common Tables of logarithmic Tangents; for if the Difference of Latitude be given, the Quantity $\overline{T-\mathcal{T}} \times 10000$, or Difference of Longitude corresponding thereto on the given Rhumb of 51° 38′ 9″ will be known also; and then, if the Difference of Longitude be given on any other Rhumb or Course, the Tangent thereof will be known likewise. For, *as the Difference of Longitude* $\overline{T-\mathcal{T}} \times 10000$ *is to the constant Tangent of* 51° 38′ 9″, *so is the given Difference of Longitudes to the Tangent of the Course on which the Ship sails* (1958, 1959).

1998. From the fluxionary Equation $\frac{rt\dot{z}}{x} = r\dot{y}$ (1970) it appears, that upon the Rhumb of 45°, when $t = r$, the Equation $\frac{r\dot{z}}{x} = \dot{y}$, will then express the *Fluxion of the nautical meridian Line*, which in its nascent State, or very beginning (where $x = r$) will become $\dot{z}$, or the Fluxion of the natural Meridian itself.

1999. Now the *nautical meridian Line* begins from the Radius with the *natural Meridian*; but the *artificial Line* of Tangents begins from the Tangent of 45°. But the Fluxion $\dot{z}$ is equal to $\frac{1}{2}\dot{t}$, or half the Fluxion of the Tangent of 45°, as is thus shewn. On the Center C describe the Circle D B E, and draw the Tangent D P,



the

N. B. Fig. 7. was forgot to be put in the Plate, and is here added.

the Secant CP, and infinitely near it CT; and deſcribe the ſmall Arch PQ. Then put Radius CD = r, DP = t, DB = z, TP = i, Bb = $\dot{z}$; and by ſimilar Triangles we have TP : QP :: CP : CD; and PQ : Bb : CP : CB = CD. Therefore TP : Bb :: CP^2 : CD^2. That is, $rr + tt : rr : i : \dot{z}$; and when $t = r$, then $2 : 1 :: i : \dot{z}$, or $i = 2\dot{z}$.

2000. Becauſe $\dot{z}$ is the Fluxion of the nautical Meridian (1998) it is the *Fluxion of the logarithm Tangents* proper to the Rhumb of 45° (1980). And ſince the Fluxion of the Logarithm of any Tangent is $\frac{i}{t}$, therefore when ſuch a Tangent becomes equal to Radius, or $t = r$, then the Fluxion of its Logarithm is barely i; but we have ſhewn that in ſuch a Caſe $i = 2\dot{z}$. *Therefore the Fluxion of our artificial Line of logarithm Tangents is double the Fluxion of the nautical meridian Line of logarithm Tangents.* And, *ſo the Fluents or Degrees of logarithm Tangents in one will be double to thoſe of the other every where.*

2001. Hence it appears, that any Line of artificial Tangents is the ſame with the nautical meridian Line, *if we take every half Degree for a whole one, and number them accordingly.* Hence any ſuch *Line of logarithm Tangents* may be uſed or ſubſtituted for a *Line or Table of meridional Parts*, and all the Problems of *Mercator's Sailing* are ſolved thereby. The Identity or Coincidence of theſe two Lines is repreſented in the double Scale (Fig. 8.) where AB is the common Line of artificial Tangents, and CD the *nautical Meridian*; one to 25 Degrees, the other to 50.

CHAP. III.

The different SPECIES *of* Loxodromic SAILING, *deduced from the preceeding* THEORY, *and illuſtrated by proper* DIAGRAMS.

2002. FROM a particular Conſideration of the Nature of what is called *Departure* will reſult another *Species of Navigation* called MIDDLE LATITUDE SAILING. It is evident, by In-

Inſpection (Fig. 1.) that which is called *Departure* is only the Arch of a Parallel contained between two Meridians, as P H, P I, paſſing through the two Latitudes L and M; and is always to be reckoned in that Parallel to which the Ship arrives. Thus if ſhe ſails from L to M, the Departure is the Arch M O, paſſing thro' the Latitude M. But if the Ship ſails from M to L, then the Departure is the Arch L *m*, ſo that there is in Reality, two different Departures belonging to the ſame Courſe, Diſtance, or Difference of Latitude.

2003. Therefore to avoid all Ambiguity and Falacy in Sailing, it is neceſſary to make conſtant Computations of the Departure upoh Parallels ſo near together, that the Arches M O and L m may differ but by an infinitely ſmall Quantity, or that the Triangle L O M may be eſteemed rectilineal; and then we ſhall have the ſeveral Departures M O, N R, &c. computed as they riſe in paſſing over the ſeveral Meridians, as from L to M, from M to N, &c. till at laſt the Ship arrives at ſome Diſtant Port W in Latitude Q W, and the Sum of all theſe Departures will be the whole Departure made in the Voyage from L to W.

2004. But ſince the particular Departures M O, N R, &c. proceed continual decreaſing from L to W, 'tis evident their Sum will be greater than W S, which is the Sum of the ſame Number of Departures each of them equal to the leaſt of thoſe made in the Voyage. And on the other Hand if the Ship ſails from Latitude W to Latitude L, the Sum of all the particular Departures will be the ſame as before, but leſs than the Arch Y L, which is the Sum of the ſame Number of Departures, each equal to *m* L, the largeſt of thoſe made in the Voyage.

2005. Conſequently, the Departure made in the Voyage from L to W, or from W to L, being the ſame; and conſiſting every where of Quantities equidifferent, the greateſt and leaſt of which are as L Y to W S; the Sum of all the Departures will be a *Mean* between the two, or as $\frac{LY + WS}{2} = de$, the correſponding Parallel, or mean Arch of the *middle Latitude* H e (136), which therefore will nearly repreſent the whole Departure of the Voyage either Way between the two Ports L and W. (318.)

2006. Hence

2006. Hence we deduce this *fundamental Theorem*, *The Co-Sine of Middle Latitude* P e *is to Radius* P H *as the Departure* d e *is to the Difference of Longitude* H Q. Note, If $\frac{1}{2}$ the Sum of the Co-Sines of the Latitudes L and S be taken, it will be nearer the Truth than the Co-ſine of Middle Latitude.

2007. Put Radius = R; Middle Lat. = M; Departure = D; Diff. of Lat. = L; Diff. of Long. = G; the Angle of the Courſe = C; and Diſtance run = S. Then by the laſt, we have cs M : R :: D : G; and by (1957) it is R : t C :: L : D. Therefore R × D = cs M × G = t C × L, which gives this Theorem, *the Co-Sine of Middle Latitude is to the Tangent of the Courſe as the Difference of Latitude is to the Difference of Longitude.*

2008. Then becauſe the Diſtance ſailed L M is to the Departure M O, as Radius to the Sine of the Courſe; that is, as S : D :: R : sC; we have R × D = sC × S = cs M × G; which gives this Theorem, *as Co-Sine of Middle Latitude is to the Sine of the Courſe, ſo is the Diſtance run to the Difference of Longitude.*

2009. Having thus premiſed the Principles of each *Species* of *Sailing*, we ſhall illuſtrate the ſame by Figures adapted to them ſeparately. And firſt, as to PLAIN SAILING, we have ſhewn (1957, &c.) that it conſiſts only in the *ſix Caſes* of a *plain right-lined Triangle*; which therefore let be denoted by A B C (Fig. 9.) in which A C is the *Difference of Latitude*; A B, the Departure; B C, the Diſtance ſailed; and A C B, the Angle of the *Courſe*; any two of which being known, the reſt are found by *plain Trigonometry* (711 to 716.)

2010. MERCATOR'S SAILING has all its Caſes expreſſed in two ſimilar Triangles in one Figure, *viz.* A C B and D C E (Fig. 10.) in which A C is the Length of the Arch of the Meridian in proper or natural Degrees of the Latitude, or Difference of Latitude between two Ports. C D is the ſame enlarged according to (1968) and is called the *meridional Difference of Latitude.* A B is the *Departure.* D E the *Difference of Longitude.* C B, the *natural*, and C E the *enlarged Diſtance* run, and C the *Angle of the Courſe.* Here are alſo *ſix Varieties* or different Caſes of Sailing from two *Data*, as before.

2011. In MIDDLE-LATITUDE SAILING, two plain right-angled Triangles are alſo uſed, but not *ſimilar* ones, *viz.* A C B

and DFE (Fig. 11.) in which AC = FD is the *Difference of Latitude*. AB the *Departure*, DE the *Difference of Longitude*, CB is the *Diſtance ſailed*; and C the *Angle of the Courſe*. Two of theſe being given, the others are found by the Theorems above (2006, &c.)

2012. But that the Reaſon of this complex Figure of Triangles may appear, it is to be obſerved, that if GH be drawn parallel to AB or DE, it will be the *Tangent* of the Courſe C, and GF the *Co-Sine* of the *Middle Latitude* to the Radius CG. For on the Center G deſcribe the Semicircle CLN; and from L ſetting off the Middle Latitude to M, we have MK the *Sine*, and KG = FG the *Co-Sine* thereof; and therefore by ſimilar Triangles GFH, DFE, we have GF : GH :: FD : DE; that is, the *Co-Sine of Middle-Latitude is to the Tangent of the Courſe, as the Difference of Latitude to the Difference of Longitude*; as before in (2007).

2013. Again; we have CG : GH :: CA : AB; and GF : GH :: DF (= AC) : DE : conſequently it is GF : CG :: AB : DE, or *Co-Sine of Middle Latitude to Radius, as the Departure to the Difference of Longitude*, agreeable to (2006). And ſo of the other Theorems.

2014. Another Species of Navigation, is called, PARALLEL SAILING; which conſiſts in conducting the Ship due EAST or WEST upon a PARALLEL to the Equator. The Diagram by which this is repreſented, is that of Fig. 12. Where P is the Pole of the World; PC, PD, two Meridians; GH an Arch of the Equator, and EF a ſimilar Arch of a Parallel of Latitude. Then we have PC : CD :: PA : AB, or *Radius to the Difference of Longitude as Co-Sine of the Latitude to the Diſtance ſailed in the Parallel* (1964).

2015. This Sort of Sailing therefore admits of three Caſes only, *viz.* (1.) When AB and AP are given to find CD. (2.) When AB and CD are given to find PA. And (3.) When AP and CD are given to find AB. All which is evident by Inſpection.

CHAP.

CHAP. IV.

The THEORY *of* NAVIGATION *by* Logarithmic TANGENTS *exemplified in a Solution of the ſeveral Caſes of* SAILING *by that* Method.

2016. AS all the above Methods of computing the Way and Courſe of a Ship conſiſt in the Solution of a plain right-angled Triangle, they require no Example; but the Manner of ſolving the Propoſitions of Sailing by the Tables of *Logarithmic Tangents* is not quite ſo common, nor ſo clear, in what we have had hitherto publiſhed on that Subject. It may be proper therefore to illuſtrate the Theory we have given by a Solution of ſuch Caſes in which the *Longitude* is concerned, eſpecially as it will from thence appear that this Invention ſuperſedes the Neceſſity of a Table of *meridional Parts*, and is when joined with *plain Sailing*, a compleat Compendium of practical Navigation.

2017. If we retain the Notation in 1987, *&c.* and, moreover, put L for the Difference of Longitude, C for the Ship's Courſe, and G for the conſtant natural Tangent of 51° : 38′ : 9″; then we have this Analogy $\overline{T-\mathcal{T}} \times 10000 : G :: L : tC$ (by 1997) which admits of *three Caſes*, beſides thoſe of plain Sailing, *viz.*

2018. CASE I.

Given the Difference of Latitude, and Difference of Longitude, to find the Ship's Courſe; the Theorem is $\frac{G \times L}{\overline{T-\mathcal{T}} \times 10000} = tC$.

2019. CASE II.

Given the Difference of Latitude and Courſe, to find the Difference of Longitude; the Theorem is $\frac{\overline{T-\mathcal{T}} \times 10000 \times tC}{G} = L$.

2020. CASE III.

Given the Courſe, and Difference of Longitude, to find the Difference of Latitude; the Theorem is $\frac{G \times L}{tC \times 10000} = \overline{T-\mathcal{T}}$.

2021. For an Example of the firſt Caſe, ſuppoſe a Ship ſails from the *Lizard* in Latitude 49° 55′ North to *Barbadoes* in Lat. 13° 10′ North; the Difference of Longitude being 53° 00′, required the Rhumb on which the Ship was ſteered?

The Solution of this Caſe is as follows:

	Lat.	Comp.	Halves	Log. Tang.
Barbadoes	13 10 —	76 50 —	38 25 —	9,8993082 = T.
Lizard	49 55 —	40 05 —	20 2½ —	9,5620477 = *t*.
The Difference is			T — *t* =	0,3372605
Multiply by				10000
The Product is $\overline{T-t}$ × 10000			— =	3372,605
Then the Tang. of 51° 38′ 09″ = G =				10,1015104
12633210				
Mult. Diff. of Long. L = 53° 00′ = 3180			—	3,5024271
The Product is G × L			—	13,6039375
Which divide by $\overline{T-t}$ × 10000 =				3,5279654
3372,605			—	
The Quotient is the Tangent of the Courſe required			*t*C = 49° 59′ 10″ —	10,0759721

2022. Then to find the Diſtance ſailed, uſe this Proportion of *plain Sailing*, viz. *As Co-Sine of the Courſe is to Radius, (or as Radius to the Secant of the Courſe)* ſo is the Difference of Latitude A C to the Diſtance run C B = 3429,38 nautical Miles or Minutes of a Degree. (See Fig. 9.)

2023. Example to Case II.

Suppoſe a Ship ſails from Latitude 49° 55′ *to Latitude* 13° 10′, *on a Rhumb or Courſe of* 49° 59′ 10″; *to find the Difference of Longitude.* The Solution will ſtand thus. (See Theorem 2019).

Multiply $\overline{T-t}$ × 1000 = 3372,605	—	3,5279654
By Tang. of the Courſe *t* C = 49° 59′ 10″	—	10,0759721
		13,6039375
Divide by conſtant Tang. G = 51° 38′ 9″	—	10,1015104
The Quotient is Diff. of Long. L = 53° 00	—	3,5024271

Thus,

Thus, by two good Obſervations of the Latitude and the Courſe ſteered, the Reckoning of a Ship's Way is beſt aſcertained, eſpecially if you ſail near the North or South.

2024. Example to CASE III.

Admit a Ship ſails from Latitude 49° 55′ *on a Courſe of* 49° 59′ 10″ *ſoutherly, till her Difference of Longitude be* 53° 00′, or 3180 *Miles; to find the Latitude ſhe is then in.* The Solution of this Caſe is thus (by Theorem 2020).

To the Log. of the conſtant Tang. G —— 10,1015104
Add the Log. of the Diff. of Long. L = 3180 —— —— } 3,5024271
The Sum is the Log. of G × L —— 13,6039375
Then to the Tang. of Courſe = 49° 59′ 10″ —— —— —— } 10,0759721
Add the Logarithm of 10000 = 4,0000000
The Logarithm of *t* C × 10000 —— 14,0759721
Then $\frac{G \times L}{t\,C \times 10000} = \overline{T - \mathcal{T}}$ = 0,3372605 —— } —— 9,5279654

Now one Latitude being given 49° 55′, the Logarithm
Tang. of half its Complement is — 9,5620477 = $\mathcal{T}$.
To which add —— 0,3372605 = T — $\mathcal{T}$.
The Sum is the Log. Tang. 38° 25′ — 9,8993082 = T.
The Double of which is 76° 50′, which is the Complement of 13° 10′, the Latitude of the Ship required. (See 2021).

2025. From hence it is evident, that theſe three Caſes, added to thoſe of *plain Sailing*, are ſufficient for all practical Computations of a Ship's Ways, or for keeping a Reckoning at Sea, which are all performed by a *Table of Logarithmic Tangents*, or by the *common Gunter* properly made, as we ſhall hereafter ſhew, when we come to give the *Theory and Conſtruction of mathematical Inſtruments.*

2026.

2026. What has been hitherto delivered is upon the common Hypotheſis, that the *terraqueous Globe* is of a *ſpherical Figure*, but that it is not ſuch is well known; and not only ſo, but the Difference therefrom is ſo conſiderable, as not to be without its ſenſible Effects in *Aſtronomy*, *Geography*, and *Navigation*, as I have at large ſhewn in my NEW PRINCIPLES of GEOGRAPHY and NAVIGATION, and have there conſtructed large Charts to 70 Degrees Latitude, and new Table of meridional Parts adapted to the true *ſpheroidical Figure* of the Earth; by which the Solutions of all Caſes of Sailing are rendered juſt as eaſy as by the falſe Charts and erroneous Tables in preſent Uſe, which are no ſmall Diminution of the Glory of *naval Science* in ENGLAND. What farther relates to this Subject we propoſe to deliver in the *Theory* and Conſtruction of Inſtruments uſed in Navigation (in which there is great Room for Improvement) but this we muſt defer to a future Part of this Work, and proceed with our Inſtitutions relative to the Theory of the Sciences, which ought to precede the mechanical and inſtrumental Part.

PHY-

PHYSICO-BALLISTICS:

OR,

INSTITUTIONS of GUNNERY.

CONTAINING

The *philoſophical* and *mathematical* ELEMENTS of that Art, connected with, and illuſtrated by the *Experiments* publiſhed by the late Mr. ROBINS.

CHAP. I.

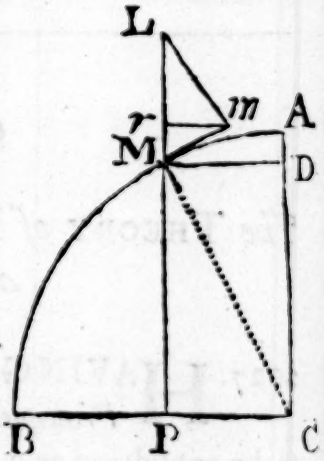
The THEORY *of* RESISTANCE *to* BODIES *moving in a* reſiſting Medium.*

2027. HAVING in the preceding Part delivered the true *Principles* of the Art of *navigating* a *Ship*, which is to be conſidered as a Part of *Military Science*, we think it may be very properly connected with its Siſter Science, BALLISTICS, or

* It appears, (ſays Mr. ROBINS) that the modern Writers on the Art of Gunnery have been very much deceived, in ſuppoſing the Reſiſtance of the Air to be inconſiderable, and thence aſſerting, that the Track of Shot and Shells of all Kinds is nearly in the Curve of a Parabola. That by this Means it has happened, that all their Determinations about the Flight of Shot diſcharged with conſiderable Degrees of Celerity are extreamly erroneous, and conſequently that the preſent Theory of Gunnery in this its moſt important Branch is uſeleſs and fallacious. *Preface to principles of Gunnery.*

or the *Doctrine* of *Projectiles* (not *in Vacuo*, but) in a *reſiſting Medium*, eſpecially as nothing conſiderable has been wrote directly on this Subject in our Language to explain the true PRINCIPLES or THEORY of GUNNERY, nor any Thing of a practical Nature, but a Treatiſe of Gunnery confirmed by Experiments by Mr. ROBINS, an excellent Performance it is true, but unleſs the *phyſical* PRINCIPLES or THEORY on which the greateſt Part depends, be explained and connected with the Experiments, the *Art* of GUNNERY will, after all, be deficient in the moſt eſſential Part, as we preſume will be ſufficiently evident from the enſuing Diſcourſes.

2028. Since the Path of Projection is through a *reſiſting Medium*, the firſt Thing neceſſary is to eſtabliſh the THEORY of RESISTANCE to Bodies moving in a *fluid Medium*, where the Particles, both in the Body and the Medium, are ſuppoſed to be abſolutely void of any elaſtic Force, and to act upon each other only by Virtue of the *Vis inſita*. We ſhall here alſo ſuppoſe the Fluid to be quite perfect, or free from all *Tenacity* of Parts, and that the Friction is of Courſe nothing at all.



2029. Therefore let A M B be any Curve moving in a Fluid in the Direction of the Axis C A; then to any Point M let be drawn a Tangent, in which take $M m = \dot{z} =$ Fluxion of the Curve; and let L M P be drawn perpendicular to the Baſe B C, and draw the Perpendiculars L m and $m r$; and put $CB = a$, $CP = x$, $PM = y$; then $M r = \dot{y}$, and $r m = \dot{x}$; and becauſe of ſimilar Triangles C M P and M r m, we have $CM\ (=a) : PM\ (=y) :: M m\ (=\dot{z}) : r m\ (=\dot{x})$; whence $\frac{a\dot{x}}{y} = \dot{z}$, and $\dot{z}^2 = \frac{a^2\dot{x}^2}{yy}$.

2030. Let the abſolute Force of a direct Stroke of a Particle of the Fluid be repreſented by L M, and ſince the Stroke at M is oblique to the Curve, it muſt be reſolved into two, *viz.* M m, and L m; the firſt of which does not affect the Curve, but the latter acts perpendicular to it. And therefore the whole Force is to the oblique Force as L M to L m, or as Radius to the Sine of In-

Incidence (1033). Wherefore the whole Force being = 1, that by which the Point M is ſtruck will be as $\frac{\dot{x}}{\dot{z}}$; for becauſe of ſimilar Trian. L*m* M, M*mr*, we have M*m* ($= \dot{z}$) : *mr* ($= \dot{x}$) :: LM ($= 1$) : L*m* $= \frac{\dot{x}}{\dot{z}}$.

2031. But this Force L*m* muſt be reduced to another acting againſt the Point M in the Direction in which it moves, that is, it muſt be reſolved into the two Forces L*r* and *mr*; of which the latter being perpendicular to P L does no Ways hinder its progreſſive Motion; but the former L*r* acts wholly againſt it. But M*m* ($= \dot{z}$) : *mr* ($= \dot{x}$) : L*m* ($= \frac{\dot{x}}{\dot{z}}$) L*r* $= \frac{\dot{x}^2}{\dot{z}^1}$, which therefore expreſſes the Reſiſtance to the Particle M moving in the Direction M L.

2032. But as this Force acts perpendicularly (not to the Curve, but) to the ordinate M D, it muſt be multiplied by the Fluxion thereof, and the Product, *viz.* $\frac{\dot{x}^2}{\dot{z}^2} \times \dot{x} = \frac{\dot{x}^3}{\dot{z}^2}$, will be the Expreſſion for the Flux. of the Reſiſtance of the Arch A M. Then ſince $a : p :: x : \frac{px}{a}$ = Circumference of a Circle whoſe Radius is x; if we multiply this Circumference by the Fluxion of Reſiſtance, the Product $\frac{p x \dot{x}^3}{a \dot{z}^2}$ will be the Fluxion of Reſiſtance to the Surface of the Solid generated by the Revolution of the Curve A M about its Axis A C.

2033. In order to find the Fluent or Reſiſtance for the Surface of a Globe, we have $\dot{z}^2 = \frac{a^2 \dot{x}^2}{yy}$ (2029) therefore $\frac{p \dot{x}^3 x}{a \dot{z}^2} = \frac{pyy x \dot{x}}{a^3}$; but $yy = aa - xx$ (828) whence $\frac{py^2 x \dot{x}}{a^3} = \frac{p}{a^3} \times \overline{aax\dot{x} - x^3\dot{x}}$, whoſe Fluent is $\frac{p}{a^3} \times \overline{\frac{1}{2}a^2x^2 - \frac{1}{4}x^4}$, for the Reſiſtance; and ſo when $x = a$, we ſhall have $\frac{1}{4}ap$ for the Expreſſion of the Reſiſtance to the Surface of the Hemiſphere, or Globe moving in the Fluid.

2034. If a Cylinder were to move in the ſaid Fluid in the Direction of the Axis, ſince the Stroke of every Particle would be direct, and the whole proportional to the Area of the Baſe; it is evident (ſuppoſing the Diameter of the Cylinder and Globe equal) that the whole Reſiſtance to the Cylinder will be (as the Area of its Baſe) $\frac{ap}{2}$ (ſee 830) *whence it appears, the Reſiſtance to the Globe is to that of a Cylinder (of equal Diameter) as* 1 *to* 2.

2035. A Cylinder falling upon ſuch a Fluid, will in the Time of paſſing through its whole Length, communicate a Motion to the Particles, as will be to the whole Motion of the Cylinder, as the Denſity of the Medium to the Denſity of the Cylinder. For let Q, M, V, B, D, be the Expreſſions for the *Quantity* of *Motion*, *Matter*, *Velocity*, *Bulk*, and *Denſity* of the Cylinder; and q, m, v, b, d, denote the ſame Things in the Fluid. Then the Motion of the Cylinder will be $Q = MV$, and that communicated to the Fluid will be $q = vm$ (970); but it is $M = BD$, and $m = bd$ (973). Whence $Q = VBD$, and $q = vbd$; and ſo $Q : q :: VBD : vbd$; but $VB = vb$, therefore $q : Q :: d : D$.

2036. If the Axis and Diameter of the Cylinder be equal to the ſame in a Globe; it is evident, ſince the Reſiſtance to the Globe is but $\frac{1}{2}$ that of the Cylinder, the Globe, that it may communicate the ſame Motion to the Fluid, muſt move through twice the Length of its Diameter. If the Velocity and Denſity of the Cylinder and Globe be equal, (and $\mathcal{Q}$, M, V, B, D, ſtand for the ſame Things in the Globe) then, ſince $V = V$, it is $\mathcal{Q} : Q :: (M : M :: B : B) :: 2 : 3$ (837). Alſo the Motion of the Cylinder is to that communicated by the Globe in paſſing through two Diameters, as the Denſity of the Cylinder (or Globe) to that of the Medium, *viz.* $Q : q :: D : d$ (2035). Laſtly, let q be the Motion communicated to the Medium by the Globe in paſſing through $\frac{4}{3}$ of it's Diameter; then $q : q :: 2 : \frac{4}{3} :: 3 : 2$. And by compounding theſe Ratios, we ſhall have $\mathcal{Q} : q :: D : d$; that is, the *whole Motion in the Globe is to that communicated to the Medium (equal to the Reſiſtance of the Medium to the Globe) in paſſing through* $\frac{4}{3}$ *of its Diameter, as the Denſity of the Globe to the Denſity of the Medium.*

2037.

2037. If the Particles of the Cylinder, Globe, and Medium be ſuppoſed perfectly elaſtic, then the Velocity of the Parts, after Percuſſion, will be double to what it is in the Medium before mentioned (1015) and therefore the ſame Effect will be produced in half the Time, or in paſſing through half the Space. Conſequently, in a perfectly elaſtic Fluid, *a Globe*, in the Time of paſſing through $\frac{2}{3}$ Parts of its Diameter, will communicate a Motion to the Particles that will be to the whole Motion of the Globe, as the Denſity of the Fluid to the Denſity of the Globe.

2038. There is no ſuch Thing in Nature as a *perfect Fluid*, or ſuch whoſe Parts are abſolutely free, and unaffected by any external Force; for all Fluids are heavy Bodies, that is, their Parts gravitate towards the Center of the Earth, and conſequently on one another. Therefore the lower Parts will be compreſſed by the Weight of thoſe above them; and therefore, when a Body is conſidered as moving through ſuch a compreſſed Fluid, the Motion or Velocity of the Body will be leſs than that with which the Particles will ruſh into the Space relinquiſhed by the Body, by Virtue of the Compreſſion, I ſay, in this Caſe we are to conceive the Force of Re-action on the fore Part of the Body, in ſome Degree abated by the Particles circulating round to fill up the relinquiſhed Space, in order to reſtore the Equilibrium (that would otherwiſe be deſtroyed) by the conſtant Influx of the Fluid behind the Body.

2039. This complexed Caſe of real Fluids makes the Computation of their Reſiſtance to Bodies very difficult and tedious, but Sir *Iſaac Newton* has fully conſidered the Thing,* and found that the Reſiſtance to a Cylinder moving in ſuch a compreſſed Fluid, is but a fourth Part of what we have ſhewed it would be in a perfectly non-elaſtic Fluid (2035.) ſuppoſing the Velocity of the Body and Denſity of the Fluid the ſame in both Caſes; that is, *the Motion imparted to ſuch a Fluid in the Time it paſſes through four Times its Length, is to the whole Motion of the Cylinder, as the Denſity of the Fluid to that of the Cylinder.*

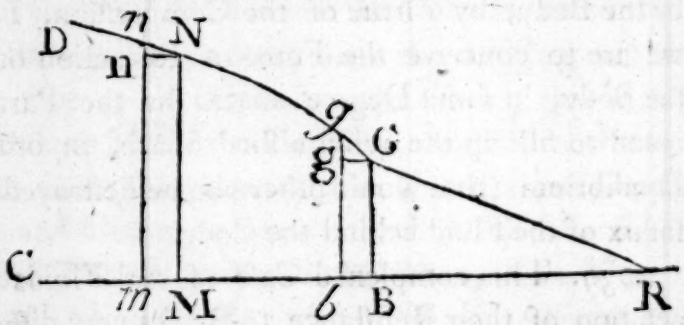
2040. In ſuch a compreſſed Fluid, Bodies of every Form (if their tranſverſe Sections are equal) will be equally reſiſted; for if a *Cylinder*, *Globe*, *Cone*, *circular Plane*, &c. of equal Diame-

* See his Principia, Lib. II. Prop. 37, &c.

ters were placed in a Canal of ſuch a *current Fluid*, it is evident the Fluid would be equally obſtructed by them all, or the increaſed Velocity of the Fluid paſſing by each will be the ſame; they therefore equally reſiſt the Fluid, and, were they to move in the Fluid, would be equally reſiſted by it. Conſequently, ſince the Motion of a Globe is but $\frac{2}{3}$ of that of the Cylinder circumſcribing it, the Force that will generate or deſtroy the whole Motion of the Cylinder while it moves through 4 Times its Length, will generate or deſtroy the whole Motion of the Globe while it moves uniformly through $\frac{2}{3}$ of 4 Diameters, or $\frac{8}{3}$ Parts of its Diameter, that is, through $2\frac{2}{3}$ Diameters.

2041. It will be neceſſary here to obſerve, that the *Theory of Reſiſtance* in a rare elaſtic Medium is, by the *Newtonian Matheſis*, arrived to its Perfection, ſince thereby it is determined what the Figure or Form of a Body muſt be, which, moving in ſuch a Medium, ſhall meet with the *leaſt Reſiſtance poſſible*. And as this Part of mathematical Philoſophy is of primary Conſideration in *Naval Architecture*, and *Military* Science, we ſhall here explain the Principles of the *Calculus*, as follows.

2042. On the Right-line B C, ſuppoſe the Parallelograms B G g *b*, M N n *m*, of the leaſt Breadth, to be erected, whoſe



Heights B G, M N, their Diſtance M *b*, and half the Sum of their Baſes $\frac{1}{2}$ M *m* + $\frac{1}{2}$ B *b* = *a*, are given: Let half the Difference of the Baſes $\frac{1}{2}$ M *m* — $\frac{1}{2}$ B *b* be called *x*: Let G and N be Points in the Curve G N D; and producing *b* g, and *m* n to *g* and *n*, (ſo that g *g* = n *n* = *b*,) the Points *g* and *n* may alſo be in the ſame Curve.

2043. Now if the Figure C D N G B, revolving about the Axis B C, generates a Solid, and that Solid moves forwards in a rare and elaſtic Medium from C towards B, (the Poſition of the Right-line B C remaining the ſame;) then will the Sum of the Reſiſtances againſt the Surfaces generated by the Lineolæ

G *g*,

Gg, Nn, be the leaſt poſſible, when $\overline{Gg}^4$ is to $\overline{Nn}^4$ as $BG \times Bb$ to $MN \times Mm$.

2044. For the Force of a Particle on Gg and Nn, to move them in the Direction BC, is as $\frac{1}{\overline{Gg}^2}$ and $\frac{1}{\overline{Nn}^2}$ (2031); and the Number of Particles that ſtrike in the ſame Time on the Surfaces generated by Gg and Nn, are as (the *Annuli* deſcribed by gg and nn, that is, as $BG \times gg$ and $MN \times nn$, or as) BG and MN; therefore the Reſiſtances againſt thoſe Surfaces are as $\frac{BG}{\overline{Gg}^2}$ to $\frac{MN}{\overline{Nn}^2}$, that is (putting y for $\overline{Gg}^2$, and z for $\overline{Nn}^2$,) as $\frac{BG}{y}$ to $\frac{MN}{z}$.

2045. But the Sum of theſe Reſiſtances $\left(\frac{BG}{y} + \frac{MN}{z}\right)$ is a *Minimum*. Therefore $-BG \times \frac{\dot{y}}{yy} - MN \times \frac{\dot{z}}{zz} = 0$, or $MN \times \frac{\dot{z}}{zz} = -BG \times \frac{\dot{y}}{yy}$: But $y = (\overline{Gg}^2 = \overline{Bb}^2 + \overline{gg}^2 =) aa - 2ax + xx + bb$; and $z = (\overline{Nn}^2 = \overline{Mm}^2 + \overline{nn}^2 =) aa + 2ax + xx + bb$; therefore $\dot{y} = 2x\dot{x} - 2a\dot{x}$, and $\dot{z} = 2a\dot{x} + 2x\dot{x}$: Conſequently $\frac{MN}{zz} \times 2\dot{x} \times \overline{a + x} = \frac{BG}{yy} \times 2\dot{x} \times \overline{a - x}$; or $\left(\frac{MN}{zz} \times \overline{a + x} =\right) \frac{MN}{zz} \times Mm = \left(\frac{BG}{yy} \times \overline{a - x} =\right) \frac{BG}{yy} \times Bb$. (2042.) Therefore $(yy)\ \overline{Gg}^4 : (zz)\ \overline{Nn}^4 :: BG \times Bb : MN \times Mm$.

2046. Conſequently, that the Sum of the Reſiſtances againſt the Surfaces generated by the *Lineolæ* Gg and Nn, may be the leaſt poſſible, $\overline{Gg}^4$ muſt be to $\overline{Nn}^4$ as $BG \times Bb$ to $NM \times Mm$.

2047. Wherefore, if gg be made equal to gG, ſo that the Angle gGg may be 45°, and the Angle $BGg = 135°$; alſo $\overline{Gg}^2 = 2\overline{gg}^2$, and $\overline{Gg}^4 = 4gg^4$; then $4\overline{gg}^4 : \overline{Nn}^4 :: GB \times Bb$

$\times B b : NM \times M m$; and ſince GR is parallel to $N n$, and BG, BR parallel to $n n$, $N n$; alſo $n n = g g = g G$; it follows, that $(n n = g G =) B b : (N n =) M m :: BG : BR$; therefore $B b = \frac{BG \times M m}{BR}$; alſo $(n n =) g G : N n :: BG : GR$. Conſequently, $\left(\frac{4\overline{gg}^4}{\overline{Nn}^4} =\right) \frac{4\overline{BG}^4}{\overline{GR}^4} = \left(\frac{BG \times Bb}{MN \times Mm} =\right) \frac{\overline{BG}^2}{MN \times BR}$. Therefore $4\overline{BG}^2 \times BR$ is to $\overline{GR}^3$ as GR to MN.

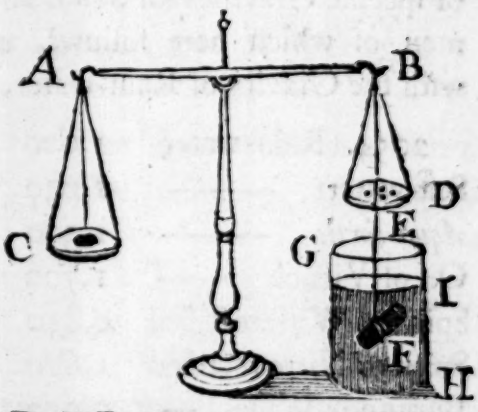
2048. How this Curve DNG is to be conſtructed by Means of the *Logarithmic Curve*, and from thence a practical Method of forming the SOLID of *leaſt* RESISTANCE, we may hereafter ſhew, in its proper Place, when we ſhall have premiſed and explained the *general Doctrine* of *Curves*.

CHAP. II.

The DOCTRINE *of* abſolute, ſpecific, *and* relative WEIGHT *in* BODIES, *explained on the* ſtatical PRINCIPLES *of* reſiſting FLUIDS, *as* WATER, AIR, *&c.*

2049. BEFORE we proceed any farther, it will be neceſſary (in the THEORY of BALLISTICS) to premiſe a few Things relating to the various Diſtinctions of GRAVITY, or Weight in Bodies, *viz.* *abſolute*, *relative*, and *ſpecific Gravity*. *Abſolute Weight* is the *whole Weight* of a Body *in Vacuo*. *Relative Weight* is that which a Body has when weighed in a reſiſting Medium. *Specific Gravity* is the *comparative Weight* of Bodies of *equal Bulk*. Thus if the Weight of a Cubic Inch of *Water* be to that of a Cubic Inch of *Copper*, as 1 to 9, we ſay the *Specific* Gravity of *Copper* is 9, or its Weight is 9 Times greater than that of Water.

2050.



2050. But to illuſtrate this Affair, let A B be a Ballance, and from the Scale D let any Body F be ſuſpended by a fine Horſe Hair E, and equiponderated in the Air with Weights put into the other Scale C. After this, immerſe it in the Fluid I H, contained in the Jar G H, and the Equilibrium will be deſtroyed. For the Body F will now appear lighter, and aſcend, while the Scale C deſcends.

2051. That the Weight of the Body F ſhould be diminiſhed in the Fluid, is hence evident, that it cannot deſcend therein without raiſing at the ſame Time an *equal Bulk of the Fluid*, which Quantity of the Fluid, thus raiſed, will reſiſt the Body F with all its Force (equal to its Weight) and therefore will deſtroy juſt as much Motion (or Weight) in the Body F (965). And therefore, by putting Weights in the Scale D, till the Equilibrium is again reſtored, we ſhall have the *Compariſon of Weight between the Body* F, *and the Fluid, in equal Bulks:* For the Weight of the Solid is in the Scale C, and that of an equal Bulk of the Fluid in the Scale D; and hence the ſpecific Gravity of the Solid and Fluid becomes known; for which Reaſon this Inſtrument is called the HYDROSTATIC BALANCE.

2052. If from the Weight of the Solid, in Air, we deduct the Weight of an equal Bulk of the Fluid, the Remainder is the *relative Gravity* of the Body; and is all that Force by which the Body *ſinks* or deſcends in the Fluid. But if the Weight of the Solid be leſs than that of an equal Bulk of the Fluid, it is evident ſuch a Solid cannot be totally immerſed by its Weight, but will *ſwim* with a Part extant above the Fluid. If the ſpecific Gravity of the Solid and Fluid be equal, the Solid will retain any Poſition in the Fluid, and neither *ſink nor ſwim*. Hence the *Rationale* of SINKING and SWIMMING is evident.

2053. Not only the ſpecific Gravity of Solids, but alſo of Fluids become known by the ſame Balance; for if the Solid F be weighed in two different Fluids, the Weights put into the Scale D to reſtore the *Equilibrium*, each Time will expreſs the Gravities of equal Bulks of thoſe Fluids; and hence a Table

of ſpecific Gravities for Solids and Fluids may be made, a Specimen of which here follows, where the Comparifon is made, with the Gravity of Rain-water, put = 1,000.

2054. Rain-water	1,000	Ebony	1,177
Sea-water	1,030	Cork	0,240
Aqua-fortis	1,300	Good Wheat	0,757
Oil of Vitriol	1,700	White Peaſe	0,807
Spirit of Wine rectified	0,840	Bone	1,656
Spirit of Nitre rectified	1,610	Ivory	1,826
Burgundy Wine	0,953	Horn	1,840
Canary	1,033	Adamant	3,400
Red Wine	0,993	Glaſs	2,666
Diſtilled Vinegar	1,030	Flint	2,542
Cow's Milk	1,030	Marble	2,700
Urine	1,030	Mundick	4,430
Mercury Crude	13,593	Chalk	2,370
Amber	1,040	Magnet	1,840
Sulphur	1,800	Newcaſtle Coal	1,272
Borax	1,720	Oil-ſtone	2,380
Red Coral	2,689	Slate	2,740
Cinnabar natural	7,300	Alabaſter	1,875
Tartar common	1,849	Copperas Stone	4,300
Camphire	0,995	Copper Ore	3,775
Vitriol of *Dantzick*	1,715	Lead Ore	6,800
Sal Gemmæ	2,143	Biſmuth	9,700
Allum	1,714	Speltar	7,065
Nitre	1,900	Tin	7,550
Gum Arabic	1,375	Iron	7,645
Verdigreaſe	1,714	Braſs, wrought	8,000
Bees Wax	0,960	Copper	9,000
Pitch	1,190	Lead	10,131
Roſin	1,100	Silver, Sterling	10,000
Honey	1,450	Pure	11,091
Dry Box Wood	0,950	Gold, Sterling	17,150
Dry Fir	0,546	Pure	19,640
Lignum Vitæ	1,327		

2055. Becauſe it is found by Experiment, a Cubic Foot of Rain-water weighs very exactly 1000 Ounces *Averdupois*, therefore the

the Numbers in the Table will expreſs the Ounces contained in a Cubic Foot of any of the other Subſtances. An Ounce Averdupois $= 437\frac{1}{2}$, and an Ounce Troy $= 480$ Grains. But $437\frac{1}{2} : 480 :: 51 : 56$, nearly. The *Averdupois Pound* is *to the Troy Pound*, as $437\frac{1}{2} \times 16 = 7000$ to $480 \times 12 = 5760$; that is, as 17 to 14 nearly. Therefore a Cubic Foot of Water weighs $62\frac{1}{2}$ *lb.* Averdupois, and 76 *lb.* Troy nearly.

2056. What relates to the *abſolute* and *ſpecific Gravities*, the *Magnitudes*, *Denſity*, *&c.* of Bodies, will beſt be underſtood by ſymbolical Computation; in order to which let A and B be two Bodies of equal Bulk, but different Quantities of Matter; and let B and C be two other Bodies with equal Quantities of Matter, but of different Bulks:

And let $\left\{\begin{array}{l} D = \text{Denſity} \\ B = \text{Bulk} \\ M = \text{Quantity of Matter} \end{array}\right\}$ in the Body A.

Alſo $\left\{\begin{array}{l} D = \text{Denſity} \\ B = \text{Bulk} \\ M = \text{Matter} \end{array}\right\}$ in the Body B.

And $\left\{\begin{array}{l} d = \text{Denſity} \\ b = \text{Bulk} \\ m = \text{Matter} \end{array}\right\}$ in the Body C.

2057. Then, becauſe the Denſity of any Body is proportional to the Quantity of Matter under equal Bulks, we ſhall have $D : \mathrm{D} :: M : \mathrm{M}$; and, becauſe when the Quantities of Matter are equal, the Bulks muſt be reciprocally as the Denſities, therefore we have $D : d :: b : B$. Whence $D = \frac{\mathrm{D}\,M}{\mathrm{M}} = \frac{db}{\mathrm{B}}$; conſequently $\mathrm{D}\,M\,B = d\,b\,\mathrm{M}$. But $B = \mathrm{B}$, and $M = m$; therefore $\mathrm{D\,B}\,m = d\,b\,\mathrm{M}$. Whence we have $\mathrm{D} : d :: b\,\mathrm{M} : m\,\mathrm{B}$; and $\mathrm{B} : b :: d\,\mathrm{M} : \mathrm{D}\,m$; and $\mathrm{M} : m :: \mathrm{D\,B} : d\,b$.

2058. The *ſpecific Gravity* of Bodies being as the Weights, that is, as the Quantities of Matter, in equal Bulks, will be as the Denſity: Therefore $\mathrm{D} : d :: \mathrm{S} : s$; and by Subſtitution of Ratios, we have the general Theorem above become $\mathrm{S\,B}\,m = s\,b\,\mathrm{M}$. And ſince the abſolute Weights (A, *a*,) of any two Bo-

dies are as the Quantities of Matter, we have $SB\,a' = A\,s\,b$. Wherefore $S : s :: A\,b : a\,B$; that is, the ſpecific Gravities will be as the abſolute Weights directly, and the Bulks inverſely, or as the abſolute Weights divided by the Bulks.

2059. Alſo $A : a :: SB : s\,b$; that is, the abſolute Weights of Bodies are in the compound Ratio of their ſpecific Gravities and Bulks. Or the abſolute Weight of any Body is had by multiplying its Bulk and ſpecific Gravity together.

2060. Again; becauſe $B : b :: A\,s : a\,S$, it appears that the Bulk or Magnitudes of Bodies will be as the abſolute Weights directly, and ſpecific Gravities inverſely. Or the Magnitude of any Body is had by dividing its abſolute Weight by its ſpecific Gravity.

2061. From what we have ſaid, it is evident we have not the *abſolute Weight* of Bodies in the Air, but only the Relative. Let A = abſolute Weight, and B = relative Weight; then $A - B = B$ the Weight of an equal Bulk of Air, (2051) and the ſame may be ſaid for any other Fluid. Wherefore let the Motion of a Globe moving in a Fluid be ſuch as will be generated by the Force of its relative Weight falling *in Vacuo* through a Space (S) that ſhall be to $\frac{4}{3}$ of its Diameters ($\frac{4}{3}$ D) as the Denſity of the Globe (D) to the Denſity of the Fluid (d), *then will the Velocity it will acquire by the Fall be the greateſt it can poſſibly acquire by deſcending in the Fluid; and the relative Weight of the Globe will be equal to the Reſiſtance, ariſing from the Medium to the Globe*, as mentioned in (2039, 2040).

2062. For let R = Reſiſtance, F = Force that will generate or deſtroy the Motion of the Globe while it deſcribes $\frac{8}{3}$ Parts of its Diameter; then $F : R :: D : d$ (2039) and ſo $F = \frac{R\,D}{d}$. And ſince S = Space deſcribed in the Fall, the Globe by an uniform Motion with the Velocity acquired by the Fall, will deſcribe a Space = 2 S in the ſame Time (993) and ſo, becauſe $S : \frac{4}{3}D :: (2S : \frac{8}{3}D)\ D : d$. But the Times T, t, in which the Spaces 2 S and $\frac{8}{3}$ D are uniformly deſcribed are as thoſe Spaces, *viz.* $T : t :: 2S : \frac{8}{3}D$, hence $T : t :: D : d$.

2063. Again, ſince in the Times T and t equal Quantities of Motion are produced (for F produces the whole Motion of the Globe in the Time t, and the relative Weight (B) of the Globe

pro-

produces the ſame in the Time T, (2061). And ſince the leſs the Time is, the greater muſt be the Force to produce a given Effect; therefore F : B :: T : t :: *D* : *d*. Conſequently F = $\frac{BD}{d} = \frac{RD}{d}$; whence R = B, and the *Reſiſtance being equal to the relative Weight*, the Body can be no longer accelerated, but muſt deſcend with an uniform Motion in the Medium, with a Velocity equal to that acquired by the Fall through S *in Vacuo*.

2064. Hence if the Denſity of the Globe and of the reſiſting Medium be given, as alſo the Velocity of the Globe in the Beginning of its Motion; then the Reſiſtance it meets with may be computed as follows. Let A = abſolute Weight of the Globe *in Vacuo*, and *B* = Weight of an equal Bulk of the Medium; then A — *B* = B = the relative Weight of the Globe in the Medium, (2061).

2065. And ſince the Spaces deſcribed in the ſame Time *in Vacuo* are as their accelerating Forces (999), therefore as the Space deſcribed *in Vacuo* by the Weight A is 16,2 Feet in one Second, the Space deſcribed *in Vacuo* by the Weight B in one Second will be known, for as A : B :: 16,2 : $\frac{16,2\,B}{A}$ = to the ſaid Space.

2066. Moreover, the Time of deſcribing the Space S *in Vacuo* by the Weight B is thus found; as $\frac{16,2\,B}{A}$: S :: 1″ : $\frac{A\,S}{16,2\,B}$ the Square of the Time required; or $\sqrt{\frac{A\,S}{16,2\,B}}$ = T the Time. But S = $\frac{4\,D\,D}{3\,d}$ (2061) whence T = $\sqrt{\frac{4\,A\,D\,D}{48,6\,B\,d}}$.

2067. The Velocity of the Fall will carry the Globe with a uniform Motion over a Space = 2S in the Time of the Fall T; and the Velocity of uniform Motion is always as the Space divided by the Time (991); therefore $\frac{2\,S}{T}$ = V = the Velocity acquired by the Fall.

2068. The Globe moving in the reſiſting Medium meets with a Reſiſtance = B (2063); if any other Velocity (*V*) be given

the Reſiſtance agreeing to that Velocity will be thus found. As $V^2 : v^2 :: B : \frac{v^2 B}{V^2}$ = Reſiſtance to the Velocity v. For that the Reſiſtance is as the Square of the Velocity in general, is hence evident; becauſe the Reſiſtance will be greater in Proportion to the Number of Particles which ſtrike againſt the Globe in a given Time, and alſo to the Intenſity of the Stroke of each Particle; and each of theſe will be as the Velocity, and therefore both together as the *Square of the Velocity*.

2069. But, as I ſaid, this Ratio of the Reſiſtance to the Velocity is only general, and agreeing to *ſlow Motions*; becauſe in very ſwift Motions the Circumſtances will be altered, on which the Reſiſtance depends, both in regard to the Medium, and to the Figure of the Body moving in it. For with reſpect to the Medium, though compreſſed, yet if the Velocity of the Body be ſo much greater than that of the Particles ruſhing in behind the Body, that a *Vacuum* is left, the Caſe of this compreſſed Fluid will be nearly the ſame with one that is free (2028); and conſequently the Reſiſtance to a Cylinder moving ſo very ſwift as to leave a vacuous Space behind, will be near 4 Times as great as when it moves ſlowly through the ſame compreſſed Fluid (2039).

2070. Again, the Reſiſtance will vary in ſwift Motions according to the Figure of the Body. In ſuch a Caſe the vacuous Space will be more or leſs, and the *compreſſed Fluid* approaches to the Nature of one that is free, wherein the Reſiſtance to a Globe has been ſhewn to be but half what it is to a Cylinder (2034); therefore the Reſiſtance to a Globe moving very ſwiftly, even in a compreſſed Fluid, (becauſe of the *Vacuum* behind) may meet with little more than half the Reſiſtance of the Cylinder moving with the ſame Velocity.

2071. But we cannot ſuppoſe the Compreſſion of the Medium not at all to affect the Body, for even to a Globe, the Particles which ſtrike it obliquely, will be in ſome Degree confined by the compreſſing Force, and therefore give more Reſiſtance than when they are free to move. Alſo if the Medium be *Elaſtic*, this Reſiſtance will be thereby farther increaſed (2037), ſo that upon the Whole we may conclude that the Reſiſtance to a Globe moving ſwiftly in a compreſſed elaſtic Medium (ſuch as our Air) will

will be between that of a Globe and of a Cylinder in a free and perfect Fluid, and that arising to either in a slow Motion from a compressed Medium; that is, it is more than *twice* and *less than 4 Times the Resistance* the same Globe would meet with moving slowly in a compressed Medium.

2072. Hence then we may take it for granted, that in very swift Motions, a Globe is resisted about *three Times more*, in Proportion to its Velocity, than when its Motion is slowest; and that the Resistance will decrease as the Velocity decreases, and as the Circumstances of the Medium return to their natural State. Let this suffice at present, for the *Theory of the Resistance of Fluids*, which is a *fundamental Doctrine* in the Science of BALLISTICS.*

CHAP. III.

Of the NATURE *and* PROPERTIES *of* AIR; *its* ELASTICITY, DENSITY, absolute *and* specific GRAVITY *explained by* COMPUTATION *and* EXPERIMENTS.

2073. THE Nature of ELASTICITY, and whence it arises has been already, in general, shewn (975—980). We now propose to consider the *Physico-mathematical Theory* thereof; and apply it to the Air, and other elastic Fluids, particularly that which is generated by *firing* GUNPOWDER, that the ENGINEER may be fully acquainted with the Philosophy of this important Subject.

2074. In order to this, let there be placed any Number of Particles in the right-lined Distance AB at an equal

A . . . . . B
C . . . . . D

Distance

* It is necessary here to correct an Error in the Table of specific Gravities, Pape 296, where that of a *Magnet* is said to be 1,840, whereas it should be 4,840, as I have found by Experiment. This Error also passed unobserved in the larger Table of the PHILOSOPHIA BRITANNICA, 2d Edition.

Diſtance from each other; and in any other equal Diſtance C D let there be placed twice as many Particles, at equal Intervals alſo, then it is plain the Intervals between the Particles in C D will be but half ſo great as thoſe between the Particles in the Line A B. Hence the Number (N) of Particles in the Lines A B, C D, will be inverſely as the Interval (L) between each, that is N will be always as $\frac{1}{L}$.

2075. In a *Superficies*, ſuppoſing the Intervals of all the Particles equal, we ſhall have N^2 as $\frac{1}{L^2}$ (670); alſo in a Solid it will be N^3 as $\frac{1}{L^3}$ (675). But N^2, and N^3 is as the Denſity (D) of the Surface, and of the Solid (973); therefore D is as $\frac{1}{L^2}$ in a Surface, and as $\frac{1}{L^3}$ in a Solid.

2076. Let us now ſuppoſe each Particle repels the Particles next to it (and thoſe only) with a Force as (F) which is as any Power (n) of the Interval inverſely; that is, let F be as $\frac{1}{L^n}$. Now the Sum or whole Force in the Superficies will be as the Denſity D, and as the centrifugal Force F; or as $D \times F = \frac{1}{L^2} \times \frac{1}{L^n} = \frac{1}{L^{n+2}}$. Which therefore will expreſs the *elaſtic Force* of the Fluid.

2077. But in the Body of the Fluid the Denſity being as $\frac{1}{L^3}$ (2075), we have $D L^3$ as 1, L^3 as $\frac{1}{D}$, and L as $\sqrt[3]{\frac{1}{D}}$ which ſubſtituted in the Expreſſion of the elaſtic Force $\frac{1}{L^{n+2}}$ gives $D^{\frac{n+2}{3}}$; *therefore the elaſtic Force of the Medium is as the Cube Root of that Power of the Denſity whoſe Index is* n + 2.

2078. Hence if the elaſtic Force (E) in any Fluid be as the Denſity (D) then the Expreſſion $D^{\frac{n+2}{3}} = E^{\frac{n+2}{3}} = E$; and ſo $\frac{n+2}{3} = 1$, whence $n + 2 = 3$, and therefore $n = 1$;

where-

wherefore in such a Fluid F will be as $\frac{1}{L}$ (2076); that is, the *Particles repel each other with Forces that are inversely as the Distances of their Centers.*

2079. And this is the Property of our Air whose Density is always as the compressing Force, as is proved by the following Experiment. Let Mercury be poured into an inflected Tube A B C D, open at both Ends, to a small Height B C; then stopping the Orifice D very close, measure the Length of the confined Air D C very nicely; this done, pour Mercury into the other Leg A B till its Height above the Surface of that in C D be equal to the Height at which it stands in the Barometer. Then it is plain the Air in the shorter Leg will be compressed with a Force twice as great as at first; for then it was pressed only with the Weight of the Atmosphere, but now it is pressed with that Weight, and an additional equal Weight of Mercury. With this double Force the Air in D C is now compressed into the Space D E = $\frac{1}{2}$ D C, as appears by measuring it.



2080. Hence it appears, that the Spaces S = D C, and *s* = D E, which the same Quantity of Air possesses under different Pressures *p* and P, are as those Pressures inversely, *viz.* S : *s* :: P : *p*. And because the Densities *d* and D (where the Quantity of Matter is given) are inversely as the Bulks (973); therefore *d* : D :: *s* : S :: *p* : P; that is, *the Density of the Air is directly as its compressing Force, or as the Elasticity*, which is equal thereto (982).

2081.

2081. The ſpecific Gravity of Air, and thence its abſolute Weight may be diſcovered by the following Experiment. Let A *B* be a Glaſs Tube, open at both Ends, and the End B immerſed in the Water E D of the Phial C D, and let the Mouth of the Phial C be ſealed cloſe. Then let a little Air be blown through the Tube A B, and it will condenſe the Air C E in the Phial, and thereby encreaſe its Elaſticity, which will raiſe a Column of Water B F in the Tube, whoſe Weight together, with that of the Air, preſſing on the Surface F, will be equal to the encreaſed Spring of the Air C E. If now this Phial be carried to the Height of 72 Feet, or 864 Inches above the Earth's Surface, the Water will riſe from F to G, through a Space F G = 1 Inch; whence it appears, that 1 Inch of Water is equivalent in Weight to 864 Inches of Air, of the Denſity it has at the Earth's Surface; and ſo 1 Inch of Air is equivolent, or of the ſame Weight with $\frac{1}{864}$ Part of an Inch of Water. Whence the Weight of Air is to the Weight of Water (in equal Bulks) as $\frac{1}{864}$ to 1, or as 1 to 864. Sir *Iſaac Newton* has ſtated it as 1 to 860.



2082. Now becauſe it is found by Experiment, a Cubic Inch of Water weighs 253,3 Grains; therefore ſay, as 864 : 1 :: 253,3 : 0,293 = $\frac{3}{10}$ of a Grain nearly, for the Weight of a cubic Inch of Air; and becauſe a Cube is to its inſcribed Sphere as 1 : 0,524 (847); therefore ſay, as 1 : 0,524 :: 0,293 : 0,1535 = $\frac{15}{100}$ of Grain, the Weight of a Globe of Air 1 Inch in Diameter.*

2083. The ſame Inſtrument (*viz.* the Phial with the Tube of Water (2081) being held near the Fire, the Water will aſcend to the Height of ſeveral Inches in the Tube, which ſhews the *Expanſion of the Air*, or its *increaſed Elaſticity* by HEAT. On the other Hand, if the Phial be immerſed in cold Water, the Water in the Tube will deſcend ſome Inches below F, which will demonſtrate the *Contraction of the Air*, or the *Diminution of its Elaſticity*, by COLD.

2084.

* The Weight of a Pint of Air, Wine Meaſure, or 231 cubic Inches, I have conſtantly found by the *Ballance* to be 8 $\frac{1}{2}$ Grains, which is at the Rate of $\frac{3}{10}$ of a Grain, *per* cubic Inch, as above.

CHAP. IV.

Of the NATURE *and* PRODUCTION *of* artificial AIR; *and a* COMPUTATION *of the* explosive FORCE *of* GUNPOWDER *thence deduced.*

2084. THE Properties of common Air hitherto considered are not so much the Subject of our present Speculation, as the same Properties in another Subject which the modern Philosophy has discovered, and which from its perfect Similarity to common Air, is usually called *artificial* or *factitious* AIR. This Substance, in its natural State, is no other than the common Substance of all Sorts of Bodies, and which being by chymical Operations, or otherwise, set at Liberty, acquires the Form of Air in every Respect, *becoming a fine, transparent, invisible, elastic, ponderous Fluid.*

2085. One easy Method of obtaining this elastic Fluid is thus; let A B be a tall cylindric Glass, with a very small Hole E at the Bottom; in the Glass put any alkalinous Substance D as a Piece of Chalk, Tartar, *&c.* then fill the Glass to the Top with common Water (stopping the Hole E with the Finger) and to the Water put a little *Aqua Fortis*, and then stop the Orifice A close with the Cork C; this done, you will see the Acid act upon the Alkali, and produce a Fermentation which consists in the violent Eruption and Ebullition of this new generated Fluid, which will keep constantly ascending to the Top of the Vessel, in Form of small Bubbles of Air; and by its Elasticity, it will, by Degrees, force all the Water out of the Tube through the Hole E.



2086. By this Experiment the Elasticity of this new generated Air is shewn to be greater than that of the Atmosphere; and by many other Experiments, it appears to be extremely great, and to produce Effects thereby superior to those of any other Force we know of in Nature; but in order to make Estimations of this Kind, we must first know what Quantities and

Weights of this Fluid are produced from Bodies of ſeveral Sorts; and as theſe have been determined by Experiments, by ſeveral eminent Philoſophers, I ſhall here ſubjoin a Specimen thereof from the late Dr. *Hale*'s *Vegetable Statics*.

Several Sorts of Matter.	Cubic Inch of Matter.	Cubic Inches of general Air.	Weight of the Body.	Weight of general Air.	Proport. Part of the Whole.
2087. Dear's Horn	$\frac{1}{2}$	117	241	33	$\frac{1}{7}$
Oyſter Shell ——	$\frac{1}{2}$	162	266	46	$\frac{1}{6}$
Heart of Oak ——	$\frac{1}{2}$	108	135	30	$\frac{1}{4}$
Indian Wheat ——		270	388	77	$\frac{1}{3}$
Peaſe —— ——	1	396	318	113	$\frac{1}{2}$
Muſtard Seed ——		270	437	77	$\frac{1}{3}$
Amber —— ——	$\frac{1}{2}$	135	135	38	$\frac{1}{3}$
Dry Tobacco ——		153	142	44	$\frac{1}{3}$
Honey with Calx of Bones	1	144	359	41	$\frac{1}{3}$
Yellow Wax ——	1	54	243	15	$\frac{1}{15}$
Coarſe Sugar ——	1	126	373	36	$\frac{1}{10}$
Newcaſtle Coal ——	$\frac{1}{2}$	180	158	51	$\frac{1}{3}$
Nitre with Bone Calx	$\frac{1}{2}$	90	211	26	$\frac{1}{8}$
Rheniſh Tarter ——	1	504	443	144	$\frac{1}{3}$
Calculus Humanus —	$\frac{3}{4}$	516	230	147	$\frac{2}{3}$

2088. By this Table it appears how greatly this Fluid is condenſed in its natural State in Bodies; thus $\frac{3}{4}$ of an Inch of Calculus Humanus, produced 650 Times its own Bulk of Air, which made nearly one Half of its fixed Subſtance. When this Air, therefore, comes to be ſet at Liberty, no wonder we ſee ſuch violent Expanſions, Exploſions, Incaleſcences, *&c.* as happen in firing the *Pulvis fulminans*, GUNPOWDER, and in the Mixtures of various Sorts of Fluids. For ſince it has been found by many Experiments, that the Elaſticity and Weight of this Air is the ſame with that of the common Air; therefore its *Elaſticity being as its Denſity* (2078, 2079) will be as much greater in its fixed State in Bodies than that of the common Air, as its

Bulk

Bulk when ſet at Liberty (or in a State of Expanſion) exceeds the Bulk of the Body which produced it (973).

2089. But to apply this to GUNPOWDER (with which we are here more immediately concerned) we muſt firſt conſider its conſtituent Parts which are *Nitre*, or *Salt-petre*, *Sulphur*, and *Charcoal Powder*. As to Charcoal, it appears, by Experiment, to afford none of this fluid elaſtic Air; and as to *Sulphur*, it is ſo far from yielding any of this Fluid, that on the contrary, it abſorbs or attracts, and fixes the common Air in which it is fired, and thereby diminiſhes its Quantity and Elaſticity, as is well known by Experiment. The elaſtic Fluid, therefore, that is produced, by firing Gunpowder, muſt be derived chiefly from the *Salt-petre*; and this is known to be a Subſtance imbibed from the Air by the Earth, becauſe thoſe who make it, extract it from the *ſame Parcel of Earth* many Times one after another, after it has been duly prepared, and expoſed to the Air for a proper Time.

2090. But becauſe Nitre produces only 180 Times its Bulk of Air (2087) and Mr. *Hauksbee*, by Experiment, found Gunpowder produced 232 Times its Bulk; and by moſt accurate Experiments, Mr. *Robins* has found it to produce 244 Times its Bulk; therefore it ſeems, that ſome additional Quantity of Air is produced by compounding theſe different Subſtances together in this Sort of Powder. This Quantity of Air, from Gunpowder, was determined in the following Manner: Mr. *Robins* fired $\frac{1}{16}$ of an Ounce, Averdupois, in a Receiver of about 520 cubic Inches Capacity; this ſunk the Mercury in the Gage 2 Inches; and as the Height of the Mercury was then near 30 Inches, therefore $\frac{15}{16}$ of an Ounce would have produced ſo much Air as would have preſſed all the Mercury out of the Gage; that is, $\frac{15}{16}$ of an Ounce would produce 520 cubic Inches of Air of the ſame Elaſticity of common Air. But $\frac{15}{16} : 520 :: 1 : 575$, the cubic Inches that *one Ounce* of Powder would produce.

2091. But as Heat augments the Elaſticity of Air, and this Powder was fired upon a red-hot Iron; Part of the Expanſion was owing to that Heat. But this Heat was leſs than that of boiling Water which encreaſes the Expanſion of Air to more than a $\frac{1}{4}$ Part of the Whole; if, therefore, the Number 575 be leſſened by $\frac{1}{5}$ Part, it may be nearly the ſame as would be

produced in the Receiver not heated, *viz.* 460 cubic Inches; but 17 Drams of Powder will fill 2 cubic Inches, therefore 16 : 17 :: 460 : 488 $\frac{3}{4}$ = cubic Inches of Air from 2 cubic Inches of Powder; therefore 1 cubic Inch of Powder will produce 244 of Air.

2092. Mr. *Robins*, to try how much this Elaſticity or Expanſion of Air would be augmented by the Flame of the fired Powder, ſuppoſes the Heat of this Flame nearly the ſame with that of the extreme (or white) Heat of red-hot Iron; and this he found to augment the Elaſticity of common Air in Proportion of 194 $\frac{1}{3}$ to 796. If then we ſay, 194 $\frac{1}{3}$: 796 :: 244 : 999 $\frac{1}{3}$; this laſt Number will ſhew how much the Elaſticity of the Air produced from Powder, when inflamed, is greater than that of the Air in its natural State.

2093. Since 1 cubic Inch of Mercury weighs very nicely 148,1 Ounce Averdupois Weight, a Pillar of Mercury, whoſe Baſe is one Square Inch, and Height 29 $\frac{1}{2}$, will weigh 14 Pound 15 Ounces; therefore as the Air ſuſtains by its Preſſure or natural Elaſticity, a Weight of 15 Pound (at a Medium) upon a Square Inch; the Elaſticity of the inflamed elaſtic Air of Gunpowder, which is 1000 Times as great, will act upon every Square Inch (at its firſt Accenſion) with a Force which is = 15000 *lb.* or ſomewhat above 6 Ton Weight.

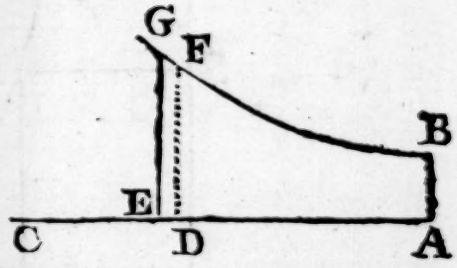
CHAP. V.

From the given DIMENSIONS *of any* PIECE *of* ARTILLERY, *the* DENSITY *of its* BALL, *and the* QUANTITY *of its* CHARGE, *are determined the* VELOCITY *which the Ball will acquire from the Exploſion; and alſo the* Length of the Charge *producing the* greateſt Velocity poſſible.

2094. IN the preceeding Article we have ſhewn what the Force of the Powder is at the Inſtant of its Accenſion or taking Fire, but ſince this Action of the Powder is not inſtantaneous,

neous, but continues to urge the Bullet all the Time it is in the Barrel; and ſince this Force is not uniformly the ſame, but decreaſes from firſt to laſt, and is every where inverſely as the Space which the elaſtic Fluid fills, that is, as its Bulk, (2080) it will be neceſſary in order to make a Computation of the Motion or Velocity of the Bullet, to premiſe the following *Lemmata*.

2095. Let AC be the Space thro' which a Body is propelled by any Force whoſe Action is continued and determined, and which may be repreſented by the Ordinate GE moving on the Abſciſs AC, and let BG be the Curve deſcribed by the Point G; draw DF indefinitely near to EG, and put AE = x, EG = y; the Velocity of the Body in E = v, the Time in which AE is deſcribed = t; then will $\dot{x}$ (= ED) $\dot{v}$ and $\dot{t}$ (the Fluxions of the Space, Time, and Velocity) be as the naſcent or evaneſcent Increments of thoſe Quantities x, v, and t (788).

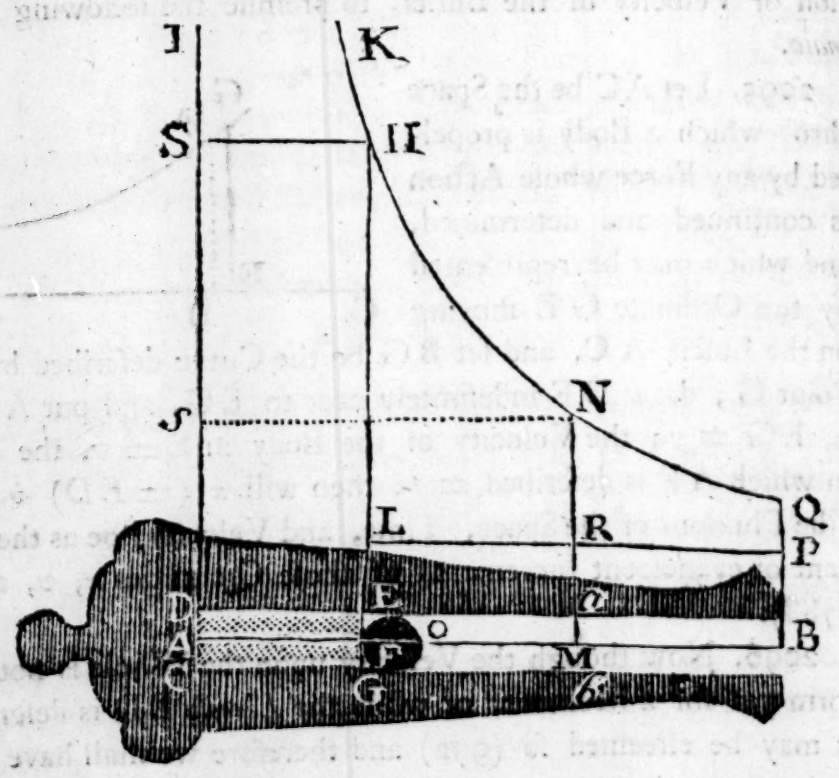


2096. Now though the Velocity upon the Whole is not uniform yet for a Moment, or while the *Lineola* ED is deſcribed, it may be eſteemed ſo (992) and therefore we ſhall have $v = \frac{\dot{x}}{\dot{t}}$ (by 991). Alſo we have the moving Force $y = \frac{\dot{v}}{\dot{t}}$ (998); wherefore it is $\dot{t} = \frac{\dot{x}}{v} = \frac{\dot{v}}{y}$, whence $y\dot{x} = v\dot{v} =$ EG × ED = the Fluxion of the Space ABGE; therefore $\frac{1}{2}v^2 =$ ABGE (804), and ſo we have $v = \sqrt{2\,\text{ABGE}}$; that is (ſince 2 is a conſtant Quantity) *the Velocity will be every where in the ſubduplicate Ratio of the Area* ABGE.

2097. Since the Quantity of Motion Q is always proportional to the moving Force GE, or y; therefore $Q = \frac{v}{t} = vm$ (970) whence $t = \frac{1}{m}$; alſo, becauſe when Q = 1, a given Quantity; then, alſo $v = \frac{1}{m}$, and t as v; therefore $\frac{1}{m} = t \times v = v^2$;

con-

conſequently $v = \sqrt{\frac{1}{m}}$, *that is, the Velocity of any Body urged through a given Space by a given Power acting with a determinate Force in every Part of that Space will be reciprocally in the ſubduplicate Ratio of the Quantity of Matter.*



2098. Theſe Things premiſed, let A B repreſent the Axis of any Piece of Artillery, A the Breech, and B the Muzzle; D C the Diameter of the Bore, and D E G C a Part of its Cavity filled with Powder; O the Ball that is to be impelled thereby, lying with its hinder Surface at the Line G E. Then, becauſe the Force (F) of the Powder againſt an Inch Square is known (2092, 2093) the Force (*F*) exerted on the Area of a Circle, one Inch in Diameter, will be known alſo; for it will be F : *F* :: 1 : 0,7854, whence *F* = 0,7854 F (269). Now let D = 1 Inch, and *d* = Diameter of any other Circle as that of the Ball O, that is, let *d* = E G, and (*f*) the Force of the Powder exerted on this laſt Circle; then *F* : *f* :: D² : *d*² (840) therefore *f* = *F d*² = 0,7854 F *d*² = the Force of the Powder acting at the firſt Inſtant of its Accenſion on the Ball O in the Direction of the Axis A B.

2099.

2099. Let this Force then be repreſented by any Line F H, perpendicular to A B in the Point F. Then, becauſe the Denſity of the elaſtic Fluid confined in the Space E D C G is to the Denſity thereof when expanded into any other Space *a* D C *b*, as thoſe cylindric Spaces inverſely (2072) or as the Lengths A M, A F; and ſince the elaſtic Force is in the ſame Ratio with the Denſity; therefore by making M N : F H :: A F : A M; and B Q : F H :: A F : A B; then ſhall M N and B Q be as the Force of the Powder upon the Ball at M and at B, and the Curve which connects the Points H, N, Q, will be an *Hyperbola* (by 778).

3000. To the Point A draw the Right-line A I parallel to F H, and S H parallel to A B; then will A I, A B be the Aſymptotes of the Hyperbola, and the Rectangles A F × F H = A M × M N = A B × B Q = 1, the Power of the Hyperbola (779); and from what has been ſaid, it is eaſy to underſtand that the whole Force of the Powder exerted on the Ball while it moves from F to B, is as the hyperbolical Space H F B Q, and therefore as the hyperbolical Logarithm of the Ratio $\frac{A B}{A F}$ (829, 850).

3001. Since the Force impelling the Ball at F is known (2098) and the Weight of the Ball is ſuppoſed given, the Ratio between the ſaid Force and the Weight of the Ball is known, which let be as F H to F L; then if the Ball were to be impelled to the Diſtance F B by a Force equal to its Gravity, becauſe the Force of Gravity through ſo ſmall a Space is uniform, or acts in every Point with the ſame Tenour, therefore in any Point M or B the Force will be as M R = F L, and B P = F L, and conſequently the Line which joins the Points L, R, P, is a Right-line, and parallel to the Axis A B, and the whole Action or Force of Gravity impelling the Ball through A B will be as the Rectangle F L P B = F B × F L.

3002. But the Velocities acquired by the Ball when propelled by the Force of Gravity, and by the Force of the Powder, thro' the Space F B, are as $\sqrt{F L B P}$, and $\sqrt{F H Q B}$ (2096). And ſince F B is a given Length, the Velocity acquired in falling through that Space is known. Thus ſuppoſe the Length of the Bore A B = 45 Inches, and the Length of the Charge A F =

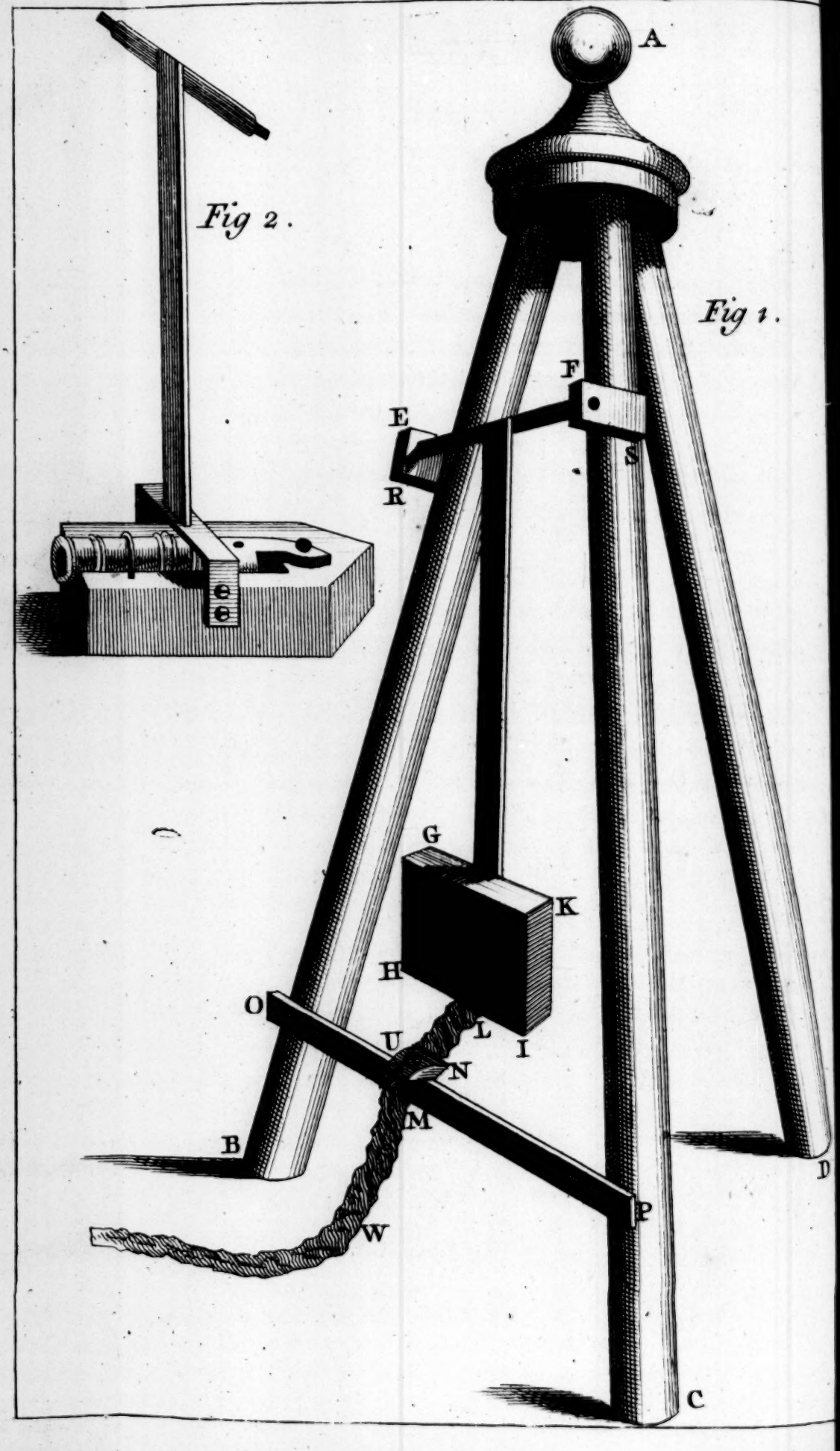
$2\frac{5}{8}$ Inches; then AB — AF = FB = $42\frac{3}{8}$ Inches (as in Mr. *Robins*'s Example). Then 16,2 Feet : $42\frac{3}{8}$ (=3,53 Feet) :: $\overline{32,4}^2$: V^2 (by 996). Whence $\sqrt{64,8 \times 3,53} = V = 15,07$ Feet *per* Second, the Velocity acquired by the Body falling thro' the Space FB.

3003. Again, to find the Ratio of FL to FH, ſuppoſe the Ball O be of Lead, whoſe Diameter $d = \frac{3}{4}$ of an Inch; then will its Weight be $\frac{1}{11\frac{1}{2}}$ of a Pound Averdupois. Therefore the Gravity of the Ball is to the Force of the Powder at F, as $\frac{1}{11\frac{1}{2}}$ to 0,7854 F d^2 (by 2098), that is, as 1 to 79521,6; and ſuch is the Ratio of FL to FH, ſuppoſing the Preſſure of Air on a ſquare Inch juſt equal to 15 *lb.*

3004. But the Number 79521,6 is ſomewhat too large, ſince a Column of Mercury, whoſe Baſe is 1 ſquare Inch, and Altitude $29\frac{1}{2}$, weighs but 14 *lb.* 15 Ounces; and I find Mr. *Robins* has taken for a mean Altitude 28,2 Inches of Mercury, or 33 Feet of Water, whoſe Weight is but about 14 *lb.* 4 Ounces, and he alſo makes the ſpecific Gravity of Lead to that of Water, as 11,345 to 1, conſequently a Column of Lead, 34,9 Inches Altitude, will have the ſame Weight or Preſſure; and multiplying this by 1000 (2093) the Product 34900 Inches, will be the Height of a Column of Lead, whoſe Preſſure is equal to that exerted on the Bullet at the Moment of Accenſion.

3005. Now the Bullet being $\frac{3}{4}$ of an Inch in Diameter, is equal to a Cylinder on the ſame Baſe, and $\frac{1}{2}$ Inch in Height; for the Solidity of the Sphere is $\frac{p d^2}{6}$, and that of a Cylinder is $\frac{p d h}{4}$ (by 836, 831). Wherefore if $\frac{1}{6} p d^2 = \frac{1}{4} p d h$, then $4 d = 6 h$; whence $h = \frac{2}{3} d = \frac{2}{3}$ of $\frac{3}{4} = \frac{1}{2}$, the Height of the Cylinder, equal to the Ball in Weight and Magnitude. Therefore 34900 × 2 = 69800; whence the Ratio of the Force of Powder is to the Weight of the Bullet, as 69800 to 1. And ſo FL : FH :: 1 : 69800.

3006. But we have FB : FA :: $42\frac{3}{8}$: $2\frac{5}{8}$: 339 : 21. Hence the Rectangle FL BP is to the Rectangle AFSH, as 1 × 339 : 21 × 69800 :: 1 : 4324. And ſince the Rectangle AFSH is equal to the Power of the Hyperbola (853) it is to any hyperbolic Logarithm FHQB, as 0,43429, &c. to the tabu-


Fig 2.
Fig 1.
A
F
E
S
R
G
K
H
O
U
L
I
N
M
B
D
P
W
C
e
e

tabular Logarithm of the ſame Ratio $\frac{AB}{AF} = \frac{360}{21}$, which is 1,2340579 (by 854). Therefore the Rectangle FLBP : Space FHQB :: 1 × 0,43429 : 4324 × 1,2340579 :: 1 : 12263. Conſequently $v : V :: \sqrt{FLBP} : \sqrt{FHQB}$ (2096): $\sqrt{1} : \sqrt{12263}$ :: 1 : 110,7 :: 15,07 Feet : 1668 Feet = V = Velocity of the Ball required; that is, the Velocity communicated to the Ball by the whole Force or Action of the Powder will be that of 1668 Feet *per Second*, of uniform Motion.

CHAP. VI.

The THEORY *for determining the* VELOCITY *of the* BULLET *at the* MUZZEL *of the* GUN *ariſing from the* QUANTITY *of the* CHARGE, *the* LENGTH *of the* PIECE, *and* DIAMETER *of its Bore; alſo the* CHARGE *producing the* GREATEST VELOCITY *in a Piece of given Dimenſions.*

3007. AS the THEORY of *military* PHILOSOPHY conſiſts chiefly in two great Points, *viz.* (1.) To ſhew how far the Velocity of the Bullet or Cannon-ball is affected, whilſt in the Piece, by the *Force of the Powder*, the *Diameter of the Bore*, and the *Quantity of the Charge*, under any given Variations; or (2.) after it is out of the Piece, by the *Reſiſtance of the Air*; and having premiſed the neceſſary geometrical Principles, we ſhall now proceed directly to an Illuſtration of them both.

3008. Let a = AB, the LENGTH *of the* PIECE under Conſideration, and b = Length of any other Piece of Artillery. Then if the Length of the Charge A F and the Bore be the ſame in each, we ſhall have the Area FHQB, or $L\frac{a}{AF} : V^2 ::$

L $\frac{b}{AF}$: V^2 (3006). But the three firſt Quantities are known, and therefore V = Velocity of the Bullet through the Length of the Bore denoted by (b) will alſo be known. Hence it appears, *that the longer the Barrel is, the greater will be the Velocity of the Bullet.*

3009. If the BORE *of the* GUN be varied, all other Things remaining the ſame; then, putting D = Diameter of the Bore in the preſent Inſtance, and D = Diameter of any other Bore; it is evident the *Quantity*, and, conſequently, the *Force of the Powder* in each will be in the Ratio of D^2 to D^2, and the Force of the Powder in the Bore, whoſe Diameter is D, will be reſiſted in the Ratio of the Weight of the Ball or of its Magnitude, that is as D^3 (by 841). Wherefore the Area F H Q B will be varied in the Ratio of $\frac{D^2}{D^3}$, or $\frac{1}{D}$; that is, D : D :: V^2 : V^2; and ſo $\sqrt{D}$: $\sqrt{D}$ :: V : V = the Velocity of the Bullet ſought. *Hence the ſmaller the Bore is, the greater will be the Velocity of the Bullet.*

3010. If A F, or the QUANTITY *of the* CHARGE be varied, other Things remaining the ſame; then the Rectangle A F S H, and, conſequently, the Area F H Q B will alſo vary, but not in the ſame Proportion; for there is a certain Quantity of Powder or Length of A F, which with Reſpect to the Length of the Piece A B, will give a greater Velocity to the Bullet than any other.

3011. Therefore let A B = a, and A F = x. Then will the Velocity be as the Force of the Powder at the firſt Inſtant of Accenſion, *viz.* as A H, or A F = x; and alſo as the whole Action of the Powder upon the Ball while in the Barrel, or as the Area F H Q B = L $\frac{a}{x}$; conſequently, the *Velocity will be* as x L $\frac{a}{x}$, or is x L a — x L x. Let z = any other Length of the Charge, then we ſhall have x L $\frac{a}{x}$: z L $\frac{a}{z}$ :: V^2 : V^2.

3012. Hence the Velocity of the Bullet at the Muzzle of Guns of a different Length, Bore, and Charge, will be as in the following Analogies, *viz.*

$V^2 : V^2 :: L\frac{a}{x} : L\frac{b}{x}$, for Lengths as a to b.

$V^2 : V^2 :: D : D$, for Diameters of Bore as D to D.

$V : V :: xL\frac{a}{x} : zL\frac{a}{z}$, for Lengths of Charge as x to z.

Conſeq uently, $V : V :: \sqrt[5]{DxL^2\frac{a}{x}} : \sqrt[5]{DzL\frac{b}{x}L\frac{a}{z}}$.

3013. But ſince it is evident from the Expreſſion of the Velocity $xL\frac{a}{x} = xLa - xLx$, there will be one determinate Value of x which will make it a *Maximum*, therefore by putting its Fluxion $= 0$, we have $\dot{x}La - \dot{x}Lx - \dot{x} = 0$ (becauſe the Fluxion of $Lx = \frac{\dot{x}}{x}$, (849); wherefore $La - 1 = Lx$. Now let $a = 10$; then, becauſe, the hyperbolical Logarithm of 10 is 2,302585; therefore $Lx = 1,302585$; but as $1 : 0,4342448 :: 1,302585 : 0,565710$ = tabular Logarithm of x (853). Therefore $x = 3,679 = AF$, when the Velocity is a *Maximum*.

3014. And becauſe in this Caſe $L\frac{a}{x} = 1$ (for $La - Lx = 1 = L\frac{a}{x}$) the Velocity $xL\frac{a}{x}$ when a *Maximum*, will be as x or 3,679. Alſo when x or $AF = \frac{1}{10}$ of AB, then becauſe $x = 1$, we have the Velocity $xL\frac{a}{x} = La = 2,302585$, which is not quite $\frac{2}{3}$ of the *Maximum* Velocity. And thus the *Velocity acquired from any other Charge of Powder may be compared with the greateſt Velocity.*

3015. It is obſervable, that when the Velocity is greateſt, then $AF : AB :: (3,679 : 10 ::) 1 : 2,71828$; which Ratio is called the *modular Ratio*, becauſe its Meaſure $\left(L\frac{a}{x}\right)$ is $1 =$ the *Module* of the *Syſtem*, or *Power of the Hyperbola* (779).

3016. This *Maximum* Charge is to that in common Uſe, (in Art. 3003) as $\frac{1}{2.7}$ to $\frac{21}{360} = \frac{1}{17}$, that is near 7 Times as large. The Reaſons why we do not, or rather care not to uſe

this Charge for the greateſt Velocity, are many; for *firſt*, it would make too great a Conſumption of Powder, *viz.* 6 or 7 Times more than what we now uſe. *Secondly*, the Force of the Powder in the Gun, would in that Caſe, be ſo great, that the Barrels muſt be 5 or 6 Times as thick as they now are to avoid burſting; by which Means they would be rendered too expenſive and unweildy. *Thirdly*, the Velocity of the Bullet, from common Charges, is ſufficient in long Guns; and the greateſt Velocity would be much too great. *Laſtly*, in ſhort Guns, as Piſtols, *&c.* we may uſe this *Maximum* Charge very well on many Occaſions, for $1 : 2,7 :: 2\frac{5}{8} : 7$ Inches, the Length of the Barrel, whoſe Charge $AF = 2\frac{5}{8}$ Inches.

3017. If inſtead of one Bullet, there be 2, 3, or 4, laid before the Charge, then will the Velocities of the Bullets, in each reſpective Caſe, be as $\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{4}}$, (574). Therefore, ſince $\sqrt{\frac{1}{4}} = \frac{1}{2}$, the Velocity of a Bullet when there is 4, will be but $\frac{1}{2}$ that of a ſingle one. Hence alſo the Velocity of a leaden Bullet is to that of an Iron one, as $\sqrt{11,345}$ to $\sqrt{7,645}$, or $\sqrt{3}$ to $\sqrt{2}$.

3018. In this Theory, Mr. *Robins* has ſuppoſed two Things, *viz.* (1.) That the Action of the Powder upon the Bullet ceaſes as ſoon as it is got out of the Piece; and (2.) That all the Powder of the Charge is fired before the Bullet is ſenſibly moved out of its Place. The firſt of theſe *Poſtulates* is ſelf evident; and the Second is very nearly true, that the few Grains uſually thrown out unfired, can but little affect the Theory; and he found it ſo by many Experiments upon Guns of different Lengths, and diſcharging a different Number of Bullets from the ſame Gun. But this was more particularly confirmed afterwards by many Experiments made by Appointment of the *Royal Society*. The general Reſult of which proved, that not above $\frac{1}{6}$ Part of the whole Charge was collected in Grains unfired; and this, in Reality, could be reckoned no more than a $\frac{1}{12}$ Part. For theſe Experiments were made with common Powder, which produced a larger Quantity unfired, than the fineſt grained Powder will do (ſuch as Mr. *Robins* uſed) in the Proportion of 5 to 3, as was found by Experience. And again, the *Saltpetre* extracted from the Powder, thrown out unfired, compared with that of

the

the Charge, was found to be nearly the Ratio of 7 to 9; and, therefore, though 2 Penny-weight of unfired Grains were collected from a Charge of 12 Penny-weight, which was $\frac{1}{6}$ Part, yet this reduced in the Proportion of 5 to 3, will be but $\frac{1}{10}$; and this again reduced in the Ratio of 9 to 7, is nearly $\frac{1}{12}$ of the Whole. In ſome Trials, the Deficiency was but the $\frac{1}{18}$ Part of the whole Charge, from whence it will appear, how inſtantaneouſly the whole Body of the Charge is fired, and how little the Velocity of the Bullet muſt be affected by the ſmall Part unfired, as will be evident from the enſuing Experiments in Chap. VIII.

CHAP. VII.

The THEORY *of the* MACHINE *contrived by Mr.* ROBINS, *for determining the* VELOCITY *which any* BALL *moves with at any Diſtance from the* Piece *it is diſcharged from.*

3019. HAVING aſcertained by the THEORY what the Velocity of a Bullet ſhould be under any given Quantities of *Charge*, *Length of Barrel*, and *Diameter of Bore*; it remains in the next Place to confirm this excellent Theory by Experiments. For this Purpoſe, Mr. *Robins* invented a Method, which by Means of a *compound* PENDULUM, and its *Apparatus*, gives the Velocity of any Bullet iſſuing either from the Muzzle of the Gun, or at any Diſtance from it. The *Rationale* of which Machine, and the Manner of applying it for ſuch Uſes, are now to explained.

3020. To this End the Machine muſt be firſt of all deſcribed, of which the Body conſiſts of three ſtrong Poles or Legs B, C, D, ſpreading at Bottom, and joining at Top in the Head-piece A. On two of theſe Legs, towards the Top, are ſcrewed on two Sockets R, S, in which a Pendulum E F H I is hung, by Means of the Croſs-piece E F, which is the Axis of Suſpenſion on which the

the Pendulum vibrates very freely. The lower Part of the Pendulum is a thick ſquare Piece of Wood G H I K, faſtened to the Back-part (which is of Iron) by Screws. A little below the Bottom of the Pendulum, there is a Brace O P fixed to the Legs B, C; and to the Brace is fixed a Part M N U made with two Edges of Steel, bearing on each other in the Line U N, ſomewhat in the Manner of a Drawing-Pen; the Strength with which they preſs on each other being diminiſhed or encreaſed at Pleaſure by a Screw Z going through the upper Piece. At the Bottom of the Pendulum is faſtened a narrow Ribbon L N W, which paſſes between the Steel-Edges, and paſſing thro' a Hole in the lower Piece of Steel hangs looſely down as at W.

3021. The Artifice of this Contrivance, tho' ſimple, is admirable; the Bullet diſcharged from the Gun againſt this Pendulum puts it into Motion; and both the Bullet and Pendulum are to be conſidered as *Non-Elaſtic Bodies*; the Bullet being of Lead, is evidently ſo; and the Wood tho' to ſlow Motions it may be conſidered as ſomewhat elaſtic, yet with reſpect to very ſwift Motions it cannot be conceived in the leaſt Degree ſo; ſince there is no Time for the Parts to act by their natural Reſort.

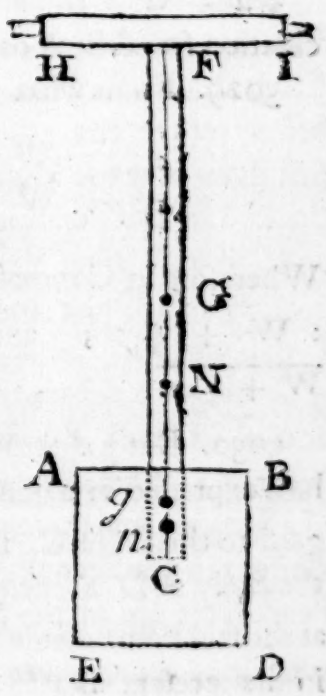
3022. The Bullet and Pendulum, therefore, will in their Motions and Action on each other, obſerve the Laws of Percuſſion laid down for Non-elaſtic Bodies (from Art. 1003, 1012.) But before we can come to apply them in the preſent Caſe, we muſt firſt ſhow how the Momentum or Quantity of Motion is to be eſtimated in the Pendulum when put into Motion by the Bullet, and alſo with what Velocity it moves after the Stroke.

3023. In Order to this we muſt know that the *Weight* of the Pendulum (Mr. *Robins* made Uſe of) was 56*lb.* 3*oz.* The *Center of Gravity* was diſtant from the Axis of Suſpenſion 52 Inches. And 200 of its ſmall Vibrations were performed in the Time of 253 Seconds; wherefore the Time of one Vibration is $\frac{253}{100}$ Parts of a Second; and therefore ſince the Length of a Pendulum which vibrates in one Second is 39,2 Inches (1125) and the Lengths of Pendulums are as the *Squares of their Time of Vibration* (1116.) Therefore ſay, as $1''^2 : \frac{253''^2}{200}$ :: 39.2 Inches : 62,73 Inches, the Diſtance of the Center of Oſcillation from the Axis of Vibration.

tion. Laſtly the Center of the Piece of Wood GHIK (on which the Bullet is ſuppoſed to impinge) is diſtant from the ſaid Axis 66 Inches.

3024. Now if all the Matter of the Pendulum were concentered in the Center of that Piece or Wood, it would reſiſt the Bullet with all its Force, viz. of 56*lb.* 3*oz.* becauſe that would then be the Center of both *Gravity*, and *Oſcillation*; but ſince the Center of Gravity is ſhort of this Point, the Reſiſtance will be diminiſhed in Proportion, *viz.* in the Ratio of 66 to 52. And again, another Diminution of the Force will ariſe on Account of the Velocity of the Motion, becauſe alſo the Center of Oſcillation (by which that is eſtimated) is ſhort of the Center of the Wood where the Stroke is made in the Ratio of 66 to $62\frac{2}{3}$. Therefore the Force or Reſiſtance will be diminiſhed in the Ratio of 66×66 to $62\frac{2}{3} \times 52$; that is, it will be $66^2 : 62\frac{2}{3} \times 52 :: 56lb.\ 3oz. : 42\ lb.\ \frac{1}{2}oz.$

3025. What has been hitherto ſaid, is rather by Illuſtration deduced from the Conſideration of a *compound Pendulum* of the *moſt ſimple Form*, but ſuch was not the Pendulum which Mr. *Robins* uſed in his Machine, the *Theory* of which we ſhall therefore give with its Demonſtration, as follows:



3026. Let FC be the Shaft, and ABED the Weight of the Pendulum, both Parallelopipeds, and HI the Axis of Motion; then is FG ($= \frac{1}{2}$ FC), the Diſtance of the *Center of Gravity* of the Shaft, and FN ($= \frac{2}{3}$ FC), the Diſtance of the *Center of Oſcillation*, and put the whole Length FC $= a$. And let the End C be the Center of the ſquare Parallelopiped ABDE, whoſe Weight let us call W, and the Weight of the Shaft w.

3027. The two Centers of Gravity, G and C, will have a common Center at g; and it will be $w : W :: Cg : Gg$. Put $Cg = d$; then is $Gg = \frac{W\,d}{w}$

$\frac{W d}{w} = \frac{1}{2} a - d$; therefore $W d = \frac{1}{2} a w - w d$; and $W d + w d = \frac{1}{2} a w$, whence $d = \frac{a w}{2 \times \overline{W + w}}$. Consequently, $F g = a - d = a - \frac{a w}{2 W + 2 w} = g = \frac{2 a W + a w}{2 \times \overline{W + w}}$, the Distance of the *compound Center of Gravity* from the Axis of Motion.

3028. By the Addition of the Weight A D, the Center of Oscillation will be drawn down from N to some other Point n; to determine which, we proceed thus. Since $F N = \frac{2}{3} a$ (1097) and the *Momenta* of the Weights in the Shaft is $\frac{1}{2} a w$; therefore $\frac{2}{3} a \times \frac{1}{2} w a = \frac{1}{3} a a w =$ *Momenta*, or Sum of all the Forces in the Shaft. Also the whole Force of the Body A D (considered alone, vibrating at the Distance F C $= a$) is $a a W$, and the *Momenta* of its Weight will be $a W$. If therefore the *Sum of the Forces* of the Shaft and Body A D, be divided by the *Sum of the Moments*, we shall have $\frac{\frac{1}{3} a a w + a^2 W}{\frac{1}{2} a w + a W} = \frac{\frac{1}{3} a w + a W}{\frac{1}{2} w + W} = n = F n$, the Distance of the Center of Oscillation from the Axis H I (1094).

3029. From what we have shewn, it appears that

$$a : g :: W + w : W + \tfrac{1}{2} w \text{ (3023).}$$
$$a : n :: W + \tfrac{1}{2} w : W + \tfrac{1}{3} w \text{ (3024).}$$

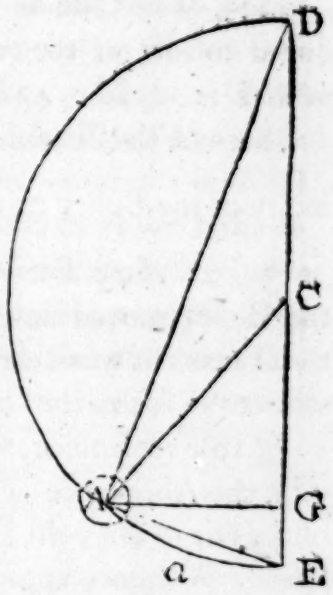
Wherefore by Composition of Ratios, we have $a a : g n :: W + w : W + \frac{1}{3} w$; and therefore $g n \times \overline{W + w} = a a \times \overline{W + \frac{1}{3} w}$.

3030. But $g n \times \overline{W + w} = F$, the Force of the Pendulum expressed in a general Way (1095) and therefore with Regard to the Point C, the Force is $a a \times \overline{W + \frac{1}{3} w}$. But were the Body A D a *simple Pendulum* of the same Weight, its Force at the said Point would be $a a \times \overline{W + w}$, which is to the Force in its present State, as $W + w$ to $W + \frac{1}{3} w$; and which, therefore, is the Ratio of Diminution.

3031.

3031. To apply theſe Theorems to the Machine here uſed, we have $a = 66$, $g = 52$, $n = 62{,}73$, (or $62\frac{2}{3}$ to uſe Mr. *Robins*'s Number) and the Weight of the whole $W + w =$ 56*lb.* 3*oz.* = 899 Ounces. Then we have 66^2 (= 4356) : $62\frac{2}{3} \times 52$ (= $3258\frac{2}{3}$) :: 56*lb.* 3*oz.* (= 899*oz.*) : 671 Ounces, or 42*lb.* nearly (3025). As $899 - 671 = 228$*oz.* $= \frac{2}{3}w$; therefore $w = 342$*oz.* and $W = 557$*oz.*

3032. This compound Pendulum, then, is reduced to the Caſe of a ſimple Pendulum whoſe Length is 66 Inches, and Weight = 42*lb.* $\frac{1}{2}$*oz.* which is to the Weight of the Bullet = $\frac{1}{12}$*lb.* as 504 to 1. Let this equivalent ſimple Pendulum then be repreſented by AC = 66; and on the Center C and Diameter DE deſcribe the Semicircle DAE. Let A*a*E be the Arch deſcribed by the Pendulum impelled by the Bullet; and AE the Chord of that Arch. Then we have ſhewn (1113) that the Velocity of the Pendulum by which it ſhall deſcribe the Arch EA is the ſame as would be acquired by an heavy Body falling through the ſame perpendicular Height GE; which Velocity is the next Thing to be determined. In order to this, it is eaſy to underſtand, that when the Pendulum is in its perpendicular Poſition in a State of Reſt, and the Ribbon drawn ſtrait, with a Pin put through that Part which is contiguous to the Edges UN, that when the Bullet impinges on the Pendulum, the Ribbon will be drawn out in ſuch Manner as to meaſure the Chord of the Arch deſcribed by the Point L; for it will be equal to the Interval between the Pin, and the Edges UN.



3033. Now by an Experiment made with a Gun 45 Inches in Length, $2\frac{5}{8}$ Inches Charge of Powder (weighing 12 Pennyweight) the Bullet $\frac{3}{4}$ Inch-diameter, and $\frac{1}{12}$*lb.* Weight; and the Muzzel of the Gun at the Diſtance of about 16 or 18 Feet; it was found that the Ribbon was drawn out $17\frac{1}{4}$ Inches. Now the Diſtance of the Point L is $71\frac{1}{8}$ Inches from the Axis, and therefore if we ſay, as $71\frac{1}{8} : 17\frac{1}{4} :: 66 : 16$ Inches nearly; it

will appear, that the Chord of a ſimilar Arch deſcribed by the Center of the Piece G H I K was 16 Inches nearly.

3034. Wherefore (in the Fig. 3026) A E = 16, and D E = (2 A C =) 132. And drawing A D, we have D E : A E :: A E : G E, that is 132 : 16 :: 16 : 1,939 = G E. If then we ſay, as $\sqrt{193\frac{1}{3}} : 32 :: \sqrt{1,939} : 3\frac{1}{4}$ = Number of Feet *per* Second, that a Body would deſcribe by an uniform Motion with the Velocity acquired in falling through G E. The Velocity therefore with which the Pendulum moved after the firſt Moment of Impact was that of $3\frac{1}{4}$ Feet *per* Second.

3035. The Caſe of the Bullet and Pendulum is therefore reduced to that of the two Bodies B and A (in 1003); and ſince $m : M :: \frac{1}{112}$ *lb.* : 42 *lb.* $\frac{1}{2}$ *oz.* :: 0,083 : 42,031 : 1 : 504,6; and becauſe the Pendulum was at Reſt, we have (1009) the Velocity of the Bullet $= v = \frac{M + m}{m}$ (= 505,6) × V = 505,6 × $3\frac{1}{4}$ = 1643 Feet *per* Second. That is, by Experiment, the Bullet moved at the Rate of 1643 Feet *per* Second, and by the Theory it was determined 1668 Feet (in 3006); and as this was at the Muzzel of the Gun, and the other at the Diſtance of 16 or 18 Feet from it, the ſmall Difference of 15 Feet is no more than the Allowance which ought to be made for the Reſiſtance of the Air to the ſwift Motion of the Bullet paſſing through that Space. Whence appears the wonderful Degree of Exactneſs between the Theory and Experiments made quite independent of it.

3036. Having thus determined the Velocity for the Length of one Chord; the Velocity of the Pendulum for any other Length of a Chord (or Ribbon drawn out) will be known; becauſe the Velocities have the ſame Ratio with thoſe Chords, for ſince D E × G E = A E², and D E is a ſtanding Quantity; it will always be G E as A E², and conſequently A E as $\sqrt{G E}$, that is, as the Velocity acquired in falling through G E (by 991). And hence, becauſe the Velocity of the Bullet $v = V \times 505,6$; and this Number 505,6 (for the ſame Bullet and Pendulum) is conſtant, we have the Velocity (v) of the Bullet always proportional to (V) the Velocity of the Pendulum, and conſequently to the Chord of the Arch in any Vibration.

CHAP.

CHAP. VIII.

An Account of the EXPERIMENTS *made for comparing the* actual VELOCITIES *with which* BULLETS *of different Kinds are diſcharged from their reſpective* PIECES, *with their* VELOCITIES *computed from the* THEORY.

3037. THE THEORY for determining the Velocities of Bullets diſcharged from given Pieces of Artillery having been fully deſcribed; and in the laſt Chapter it has been ſhewn how the *actual Velocities* of Bullets are meaſured and aſcertained by Means of a Machine; it now remains to compare the Reſult of the Theory with Experience, and thereby to evince how accurately this Theory agrees with the real Motions of Bullets though founded on Principles no ways connected with thoſe Experiments. Some of theſe we ſhall ſelect for illuſtrating the Theory by a particular Application; and then give a general View of the Experiments as we find them tabulated by Mr. *Robins* himſelf.

3038. By Experiments made on a Barrel only 12,375 Inches, the Theory was farther confirmed. For (*cæteris paribus*) we have by the Theory (3008) the L $\frac{45}{2,625}$: $\overline{17,25}^{2}$ :: L $\frac{12,375}{2,625}$: 162,37 = Square of the Chord whoſe Length is ſought in the preſent Caſe, which therefore would have been 12,74 Inches, had the Weight of the Pendulum been the ſame, but as it was ſome ſmall Matter lighter, than that in (3020), the Chord was by the Theory a little larger, *viz.* 12,8 Inches; and by Experiment it was 12,7, 12,6, 12,4; on three different Trials. Here the very ſmall Differences are little more than what muſt reſult from the Bullets lodging in the Pendulum and making it thereby heavier each Time.

3039. By Experiments made with a Barrel 24,312 Inches in Length, all other Things the ſame; the Length of Chord, by the Theory, agreed exactly with the Length of the Ribbon drawn

out being both = 14,4 Inches. Whence the Velocity of the Bullet from the preſent Gun was to that from the Gun of 45 Inches, as 14,4 to 17,1; that is, as 17,1 : 14,4 :: 1643″ : 1544″ nearly. And for the Velocity of the Bullet from the Gun of 12,375, we have by Theory 17,1 : 12,8 :: 1643″ : 1432″. Whence it appears, the Velocity in a ſhort Gun is much greater (*cæteris paribus*) than in a long one.

3040. Again, by varying the Quantity of Powder in the ſame Space or Cavity D E G C the Theory is ſtill farther confirmed; for if inſtead of 12 *dwt.* you put only half that Quantity, *viz.* 6 *dwt.* Then by the Theory (2096) the Velocity by 6 *dwt.* is to that of 12 *dwt.* in the ſubduplicate Ratio of 6 to 12 (becauſe thoſe Numbers are as the Forces or Elaſticities impelling the Ball) that is, as $\sqrt{12} : \sqrt{6} :: 17,1 : 12$; the Chord or Length of Ribbon therefore ſhould be nearly 12 Inches when the Bullet was diſcharged with 6 *dwt.* of Powder, and by two Experiments the Length of the Ribbon drawn out was 11,2; 12,2; and the Deficiency was owing to this, that the Heat in firing ſmall Quantities of Powder is not proportionally ſo great as in firing larger Quantities; and therefore the Impulſe is not to be expected quite ſo great as by the Theory. Thus 1 *dwt.* of Powder by the Theory gives a Velocity of 482 Feet *per* Second, but by repeated Trials with that Quantity, the Velocity was not quite 400 Feet *per* 1″.

3041. The leſs Space the Fire or inflamed Powder has to act in the more Denſe it is, and conſequently the ſtronger its Action; therefore when the 6 *dwt.* of Powder was rammed into but half the Space D E G C, it was found in two Experiments to produce Lengths of Ribbon which meaſured 13,2 and 13,9 Inches, whereas by the Theory it ſhould be 13,6 Inches.

3042. Let us now apply the Theory to determine the Velocity of a Ball of 24 *lb.* ſhot from a Cannot of 10 Feet = 120 Inches Length, with $\frac{2}{3}$ of its Weight (*viz.* 16 *lb.*) of Powder. The Diameter of an Iron Globe of that Weight (and conſequently of the Bore of the Piece) will be found to be 5,325 Inches, allowing the ſpecial Gravity of Iron to be as in (2054). Alſo 16 *lb.* of Powder will fill a Cylinder of that Diameter to the Height of 21,63 Inches, allowing 8 $\frac{1}{2}$ Drams to a Cubic Inch.

Inch. Hence (in the Fig. of Art. 2098.) we have A B = 120; A F = 21,63; and F B = 98,37. Whence (computing as in 3002, 3003,) we ſhall find F L : F H :: 1 : 14732,4.

3043. Therefore F P : F S :: 1 × 98,37 : 21,63 × 14732,4 :: 1 : 4078,1. Alſo, we have E S : F Q :: 0,43429 : L $\frac{AB}{AF}$ = 0,744125. Conſequently it is F P : F Q :: 1 × 0,43429 : 4078 × 0,744125 :: 1 : 6987,4. The ſubduplicate of this Ratio is that of 1 to 83,58. And the Velocity acquired in falling through the Height F B = 98,37 Inches is 23,05 Feet *per* 1″. Whence 83,59 × 23,05 = 1926,6 Feet *per* 1″ which the ſaid Bullet acquires at its Exit from the Muzzel of the Cannon. And ſince the ſpecific Gravity of Iron and Lead are (as the Quantities of Matter under equal Bulks) as 2 to 3 nearly;* therefore ſay (by 2097) as $\sqrt{3} : \sqrt{2} :: 1926,6'' : 1573''$, the Feet *per* 1″ ſuch a Bullet would move through if made with Lead.

3044. In theſe great Guns it will be neceſſary to underſtand what Space the Piece will recoil through in firing it, or rather what Space the Gun will move through Backwards, while the Ball is carried forwards to the Muzzel thereof. In order to this, we muſt conſider, that it is one and the ſame Power (*viz.* the Elaſticity of the Powder) that acts both on the Gun and Ball; and therefore in the Theorem Q V = G S M (1001) G is a given Quantity; and ſo in the preſent Caſe, we have Q V = S M, but Q = V M in every Caſe (970) therefore S = V^2; that is, *the Spaces deſcribed in a given Time, by given Forces, will be as the Squares of the Velocity.*

3045. Let S = (F B =) Space deſcribed by the Ball, and V its Velocity; and let s = Space through which the Gun recoils with the Velocity v; M = Quantity of Matter in the Gun, and m = the ſame in the Ball. Then we have S : s :: V^2 : v^2 :: $\frac{1}{m} : \frac{1}{M}$ (by 2097). Whence S : s :: M : m. Now the Weight of the Gun is about 47 *Cwt.* or 5264 *lb.* and that of the Ball 24 *lb.* alſo S = 98,37 (3042). Therefore we have 5264 *lb.* : 24 *lb.* :: 98,37

* The Reader is deſired to correct an Error in the Table of ſpecific Gravities (2054) where that of *Lead* is 10,131 inſtead of 11.131.

:: 98,37 : 0,448 = Space of the Recoil, which is not quite $\frac{1}{2}$ an Inch, when the Ball is juſt out of the Gun.

3046. Hence we ſee the uſual Methods of conſtructing Platforms for Cannon with ſo much Strength and Firmneſs, and conſequently with ſo great an Expence, are not at all neceſſary; ſince if it be but ſufficiently ſteady at the Beginning of the Recoil, the remaining Part may be much ſlighter (as Mr. *Robins* obſerves) ſince its Unſteadineſs or Shaking beyond the firſt $\frac{1}{2}$ Inch can have no Influence on the Ball, (which is then out of the Gun) nor any how alter the Direction of the Shot.

3047. Having thus finiſhed a particular Application of theſe Experiments to elucidate the THEORY of GUNNERY according to the new Principles of Mr. *Robins*; we ſhall now give a general View of the many valuable Experiments he made, and digeſted into proper Tables, by which it will appear how ſurprizingly the Theory agrees with and is corroborated and confirmed by thoſe Experiments. In theſe Trials it became neceſſary ſometimes to change the Board of the Pendulum (when too much battered and over-charged with Bullets) but when its Weight varied any Thing conſiderable, Mr. *Robins* has taken Care to acquaint us with it. The following Account we ſhall give in his own Words.

3048. The firſt Table contains three Experiments only, made with a Barrel 45 Inches in Length, and the Board on the Pendulum was 4 *lb.* lighter than that deſcribed (3023).

No.	Quantity of Powder. Dw.	Chord of aſcending Arch meaſured on the Ribbon.	The ſame by the Theory.	Error of Theory.
1	12	18,7	19,0	+,3
2	12	19,6	19,0	—,6
3	6	13,6	13,4	—,2

3049. The next Experiments were made with the ſame Barrel, but the Board on the Pendulum was now of little more Weight than that in the Example of (3023).

N°.	Length of the Cavity A F.	Quantity of Powder.	Chord of aſcending Arch meaſured on the Ribbon.	The ſame by Theory.	Error of Theory.
	Inches.	Dw.	Inch.	Inch.	Inch.
4	2 5/8	6	11,9	12,1	+,2
5	2 5/8	6	12,2	12,1	—,1
6	1 1/4	6	13,2	13,6	+,4
7	1 1/4	6	13,9	13,6	—,3
8	2 5/8	12	16,7	17,2	+,5
9	2 5/8	12	17,5	17,2	—,3
10	2 5/8	12	16,9	16,8	—,1
11	2 5/8	12	17,0	16,8	—,2
12	2 5/8	6	11,7	11,5	—,2
13	2 5/8	6	11,1	11,5	+,4
14	2 5/8	12	16,7	16,3	—,4

The laſt five Numbers reſulting from the Theory are corrected from the Quantity of Bullets lodged in the Board, which, as many other Experiments of a different Kind were tried in the Interval, amounted at laſt to above two Pounds; whence the Weight of the Pendulum being increaſed, its Vibration with the ſame Blow muſt be proportionably diminiſhed.

3050. The next Experiments were made with a Barrel of the ſame Bore with the laſt, but only 12,375 Inches in Length: To diſtinguiſh them, we ſhall for the future denominate the firſt Barrel by the Letter A, and this ſhort one by C. The Board on the Pendulum was at firſt rather lighter than in (3023).

N°.	Barrel	Extent of the Cavity containing the Powder.	Quantity of Powder.	Chord of aſcending Arch meaſured on the Ribbon.	The ſame by Theory.	Error of Theory.
		Inch.	Dw.	Inch.	Inch.	
15	C	2 5/8	12	12,7	12,8	+,1
16	C	2 5/8	12	12,6	12,8	+,2
17	C	2 5/8	12	12,4	12,8	+,4
18	A	2 5/8	12	17,0	17,3	+,3
19	A	2 5/8	12	17,2	17,2	,0
20	A	2 5/8	12	17,1	17,2	+,1
21	A	2 5/8	12	17,2	17,2	,0
22	A	2 5/8	6	12,4	12,2	—,2

3051. In ſome of the following Experiments a third Barrel was uſed of the ſame Bore with the other two, but 24,312 Inches in Length: This Barrel I denominate B; the Board fixed on the Pendulum was at firſt but little heavier than that in (3023); and when in the Courſe of the Experiments it is ſenſibly increaſed in the Weight, I diminiſh the Numbers ariſing from the Theory by a correſponding Part.

No.	Barrel	Extent of the Cavity containing the Powder. Inch.	Quantity of Powder. Dw.	Chord of aſcending Arch meaſured on the Ribbon. Inch.	The ſame by Theory. Inch.	Error of Theory.
23	A	2 5/8	12	17,1	17,2	+ ,1
24	A	2 5/8	9	15,2	15,0	— ,2
25	A	2 5/8	9	15,4	15,0	— ,4
26	C	2 5/8	12	11,5	12,8	+1,3
27	C	2 5/8	12	11,5	12,8	+1,3
28	C	2 5/8	6	8,7	9,	+ ,3
29	C	2 5/8	12	12,3	12,5	+ ,2
30	B	2 5/8	12	14,4	14,4	0,0
31	B	2 5/8	12	14,4	14,4	0,0
32	B	2 5/8	6	10,3	10,5	+ ,2
33	A	1 3/4	8	14,7	14,5	— ,2
34	A	4	12	15,7	15,3	— ,4

The Error in the 26th and 27th Experiments being much greater than what has occurred to me in any other Trials, I ſuſpect, that ſome Miſtake was made in the Weight of the Powder, or that the Barrel (which had indeed lain by in a moiſt Place) was very damp; which Circumſtance, I know by Experience, will conſiderably diminiſh the Action of the Powder.

3052. The following Experiments were made with a Pendulum much heavier, it weighing in the whole 97 *lb.* its Center of Gravity was 55,625 Inches diſtant from its Axis of Suſpenſion, and 200 of its ſmall Swings were performed in the Space of 255″ 1/4, whence its Center of Oſcillation is 63,9 Inches diſtant from the Axis of Suſpenſion. Alſo ſometimes another Barrel was uſed 7,06 Inches in Length, and ,83 in Diameter, its Ball was exactly fitted to the Bore without any Windage, ſo that it went in with Difficulty, the Weight of this Ball was 33 1/2 *dw.* This Barrel we ſhall denominate D.

N°.

No.	Barrel	Extent of the Cavity containing the Powder.	Quantity of Powder.	Chord of aſcending Arch meaſured on the Ribbon.	The ſame by Theory.	Error of Theory.
		Inch.	Dw.	Inch.	Inch.	Inch.
35	A	2 5/8	12	9,2	9,2	,0
36	A	2 5/8	12	9,5	9,2	—,3
37	A	5 1/4	24	11,7	11,3	—,4
38	A	7 7/8	36	13,2	12,6	—,6
39	A	2 5/8	12	9,3	9,1	—,2
40	A	1 3/4	8	7,6	8,1	+,5
41	C	2 5/8	12	6,1	6,6	+,5
42	C	2 5/8	12	6,5	6,6	+,1
43	B	2 5/8	12	8,0	8,2	+,2
44	B	2 5/8	12	8,3	8,2	—,1
45	A	2 5/8	12	9,5	9,1	—,4
46	A	2 5/8	12	9,1	9,1	,0
47	A	2 5/8	6	7,2	6,5	—,7
48	A	2 5/8	6	6,7	6,5	—,2
49	C	2 5/8	12	6,8	6,7	—,1
50	C	2 5/8	12	7,5	6,7	—,8
51	C	2 5/8	6	4,7	4,8	+,1
52	C	2 5/8	6	5,0	4,8	—,2
53	D	2 1/4	12	7,0	7,2	+,2
54	D	2 5/8	12	7,1	6,8	—,3
55	D	2 5/8	6	4,7	4,8	+,1
56	D	2 5/8	6	4,8	4,8	,0
57	A	2 1/16	6	6,4	6,5	+,1
58	A	2 1/16	6	6,4	6,5	+,1
59	A	2 1/16	6	6,6	6,5	—,1
60	A	2 1/16	6	6,7	6,5	—,2
61	A	2 1/16	12	9,0	9,1	+,1

The Error in the 50th Experiment, the greateſt in this Set, was doubtleſs owing to the Wind; for the 49, which was made immediately before it in the ſame Manner, and with the ſame Quantity of Powder, differs but little from the Theory. The Exceſs of the 38th Experiment above the Theory was in Part occaſioned by the Impulſe of the Flame on the Pendulum, which in this large Quantity of Powder was plainly to be diſcerned.

CHAP. IX.

The foregoing PRINCIPLES *applied to inveſtigate the* VELOCITY *which the* FLAME *of* GUNPOWDER *acquires by expanding itſelf, ſuppoſing it be fired in a given* PIECE *of* ARTILLERY *without either Bullet or any other Body before it.*

3053. IN order to experiment the Velocity with which the Particles of Gunpowder expand themſelves (in the Exploſion) at the Muzzle of the Gun, Mr. *Robins* charged the Barrel of 45 Inches (Art. 3033) with 12 *pwt.* of Powder, and a ſmall Wad of Tow only; and then placing the Muzzle 19 Inches from the Center of the Pendulum (mentioned Art. 3023) it was fired, and the Impulſe of the Flame on the Pendulum made it aſcend through an Arch, whoſe Chord was 13,7 Inches. Now ſince a Chord of 17 $\frac{1}{4}$ Inches agrees to the Velocity of 3 $\frac{1}{4}$ Feet *per* 1″, (ſee 3034) therefore ſay, as 17 $\frac{1}{4}$ Feet : 3 $\frac{1}{4}$ Feet :: 13,7 Feet : 2,6 Feet; that is, the Velocity of the Pendulum was at the Rate of 2,6 Feet *per* 1″.

3054. Now the Weight of the Powder and Wad was about 13 *pwt.* and that of the Pendulum 42 *lb.* $\frac{1}{2}$ *oz.* (when reduced to the Center 3024). Whence m : M :: $\frac{13}{350}$: 42,031 :: 1 : 1034,6; therefore the Velocity of the Powder $v = \frac{M + m}{m} \times V = 1035{,}6 \times 2{,}6 = 2692{,}56$; (ſee 1009 and 3034). Hence the Velocity of the Particles of Powder, (ſuppoſing the Whole of it (together with the Wad) impinged upon the Pendulum,) was at the Rate of 2692 $\frac{1}{2}$ Feet *per* 1″ of uniform Motion.

3055. And this is the leaſt Velocity the Particles of Powder can be ſuppoſed to acquire in the Expanſion. For firſt we may obſerve, that not more than $\frac{3}{10}$ of the Whole is converted into this elaſtic Fluid. Since 1 *oz.* = 437 *grs.* produced 460 Cubic Inches (2086) and each Cubic Inch weighs 0,293 = $\frac{3}{10}$ of a Grain (2082). Whence 460 × 0,393 = 134,78 Grains of elaſtic

elaſtic Air. But $\frac{125}{437} = \frac{3}{10}$ of the whole Powder, nearly. Therefore the other $\frac{7}{10}$ muſt, in mixing with the elaſtic Part, greatly impede the Action, and retard the Motion or Velocity thereof in Exploſion; eſpecially if it be conſidered that this inert Part is in ſome Meaſure of an unctuous Nature, and will not be thrown out, but ſtick or adhere to the Inſide of the Barrel an Impediment to the Reſt.

3056. Again, ſome Part of the Flame muſt be loſt in Expanſion Sideways through 19 Inches of Air; for an elaſtic Fluid expands itſelf equally every Way, and conſequently only a Part thereof can impinge upon the Pendulum. And even that Part will meet with a great Reſiſtance from the Air, and ſo will have its Velocity diminiſhed on that Account. The Quantity therefore of the Powder, and its Velocity, eſtimated this Way, is ſhort of what it is in the Barrel, and at the Exit from the Muzzle thereof. Now to diſcover what that is very accurately, Mr. *Robins* contrived the Experiment in a different Manner, which I ſhall here explain, with a Variation of ſome Circumſtance to render it eaſier to be underſtood and practiſed.*

3057. The Method of proceeding is this; let the Barrel be fixed to the Center of the Pendulum, and let the Weight of the Pendulum, Barrel and all, be 56 *lb.* (or 42 *lb.* reduced to the Center, as *per* Art. 596); in this Situation let it be charged with 12 *pwt.* of Powder only, put cloſe together with the Rammer, and then upon diſcharging the Piece, the Pendulum will aſcend through an Arch, whoſe Chord, at a Medium, will be $14\frac{1}{2}$ Inches. Now ſince in this Caſe the Powder (or rather its elaſtic Air) acts equally on its ſelf and on the Pendulum, therefore the Quantity of Motion or Momentum of the Powder (*viz.* $q = v\,m$) will be equal to that of the Pendulum ($Q = V\,M$) that is, $V\,M = v\,m$ (970). Alſo, ſince to $14\frac{1}{2}$ Inch Chord there correſponds the Velocity of $2\frac{3}{4}$ Feet *per* $1''$; therefore ſay, as $m = \frac{12}{3120} : M = 42 :: V = 2\frac{3}{4} : V = 3082$, nearly; ſo that the Velocity is by this Means determined for the whole Maſs of Powder to be at the Rate of 3082 Feet *per* Second.

* This Pendulum with the Barrel fixed upon it in the Manner repreſented by Fig. 2. of the preceding Plate, we apprehend is the moſt facile and ſimple of any.

3058. If now ſo light a Body as 1 *pwt.* of Tow be placed before the Powder contiguous to it, it will preſently acquire the Velocity with which the elaſtic Part of the Powder will expand itſelf when uncompreſſed, and by this Means we may be able to meaſure that Degree of Velocity pretty nearly. Thus if the Barrel be charged with 12 *pwt.* of Powder, and 1 of Wadding, and thoſe fired, the Pendulum will aſcend through an Arch, whoſe Chord is 17,3; from which, if we ſubduct 14,5 for the Powder, the Remainder 2,8 will be owing to the Wad; and therefore ſince 2,8 Inches of Chord gives the Velocity 0,53 of a Foot *per* 1″; therefore ſay, as $\frac{1}{320}$: 42 : : 0,53 : 7128 Feet, the Velocity of the Wad, or that which the ſwifteſt Part of the Flame moved with *per* 1″.

3059. In this Way alſo the Velocity of the Bullet may be determined to a greater Exactneſs than by the former Method where the Barrel was fired, and at a Diſtance from the Pendulum. Thus let the Barrel be charged with 12 *pwt.* of Powder, and Bullet $\frac{1}{12}$ *lb.* (as at 3033); then upon diſcharging it, the Pendulum will aſcend through an Arch, whoſe Chord will be 32,3 Inches. Now 14,5 of this Chord is owing to the Impulſe of the Powder (as above 3057); therefore 17,8 is occaſioned by the Motion of the Bullet, and is ſomewhat greater than that determined (in 3033). Now to a Chord of 17,8 Inches, there anſwers the Velocity 3,45 Feet (as *per* 3034); whence the Velocity of the Bullet $o = \frac{M + m}{m} \times V = 505{,}6 \times 3{,}45 = 1734$ Feet, nearly. Whence we find the Velocity of the Bullet, at the Muzzle of the Gun, is at the Rate of (at leaſt) 1700 Feet *per* 1″.

3060. From this Experiment it appears, that the Action of the Powder on the Gun is the ſame, whether it impels a Bullet before it, or whether it be fired alone; and it is therefore a convincing Proof, that the whole Quantity of Powder is fired in the latter Caſe, as well as in the former. In all the Experiments hitherto mentioned, the Bullet has been ſuppoſed contiguous to the Powder; and Mr. *Robins* found that it was laid at a ſmall Diſtance from it (as an Inch, or two, at moſt) the Theory will agree very nearly with the Experiments. But when the Bullet

is

is laid at a conſiderable Diſtance from the Powder, as 12, 18, or 24 Inches, the Caſe will be very much altered, and the Bullet will be impelled or acted upon in a Manner very different from what it was before.

3061. For now we are to conſider the Powder, by the Time it reaches the Bullet, as acting with two Forces, *viz.* one by *Pulſion* or *Percuſſion*, which it receives from the great Velocity with which it's parts expand, and ſtrike againſt the Bullet, as ſhewn above (615), and the other Force is by Preſſure; for when the inflamed Powder has expanded itſelf into all the Cavity behind the Bullet, it will, after the firſt Impact, continue to preſs on the Bullet with all the Force of its Elaſticity, till the Bullet be out of the Barrel; and hence it will be eaſy to underſtand that the Velocity of the Bullet will be in this Caſe greater than if it were impelled by its elaſtic Preſſure of the Powder only; and it is demonſtrably ſo by Experiment.

3062. For Mr. ROBINS charged the Barrel of 45 Inches with 12 *pwt.* of Powder as uſual, and then placed the Bullet at the Diſtance of $11\frac{1}{4}$ from the Breech (or $8\frac{1}{2}$ from the Powder) and upon diſcharging it againſt the Pendulum, he found that it had acquired a Velocity of about 1400 Feet *per* 1″. Whereas if it had been acted upon by the Preſſure of the Flame only, it would not have acquired a Velocity of 1200 Feet *per* 1″, as may be eaſily made appear by the Theory, thus; let M be the Place of the Bullet (ſee Fig. to 2097) then will MN repreſent the Force of Preſſure on the Ball the firſt Inſtant; but it is AM = 11,25 : AF = $2\frac{5}{8}$ :: HF = 69800 : MN = 16131, (2099). Then we find (by the Methods in 3006 and 3002) the Rectangle MRPB : Space MNQB :: MR × MB × 0,43429 : AM × MN × L $\frac{AB}{AM}$; which Quantities are all known; the Subduplicate of this Ratio is that of 1 to 86,338; now the Velocity acquired in falling through the Space = MB, is 13,51 Feet *per* 1″. Therefore ſay, as 1 : 86,338 :: 13,51 : 1166 Feet, the Velocity acquired by the Force of Preſſure only (by 2096).

3063. And to this the Experiment agrees; for the ſame Gentleman (with his uſual Sagacity) in order to ſeparate the two different Actions of Powder upon the Bullet, and to retain that only which aroſe from the continued Preſſure of the Flame, contrived

trived the following Method. He no longer placed the Powder at the Breech from whence it would have full Scope for its Expanſion, but ſcattered it as uniformly as might be through the whole Cavity of 11 $\frac{1}{4}$ Inches left behind the Bullet, conceiving that by this Means the progreſſive Motion or Velocity of the Flame in each Part would be prevented by the Expanſion of the neighbouring Parts. And he found upon diſcharging the Barrel, the Velocity of the Ball was (inſtead of 1400 Feet *per* 1″) no more than 1100 Feet *per* 1″. Which was 66 Feet ſhort of what it ſhould be by the Theory.

3064. This Deficiency he ſuppoſes was owing to ſome inteſtine Motion of the Flame; for the Powder being kindled in a Space much larger that it could fill, muſt have produced many Reverberations, and Pulſations of the Flame; and from theſe internal Agitations of the Fluid, its Preſſure on the containing Surface, he ſuppoſes, muſt be conſiderably diminiſhed; and from hence he judges it neceſſary, to avoid any ſuch Irregularity, to take particular Care to have the Powder confined cloſely in as ſmall a Space as poſſible, even when the Bullet lies at ſome little Diſtance from it.

3065. From what has been ſaid, it will be apparent, that when the Ball lies at a great Diſtance from the Charge, the Action of the Powder will be ſo greatly augmented in the Space behind, and will be ſo accumulated and condenſed by the Velocity each Part has acquired by the Time it comes to the Ball, that if the Barrel be not of an extraordinary Firmneſs in that Part, it muſt by this reinforced Elaſticity of the Power infallibly burſt. And the Truth of this Reaſoning he experienced in an exceeding good *Tower Muſquet*, forged of very tough Iron; for charging it with 12 *pwt.* of Powder, and placing the Ball at 16 Inches from the Breech, on firing it, the Part of the Barrel juſt behind the Bullet was ſwoln out to double its Diameter, like a blown Bladder, and two large Pieces of two Inches long, were burſt out of it.

3066. From what has been ſaid, we ſee the Reaſon of all the extraordinary and enormous Effects of warlike Engines in which Gunpowder is uſed; as in *Grenades*, *Bombs*, *Petards*, *Mines*, *&c.* For as we have ſhewn the Force of this Powder upon every ſquare Inch is 15000 *lb.* (2093), and that it expands itſelf with

a Velocity more than 3000 Feet *per* 1″ (3057) it is no Wonder it ſhould burſt the Shells of Bombs and Grenades with ſuch Force and Violence, and rend the Planks and Parts of Gates, Bridges, Walls, *&c.* in Petards, with ſuch ſudden inſuperable Power and Impetuoſity. For the Powder in the Petard weighing 5 *lb.* will have an Effort equal to that of a Cannon-Ball of 10 *lb.* moving with half the Velocity, *viz.* at the Rate of 1500 Feet *per* 1″. For the Momentum is the ſame in both Caſes (970).

CHAP. X.

The QUANTITY *of the* AIR'S RESISTANCE *to* PROJECTILES, *and* BULLETS *in particular, determined by* EXPERIMENTS *on the* Balliſtic PENDULUM.

3067. WE have demonſtrated the Principles on which the Computation of the Velocity and Reſiſtance of Bullets depends; and have confined the Theory by Experiments in reſpect of the former: We now proceed to do the ſame Thing for the latter, that is, we ſhall ſhew how the Quantity of the Air's Reſiſtance to Bullets is to be computed from the Theory joined with the Experiments made by Mr. ROBINS for that Purpoſe.

3068. We have already ſhewn, that when two Bodies impinge on, or ſtrike each other, the Magnitude of the Stroke is proportional to the Loſs of Motion in the percutient Body (1008). Now in the Caſe of a Shot made with a Gun, Bow, *&c.* the two impinging or percutient Bodies are the Bullet and the Air; which, indeed, are very different in their Natures; the leaden Ball being a continuous and unelaſtic Subſtance; but the Air a diſcontinued elaſtic Body. The former in Motion, the latter at Reſt. Yet this, notwithſtanding their Actions on each other, may be eaſily eſtimated both by Theory and Experiment as follows.

3069.

3069. Mr. ROBINS charged the Muſket-barrel of 45 Inches with the Bullet and Powder as uſual, and fired it againſt the Pendulum at the Diſtance of 25, 75, and 125 Feet from the Muzzle of the Piece at three ſeveral Times reſpectively; and he found that it impinged againſt the Pendulum in the firſt Caſe, with a Velocity of 1670 Feet *per* 1″; in the ſecond Caſe, with a Velocity of 1550 Feet *per* 1″; and in the third Caſe, with a Velocity of 1425 Feet *per* 1″. Therefore in ſtriking againſt 50 Feet of Air, it loſt a Velocity of about 120 Feet *per* 1″. Now ſince the whole Motion of the Bullet in the firſt Caſe was $1670 \times \frac{1}{12}$ (by 970, 3003) and in the ſecond Caſe it was $1550 \times \frac{1}{12}$; therefore the Difference $120 \times \frac{1}{12} = \frac{120}{12} = 10$ *lb.* will be the Loſs of Motion in the Bullet which it ſuſtained in paſſing through the 50 Feet of Air; but this Loſs of Motion was the Effect of the equal Reaction or Reſiſtance of the Air; conſequently the Reſiſtance of the Air to a Bullet moving with the mean Velocity of 1610 $\left(= \frac{1670 + 1550}{2}\right)$ Feet *per* 1″ is about 120 Times its Weight.

3070. To find the Time which was ſpent in paſſing through this 50 Feet of Air; ſay, as the mean Velocity 1610 Feet : 1″ = 60‴ :: 50 Feet : 1,87‴. And therefore to find the Reſiſtance of the Air to the Bullet paſſing through the ſame Space with any given Velocity, and Time will be eaſy as follows. In a ſecond Experiment made with all poſſible Care, the Mean of three Diſcharges againſt the Pendulum placed at 25 Feet Diſtance was the Velocity of 1690 Feet *per* 1″, and of 5 Shot againſt the Pendulum at the Diſtance of 175 Feet, the Mean was a Velocity of 1300 Feet *per* 1″. The Velocity loſt in this Caſe in paſſing through 150 Feet, was that of 390 Feet *per* 1″, or 130 Feet *per* 1″ for 50 Feet of Air.

3071. Now the Mean Velocity in this Caſe was $\left(\frac{1690 + 1300}{2} = \right)$ 1490 Feet *per* 1″. And ſince the Reſiſtance is at leaſt as the Square of the Velocity (2068): Therefore ſay, as $1490^2 : 1610^2$ :: 130 Feet : 152 Feet, nearly; ſo that the Loſs of Motion would be 152 Feet *per* 1″, with the Velocity 1610 Feet *per* 1″. But the Time ſpent in paſſing this 50 Feet of Air is about 2‴; and ſo this Loſs of 152 Feet in

in 2‴ will be but 142 in 1,87‴, the Time of paſſing through 50 Feet with the Velocity of 1610 *per* 1″. Conſequently the Loſs of Motion or (its Cauſe) the Reſiſtance of the Air in this Caſe was $142 \times \frac{1}{12}$ *lb.* $= 11{,}83$ *lb.* or almoſt 12 *lb. viz.* 142 Times the Weight of the Bullet.

3072. Let us next ſee what Reſiſtance a leſs Degree of Velocity met with from the following Experiment made by the ſame Gentleman with great Accuracy; he charged the ſame Gun with the ſame Bullets, but with a leſs Quantity of Powder; and the Mean of 5 Shot made againſt the Pendulum, at the Diſtance of 25 Feet, was a Velocity of 1180 Feet *per* 1″; and then of 5 others againſt the Pendulum removed to the Diſtance of 250 Feet, the mean Velocity was that of 950 Feet in 1″. Whence the Ball in paſſing through 225 Feet of Air loſt a Velocity of 230 Feet *per* 1″, or 51 Feet in ſtriking againſt 50 Feet of Air. Now the mean Velocity is that of 1065 Feet *per* 1″; whence the Time of deſcribing that Space 225 Feet was about 14‴; and that of deſcribing 50 Feet, about 3,1‴; conſequently in 1,87‴ the Loſs of Motion will be about 31 Feet *per* 1″; wherefore $31 \times \frac{1}{12} = 2$ *lb.* 10 *oz.* nearly, which is the Reſiſtance of the Air to this Velocity.

3073. Having thus determined by Experiment what Reſiſtance the Bullet meets from the Air with two different Degrees of great Velocity; let us next ſee what will reſult from a Computation made from the Theory eſtabliſhed by Sir *I. Newton* for ſlow Motions, the Principles of which we have already explained in Chap. V. and ſhall now apply them. Since the Weight of a Globe of Air, equal to the Bullet, is inconſiderable in Compariſon of the Weight of the Bullet itſelf, we have (2061) A = B. And conſequently $\sqrt{\frac{S}{16{,}2}} = T$, (2066). Alſo becauſe it is $D = \frac{3}{4}$, we have $\frac{4}{3} D = 1$; and ſo $S = \frac{D}{d}$; but Lead is 11,345 Times heavier than Water, and Water is 860 Times heavier than Air; whence $D : d :: 9756{,}7 : 1$. Conſequently S = 9756,7 Inches = 813 Feet;

whence $\sqrt{\frac{S}{16,2}} = 7'' = T$, the Time of the Fall through S.

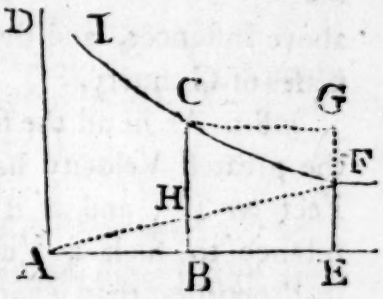
Then the Velocity $V = \frac{2S}{T} = 229,5$ Feet *per* 1″.

3074. Now this is the greateſt Velocity the Bullet can acquire by falling in the Air (2061) and the Reſiſtance it then meets with is equal to $B = A = \frac{1}{112}$ *lb.* = Weight of the Bullet; if we would have the Reſiſtance, therefore, to any other Velocity, as that of 1600 Feet *per* 1″, we muſt ſay, as $229,5^2 : 1600^2 :: \frac{1}{112}$ *lb.* : 4 *lb.* nearly. But we have before found by undeniable Experiment, that a Velocity of about 1500 Feet *per* 1″ (3072) meets with a Reſiſtance of 12 *lb.* nearly; therefore the Reſiſtance to ſwift Motions is greater than that to ſlow Motions (which is as the Squares of the Velocity) in the Ratio of 12 to 4, or 3 to 1; according to what was delivered in the Theory (2072).

3075. If we enquire by the Theory what Reſiſtance a Velocity of 950 Feet *per* 1″ will meet with, we ſhall find it to be about 1,427 *lb.* but by Experiment it was found to be in Reality 2 *lb.* 10 *oz.* which is nearly as 1 to 2. We ſhall ſee that this Reſiſtance is very ſenſibly encreaſed, even in ſo ſmall a Velocity as that of 400 Feet *per* 1″. For by Experiment, a Bullet diſcharged with that Velocity ranged but 319 Yards, or 957 Feet, on the Surface of Water. Whereas by the Theory for ſlow Motions, it ſhould have ranged to the Diſtance of 1109 Feet, as I ſhall now demonſtrate.

3076. The Reſiſtance to a Velocity of 400 Feet by the Theory is 0,253 *lb.* (found as above); then ſay, as $\frac{1}{112} = 0,083$ *lb.* : 0,253 *lb.* :: 16,2 Feet : 49,4 Feet, the Space a Body would deſcend through in one Second if urged by a conſtant uniform Force equal to the Reſiſtance 0,253 *lb.* (999). Whence $49,4 \times 2 = 98,8$ Feet is the uniform Velocity acquired in 1″. And ſince the Times are as the Velocities (1000) when the Force is given; therefore ſay, as 98,8 Feet : 400 Feet :: 1″ : 4″,049 = the Time in which the Velocity of 400 Feet *per* 1″ will be generated by the Force = 0,253 *lb.*

3077.



3077. Now let ICF be an Hyperbola, whoſe Aſymptotes are AE, AD. Let AB be the Time juſt now found = 4″,049; and BC the given Velocity of 400 Feet *per* 1″, generated in that Time by the Force 0,253 *lb.* Alſo let BE = 4″, draw CG parallel to AE, and EG to BC, then will the Rectangle BCGE be as the Space deſcribed by the conſtant uniform Velocity BC in the Time BE; and the hyperbolic Space BCFE will be as the Space deſcribed with the ſame Velocity BC decreaſing by the Reſiſtance (2096).

3078. Now AB × BC = 1 = Power of the Hyperbola, (779) whence $BC = \frac{1}{AB}$; therefore the Rectangle BCGE $= (BC \times BE =) BE \times \frac{1}{AB} = \frac{BE}{AB}$; and the Area BCFE is the hyperbolic Logarithm of the Ratio $\frac{AE}{AB}$, which is equal to the tabular Logarithm of the ſame Ratio multiplied by 2,302585 (854). But $\frac{BE}{AB} = \frac{4}{4,049} = 1$, nearly; and the Logarithm of $\frac{AE}{AB}\left(\frac{2}{1}\right) = 0,301030$; therefore 0,301030 × 2,302585 = 0,6931, *&c.* Therefore ſay, as 1 : 0,6931 :: (BC × BE =) 1600 Feet : 1108,96 = 1109 Feet nearly, as before aſſerted (3075). The Reſiſtance therefore by this Theory is much leſs than what it really is, in as much as it gives a Velocity of 152 Feet in 1109 more than the Truth, in this ſo ſlow a Motion.

3079. Where this Theory can take Place, we have FE = Velocity at the End of the Time BE; and is eaſily had thus, as AE (= 2) : AB (= 1) :: BC (= 400 Feet) : FE (= 200 Feet). Alſo, by drawing AF, it will cut BC in H; and BH will be as the Reſiſtance at the End of the Time BE, in the Hypotheſis of the Reſiſtances being ſimply as the Squares of the Velocity, which is very erroneous as we have ſhewn in all the

above Inſtances, and therefore cannot be applied to any actua Caſes of Gunnery.

3080. As in all the foregoing Caſes of ſmall Shot, the greateſt Velocity has not exceeded that of 1700 Feet *per* 1″; and as it has been ſhewn that the Reſiſtance to ſuch a Velocity is three Times more in Proportion than what it is in ſlow Motions (3074) it will be ſtill neceſſary to ſhew what Proportion of this increaſed Reſiſtance belongs to all other leſſer Degrees of Velocity in order to compleat the Theory, and render it of general Uſe. For this Purpoſe let A B be a Right-line divided into 100 equal Parts, and let C D = $\frac{2}{3}$ A D, be divided into 1700 equal Parts; then ſince D B : A B : 1 : 3 :: Reſiſtance to a Velocity of 1700 Feet *per* 1″ : Reſiſtance given to the ſloweſt Motion; it is evident, that to a Velocity of any given Number of Feet in the Line C D, there will correſpond a Number in the Line A B, which in reſpect of the whole Line, will ſhew what Ratio of the encreaſed Reſiſtance belongs to that given Velocity; thus for Inſtance, to the Velocity of 1500 Feet *per* 1″ correſponds the Ratio of $\frac{40}{100} = \frac{2}{5}$, nearly; to the Velocity of 1000 Feet *per* 1″, the Reſiſtance is nearly in the Ratio of $\frac{60}{100}$ or $\frac{3}{5}$; and that of 500 Feet *per* 1″ has the Ratio $\frac{80}{100} = \frac{4}{5}$, nearly. And ſo for any other Velocity propoſed.



3081. Hence then if the Reſiſtance for any propoſed Degree of Velocity be calculated by the Theory, the Error of the Theory in ſwift Motions may be hereby corrected, and brought pretty near the Truth. Thus, ſuppoſe I find by the Theory that to a Velocity of 1000 Feet *per* 1″, there is a Reſiſtance of 1,5 *lb.* to a Ball of $\frac{1}{12}$ *lb. wt.* if this be encreaſed in the Ratio of $\frac{3}{5}$, by ſaying, as 3 : 5 :: 1,5 *lb.* : 2,5 *lb.* this 4th Number 2,5 = 2 *lb.* 8 *oz.* will be nearly the ſame as was found by Experiment (3072) to be the true Quantity of Reſiſtance. So that by this Contrivance the Theory for ſlow Motion may ſtill be applied to very good Purpoſe.

3082.

3082. We have ſhewn that when a Cannon Ball of 24 *lb.* is impelled by a full Charge of Powder, it acquires a Velocity of 1900 Feet *per* 1″; but if it were impelled with only a common Charge (*viz.* 12 *lb.* of Powder = $\frac{1}{2}$ the Weight of the Ball) its Velocity will then be about 1700 Feet *per* 1″. With this Velocity we have ſeen a Bullet of $\frac{3}{4}$ of Inch Diameter will meet with a Reſiſtance of 10 *lb.* (3096). And ſince the Reſiſtance is (*cæteris paribus*) in the duplicate Ratio of the Diameter, or as the Surfaces; and the Square of 5,325 = Diameter of the Cannon Ball, is equal to about 50 Times the Square of $\frac{3}{4}$, the Diameter of the Bullet; therefore the ſaid Cannon Ball will meet with about 50 Times the Reſiſtance of the Bullet, or 500 *lb. wt.* or about 20 Times its own Weight.

3083. That the Reſiſtance is in the duplicate Ratio of the Diameter in Globes or Balls of different Size, is deducible from (2033) where it was ſhewn to be as $\frac{1}{4}ap$, which is half the Area of a great Circle (830) and $\frac{1}{8}$ of the Superficies of the Sphere (839); therefore the Reſiſtance is in the Ratio of the Superficies of Globes, or in the duplicate Ratio of their Diameters (842).

CHAP. XI.

The THEORY *of* RESISTANCE, VELOCITIES, TIMES *and* SPACES *deſcribed by* PROJECTILES *in their perpendicular* ASCENT *and* DESCENT *in* reſiſting Mediums.

3084. BEFORE we can proceed further, it will be neceſſary to raiſe Theorems for aſcertaining the *Spaces*, *Times*, and *Velocities* in the *perpendicular Aſcent and Deſcent of Projectiles*, in a Medium reſiſting in any multiplied Ratio of the Velocity. In order to this, let s = Space, t = Time, v = Velocity, r = Reſiſtance. Then (by 971) we have $vt = s$; and ſince this holds in every Caſe of t and s, therefore $v\dot{t} = \dot{s}$,

3085.

3085. The Reſiſtance being the Reaction of the Medium, will be proportional to the Decrement of Motion it cauſes in the moving Body; that is, $r = -\dot{q} = -\dot{v}\,m$ (970); and when m is given, it is $r = -\dot{v}$, or the Reſiſtance is as the Decrement of Velocity. The Reſiſtance (r) will alſo be inverſely as the Time (t) in which it produces a given Effect ($-\dot{v}$), therefore $r = \frac{I}{t}$; wherefore in general $r = \frac{-\dot{v}}{t}$, or $r\,t = -\dot{v}$. Hence $t = \frac{-\dot{v}}{r} = \frac{\dot{s}}{v}$ (3084) therefore $r\,\dot{s} = -v\,\dot{v}$.

3086. If the Body aſcends, it will be retarded by two Forces, the Reſiſtance (r) of the Medium, and the centripetal Force, or Gravity (g); and this Retardation will produce a Decrement of Velocity ($-\dot{v}$) which will be as the Moment of Time (t), and the Sum of the retarding Forces $r + g$, (as is evident from the Nature of the Thing); therefore in general $r\,t + g\,t = -\dot{v}$. And for a deſcending Body, if the Reſiſtance be leſs than Gravity, it will be $g\,t - r\,t = \dot{v}$. But if Gravity be leſs than the Reſiſtance, then $r\,t - g\,t = -\dot{v}$.

3087. We have ſhewn that $v\,t = \dot{s}$, whence $v = \frac{\dot{s}}{t}$; therefore for the aſcending Body, we have $\overline{r\,t + g\,t} \times \frac{\dot{s}}{t} = -\dot{v} \times v = r\,\dot{s} + g\,\dot{s}$. But for a deſcending Body, if Gravity be greater than the Reſiſtance, the Theorem will be $g\,\dot{s} - r\,\dot{s} = v\,\dot{v}$; $r\,\dot{s} - g\,\dot{s} = -v\,\dot{v}$, when the Reſiſtance exceeds Gravity.

3088. If the Body aſcends in an unreſiſting Medium (or in Vacuo) then $r = 0$; and $g\,t = -\dot{v}$, and $g\,\dot{s} = -v\,\dot{v}$; but if the Body deſcends, it will be $g\,t = \dot{v}$, and $g\,\dot{s} = v\,\dot{v}$. When the Reſiſtance becomes equal to Gravity, then $r = g$, and $g\,t - r\,t = \dot{v} = 0$; that is, the Velocity will then become conſtant or uniform. And ſince till then the Velocity is continually increaſing (for there will be $\dot{v} = g - r$, in any given Time,) it is evident, the Velocity is in that Caſe a *Maximum*, as we have before ſhewn (2063).

3089. Suppoſe the Reſiſtance (r) be as any multiplied Ratio of the Velocity $\left(\frac{v^n}{a^{n-1}}\right)$, that is, let $r = \frac{v^n}{a^{n-1}}$, where (a)

is

is a given Quantity. Then for a deſcending Body it will be g; $-\frac{v^n \dot{s}}{a^{n-1}} = v\dot{v}$ (644) and ſo $\dot{s} = \frac{a^{n-1} v\dot{v}}{g a^{n-1} - v^n}$. Alſo, becauſe $\dot{s} = \frac{\dot{s}}{v}$, we have $\dot{s} = \frac{a^{n-1}\dot{v}}{g a^{n-1} - v^n}$. And for the aſcending Body, $\dot{s} = \frac{-a^{n-1} v\dot{v}}{g a^{n-1} + v^n}$, and $\dot{s} = \frac{-a^{n-1}\dot{v}}{g a^{n-1} + v^n}$.

3090. If we ſuppoſe the Reſiſtance to be as the Square of the Velocity, then $n = 2$, and $r = \frac{v^2}{a}$ (3089) let V = greateſt Velocity acquired in the Deſcent, and becauſe in that Caſe $r = g$ (3087) and v becomes V, we have $g = \frac{V^2}{a}$, and ſo $ag = V^2$. Let z = Space which a Body falls through in Vacuo, to acquire the greateſt Velocity V; then $g\dot{z} = V\dot{V}$ (3088), and (taking the Fluents) $gz = \frac{1}{2} V^2$; therefore $2gz = V^2 = ag$. Whence $a = 2z$.

3091. Therefore in the deſcending Body, it will be $\dot{s} = \frac{av\dot{v}}{ag - vv} = \frac{2zv\dot{v}}{VV - vv}$; let $V^2 - v^2 = x^2$, and taking the Fluxions, we ſhall have $v\dot{v} = -x\dot{x}$; and ſo $\dot{s} = \frac{-2zx\dot{x}}{xx} = \frac{-2z\dot{x}}{x}$. And taking the Fluents, $s = Q - 2z\,L.\,x$. (Here Q is ſome conſtant Quantity, and L. x is the Fluent of $\frac{\dot{x}}{x}$ by (849). But $Q - 2\,L.\,x \times z = Q - z\,L.\,x^2 = Q - z\,L.\,\overline{V^2 - v^2}$. Now when $s = 0$, and then $v = c$ = the Celerity with which the Body begins to deſcend, we ſhall have Q in that Caſe $= z\,L.\,\overline{V^2 - c^2}$. And conſequently $s = z\,L.\,\frac{\overline{V^2 - c^2}}{\overline{V^2 - v^2}}$.

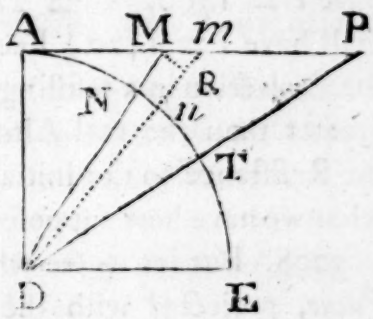
3092. Let L. $d = 1$; and it will be $s\,L.\,d = z\,L\,\frac{V^2 - c^2}{V^2 - v^2}$, and $\frac{s}{z} L.d = L.d^{\frac{s}{z}} = L\,\frac{V^2 - c^2}{V^2 - v^2}$; and therefore $d^{\frac{s}{z}} = \frac{V^2 - c^2}{V^2 - v^2}$; whence

whence we get $v^2 = \frac{V^2 \dot{z} + c^2 - V^2}{\dot{z}}$. And for the Time ($t$) of the Descent, we have $\dot{t} = \frac{a\dot{v}}{ag - v^2}$ (3089) $= \frac{2z\dot{v}}{V^2 - v^2}$ $= \frac{\frac{z}{V}\dot{v}}{V + v} + \frac{\frac{z}{V}\dot{v}}{V - v}$ (as will appear by reducing the two last Fractions to a common Denominator.) Therefore taking the Fluents, we have $t = Q + \frac{z}{V} L.\overline{V + v} - \frac{z}{V} L.\overline{V - v} =$ $Q + \frac{z}{V} L \frac{V + v}{V - v}$; let $t = 0$, then $v = c$; and we shall have $Q = -\frac{z}{V} L \frac{V + c}{V - c}$. Consequently, we have $t = \frac{z}{V} L. \frac{\overline{V + v} \times \overline{V - c}}{\overline{V + c} \times \overline{V - v}}$.

3093. If the Body falls from a State of Rest, then the inceptive Velocity $c = 0$; and the above Equations will become for the Space $s = z L \frac{V^2}{V^2 - v^2}$; for the Velocity at the End of the Fall $v^2 = \frac{V^2 \dot{z} - V^2}{\dot{z}}$; and for the Time of the Fall $t = \frac{z}{V} L \frac{V + v}{V - v}$. If we put $\dot{z} = m$, then $v = V\sqrt{1 - \frac{1}{m}}$.

3094. In the same Manner, we find the Theorems for an ascending Body are $s = z L \frac{V^2 + c^2}{V^2 + v^2}$; $v^2 = \frac{V^2 + c^2 - V\dot{z}}{\dot{z}}$; and $\dot{t} = \frac{a\dot{v}}{ag + vv} = -\frac{2z\dot{v}}{VV + vv}$ (3089, 3090). Now this fluxionary Equation is analogous to that which is found for the Fluxion of an Arch of a Circle, by Means of the Radius and Tangent of that Arch; and therefore the Fluent of the Time may

may be found in the Meaſure of the Arch of a Circle, as follows.



3095. On the Center D, with the Radius A D = V, deſcribe the Quadrant A T E; let the initial Velocity c be expounded by the given Tangent A P, the reſidual Velocity v by a Part of that Tangent A M; and draw D m infinitely near to D M; and then $\dot{v}$ = M m; draw D P; and from the Point M let fall the Perpendicular M R upon D m; then are the Triangles D N n, and D M R ſimilar; and ſo D M : D N (or D A) :: M R : N n. Alſo the Triangles m R M and M A D are ſimilar; and ſo D M : D A :: m M : M R. The reſpective Terms of theſe two Analogies being multiplied together, give $DM^2 : DA^2 :: M\dot{m} : N\dot{n}$; that is, $VV + vv : VV :: \dot{v} : N\dot{n} = \frac{VV\dot{v}}{VV + v\dot{v}}$ = the Fluxion of the Arch A N.

3096. If this laſt Equation be multiplied in each Part by $\frac{2z}{VV}$, it will become $\frac{2z \times N\dot{n}}{VV} = \frac{2z\dot{v}}{VV + vv}$; therefore $\dot{t} = -\frac{2z \times N\dot{n}}{VV}$ (3094); and taking the Fluents, it is $t = Q - \frac{2z \times AN}{VV}$. Let $t = o$, then will A M = A P, and A N = A T, and in that Caſe, therefore, $Q = \frac{2z \times AT}{VV}$. Therefore it will be $t = \frac{2z \times \overline{AT - AN}}{VV} = \frac{2z \times TN}{VV} = \frac{TN}{g}$; becauſe $2zg = V^2$, by (3090).

3097. To apply theſe Theorems to the Motion of a Bullet ſhot from a Gun in a perpendicular Direction upwards, on Supposition that the Reſiſtance is always in the *duplicate Ratio* of the Velocity. Let us take the Example of (3059) where the Bullet is projected with a Velocity c = A P = 1700 Feet *per* 1″. To determine the Height or Space of that Projection, we have s =

$\propto L \frac{V^2 + c^2}{V^2}$ (by 3094) becauſe in this Caſe $v = 0$. Now ſince $c = 1700$, $V = 229{,}5$ (3073); and $z = 813$ (*ib.*) we ſhall have $s = 1420\frac{1}{2}$ Feet, for the perpendicular Altitude of the Projection in a reſiſting Medium. Now this is conſiderably greater than the real Altitude, becauſe we have ſhewn (2072) the Reſiſtance to the initial Velocities is three Times more than what we have here ſuppoſed.

3098. But let us ſee what Height the Bullet will aſcend to *in Vacuo*, projected with the ſame Velocity of 1700 Feet *per* 1″. This is found by the Theorem $\frac{m^2}{4n} = a$ (1154) where $m = 1700$, and $n = 16{,}11$, and $t = 1''$. Whence $m^2 = 2890000$, and $4n = 64{,}44$; and ſo $a = 45675$ Feet, which is about 32 Times higher than before; and very likely 40 Times higher than it does really aſcend in Air with the given Velocity.

3099. The Time in which this perpendicular Altitude *in Vacuo* is deſcribed, is found by Theorem (1152) $\frac{s\,t\,m}{n} = T$, for in this Caſe $s = r = 1$, and $t = 1''$; therefore the Time of Aſcent and Deſcent is $\frac{m}{2n} = \frac{1700}{32{,}22} = 53''$ nearly.

3100. Now the Time in which the Bullet aſcends perpendicularly to the Height of $1420\frac{1}{2}$ Feet in the Air is equal to $\frac{TA}{g}$ (by 3096), becauſe in this Caſe T N becomes T A (ſee Fig. to 3095). To find the Arch A T we have the Radius A D = V = 229,5, and A P = c = 1700; wherefore ſay,

As A D	——	229,5	— 2.360783
Is to A P	——	1700	— 3.230447
So is Radius	——	90° 00′	— 10.000000
To the Tangent of the Angle P D A = 82° 18′			— 10.869664

3101. The Diameter is 2 A D = 459; therefore ſay, As 1 : 3,14159 :: 459 : 1442, the Circumference of the Circle. Then ſay, As the Circumference 360° is to the Arch 82° : 18′ ſo is 1442 to the Length of the Arch A T = 329,8. Now Gra-

Gravity produces an uniform Velocity $g = 32{,}22$ Feet *per* 1″. Whence $\frac{329{,}8}{32{,}22}\left(=\frac{AT}{g}\right) = 10''$ nearly, the Time of the Aſcent in the Medium of Air, reſiſting according to the duplicate Ratio of the Velocity. The Time therefore in this Caſe is not more than $\frac{1}{5}$ of that which is ſpent in the Aſcent *in Vacuo* (3099).

3102. The Time of the Deſcent through the Air from the ſaid Altitude of 1420 $\frac{1}{2}$ Feet, is $\frac{z}{V} L \frac{V+v}{V-v} = t$ (by 3093). But in order to determine this, we muſt firſt find the Velocity (v) acquired in the Deſcent from the Theorem $v = V\sqrt{1-\frac{1}{m}}$ (in Art. 3093) where $V = 229{,}5$; $d = 10$, (becauſe its Logarithm $= 1$ (by 3092), $s = 1420{,}5$; $z = 813$; and $m = d^{\frac{s}{z}} = 55{,}86$. Whence we find the Velocity $v = 227{,}44$ Feet *per* 1″. Therefore $V + v = 56{,}94$; and $V - v = 2{,}06$; and $L\frac{V+v}{V-v} = 2{,}345992$; alſo $\frac{z}{V} = 3{,}543$; therefore $t = 8''{,}31$. Whence it appears that the Times of the Aſcent and Deſcent are in the Ratio of 10 to 8,3; or nearly as 5 to 4, and both together make but 18″,3 which is 35″ leſs than the Time *in Vacuo* (3099).

3103. Hence it appears they are widely miſtaken who aſſert, that if an heavy Body be projected upwards with a Velocity greater than that which can be acquired in falling, the Time of the Deſcent will be greater than that of the Aſcent; ſince it is demonſtrated, that a longer Time is required to deſtroy a great Velocity in aſcending, than in generating a much ſmaller one in deſcending through the ſame Space.

3104. From this Theory alſo it is manifeſt, that if a Ball or Shot be projected downward with a Velocity equal to the greateſt that can be acquired in falling, the Motion will be uniform; but if it be projected with a Velocity greater than that, the Motion will be retarded; if with a leſs Velocity, it will be accelerated, but never will become equal to that greateſt Velocity, all which is evident from the Theorem in Art. 3092.

3105. Hence alſo we ſee the Reaſon why the Force acquired by a Bullet in falling is ſo much leſs than that with which it is projected; for ſince the Maſs of Matter continues the ſame, the Forces will be as the Velocities, that is, in the foregoing Example (Art. 3097, 3102). The Force of the Bullet, when projected, from the Muzzle of the Gun, is to the Force it acquires in falling, as 1700 to 227,5; or nearly as 8 to 1.

3106. Indeed, with reſpect to very light Bodies, the Caſe may be a little different; and the Times of Deſcent of an Arrow, a Ball of Wood, *&c.* may be greater than that of the Aſcent, ſince they are reſiſted by the Air in a much greater Proportion to their Quantity of Matter, on Account of their larger Quantity of Surface; what this may be, as nothing of Conſequence depends upon it, I have not here calculated; but it may be eaſily done by any one who underſtands the foregoing Theory, and has Curioſity and Leiſure for ſuch Amuſements.

3107. In all that has been hitherto ſaid of the perpendicular Projection in a reſiſting Medium, we may obſerve what an egregious Difference there is between ſuch a Shot and one made in *Vacuo* in regard of the *Height*, the *Time*, and the *Velocity* and *Force* of the PROJECTILE; and how widely the latter is different from the Truth.

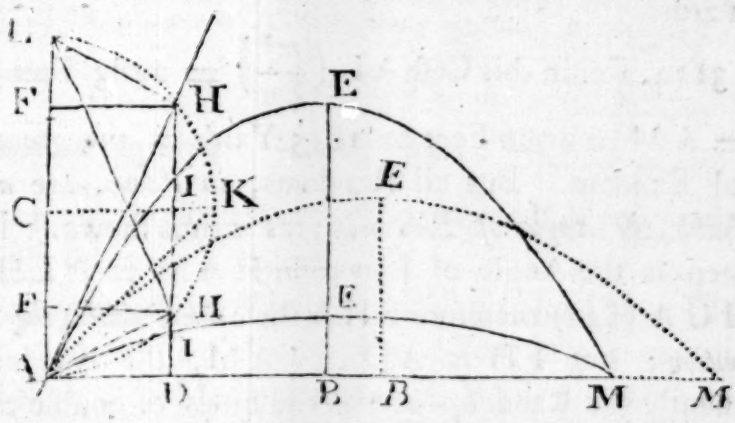
CHAP. XII.

The common practical RULES *of* GUNNERY, *derived from the* parabolical HYPOTHESIS, *compared with* EXPERIMENTS, *and thereby ſhewn to be extremely* fallacious, *and of no Uſe in* PRACTICE.

3108. WE next proceed to ſhew how very fallacious all the Rules and Caſes of *practical* GUNNERY are, as they are derived from the Principles of the *parabolical Hypotheſis*, which we have explained (in 1141, *&c.*) and ſhall here apply, in order to confute them by Experiments. After this we ſhall conſider the Nature of the Motion of a Body projected obliquely in a re-

reſiſting Medium, and the Form of the Path it deſcribes, and ſhew it to be no PARABOLA (as is always ſuppoſed) nor any other *regular* or *geometrical Curve*.

3109. It was ſhewn (1155) that the greateſt Amplitude or Random of a Shot (in Vacuo) was that which was made upon an Elevation of 45°. And it appears, that AD is equal to half AL, and therefore ſince AD = ¼ AM, we ſhall have AM = 2 AL = the *greateſt Random* poſſible. But we have ſhewn (in our common Example (3098) that AL = 45675 Feet; and therefore 2 AL = 91350 Feet = AM, the Random on 45°. Now this is about 17,3 Miles; but all experienced Gunners, and practical Writers aſſure us that this Range is actually not quite ½ a Mile; and ſo *Merſennus* found the horizontal Range of an Arquebuſe of 4 Feet to be leſs than 800 Yards, which is not quite ½ a Mile. The Range therefore by this falſe Theory is about 35 Times greater than the Truth.



3110. The Caſe is the ſame with reſpect to large Shot as in ſmall ones; for St. *Remy* (as quoted by Mr. *Robins*) tells of ſome Experiments made by Mr. *Du Metz*, in which the Range at 45°, of a Piece 10 Feet in Length, carrying a Ball of 24*lb.* and charged with 16*lb.* of Powder, was 2250 *French* Fathom, which is not quite 3 Miles. But ſuch a Ball was projected with a Velocity of about 1900 Feet *per* 1″, (Art. 3043.) and therefore *per* Theorem (1154.) $\frac{m^2}{4n}$ = AL = 56021 Feet; and ſo 2 AL = AM = 112042 Feet = 21 Miles, which is more than 7 Times the real Random, in the reſiſting Medium of Air.

3111. The ſame Difference between this Hypotheſis, and the Truth, will be found in Proportion to obtain in ſmaller Degrees of Velocity, and on any other Elevations, as is evident from Mr.

Mr. *Robins*'s Experiments compared with this erroneous Theory. For a leaden Bullet, $\frac{3}{4}$ of an Inch Diameter, discharged with a Velocity of 400 Feet *per* 1″, and in an Angle of 19° : 05′, of Elevation ranged on the horizontal Plane no more than 448 Yards, whereas by the Theory it ought to have ranged 1023 Yards.

3112. For in this Case AL $\left(\frac{m^2}{4n}\right)$ = 2483 Feet; and 2 AL = AM = 4966 Feet = 1655 Yards = the greatest horizontal Random. But all Randoms, *in Vacuo*, are *as the Sines of double the Angles of Elevation*; as is thus shewn. Draw CH,* then is the Angle of Elevation HAM = ALH (665) = $\frac{1}{2}$ HCA (642) therefore FH is the *Sine of double the Angle of Elevation*; but FH = AD = $\frac{1}{4}$ AM, the Random. Consequently the Randoms are as the Sines of double the Angles of Elevation. Therefore say,

As the Sine of double 45° = 90°	——	10.
To the Sine of double 19° : 5′ = 38° : 10′	——	9.790954
So is the greatest Random 1655 Yards	——	3.218798
To the Random on 19° : 5′ = 1023 Yards	——	3.009752

So that the Range made in the resisting Air is but about $\frac{3}{7}$ of what it would be in the *parabolic Hypothesis*.

3113. Again, a Ball was discharged with the same Velocity as in the last Experiment, but on an Elevation of 9° : 45′; and its Range on the Horizon was at a Medium 990 Feet, or 330 Yards. Now if this were to be deduced directly from the Theorem (in 1152) $\frac{c\,s\,m^2}{n}$ = AM, we should find AM = 1655 Feet, or 552 Yards nearly, as in the following Operation.

Add the Logarithms of	Sine of 9° 45′	——	9.228784
	Cosine, 80° 15′	——	9 993681
	And of m^2 = 160000	——	5.204120
From that Sum	——	——	4.426585
Subduct the Value of n = 16,11	——	——	1.207095
The Remainder is the Range AM = 1665,6	——		3.219490

3114.

* The Reader is desired to draw this Line in the preceding Figure, as it is there (by Accident) omitted.

3114. But if this Random were to be computed from the Random on an Elevation of 19° : 30′ found in (Art. 3112.) it would come out a very different Number from either of the foregoing; for ſay

As the Sine of double 19° : 5′ = 38° : 10′ —		9.790954
Is to the Sine of double 9° : 45′ = 19 : 30 —		9.523495
So is the Random on 19° : 5′ =	448 —	2.651278
To the Random required ——	242 —	2.383819

Now this is as much too little as the other was to big, and proves the Theory falſe on every Account.

3115. Again, for greater Variety, a Ball was fired at an Elevation of 8°, and with a Velocity of 700 Feet in 1″, and the horizontal Range at a Medium was 690 Yards; but if this Range be computed from the common Theory, we ſhall have $2AL\frac{m^2}{4n} = AM = 15208$ Feet, and then by (Art. 1154.) we ſhall find the Random at an Elevation of 8° to be 4192 Feet, or 1400 Yards nearly, which is more than double the real Diſtance.

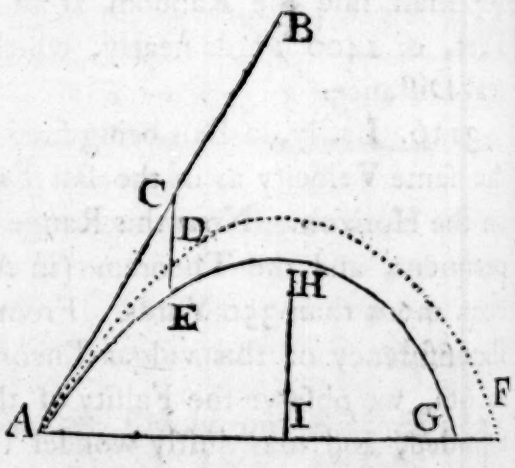
3116. Laſtly, a Ball being fired at an Elevation of 4° with the ſame Velocity as in the laſt Experiment ranged 600 Yards on the Horizon. Now this Range if deduced from the laſt Experiment, and the Theorem (in Art. 1154.) ſhould not have been more than 350 Yards. From all, which Inſtances of the Inconſiſtency of this vulgar Theory, with Facts and Experiments, we obſerve the Falſity of the Hypotheſis on which it is founded, and may juſtly wonder to ſee every Day Books publiſhed relating to GUNNERY, and the *Doctrine* of Projectiles, with as ſtrong a Preſumption of theſe falſe Principles, as though Philoſophy were not in the leaſt underſtood, and the Nature of theſe Things had never been enquired into by any one.

CHAP.

CHAP. XIII.

Sir ISAAC NEWTON's *Method of inveſtigating the* PATH *of a* PROJECTILE *in a* reſiſting *Medium illuſtrated, applied to practical* GUNNERY, *and exemplified by* EXPERIMENTS.

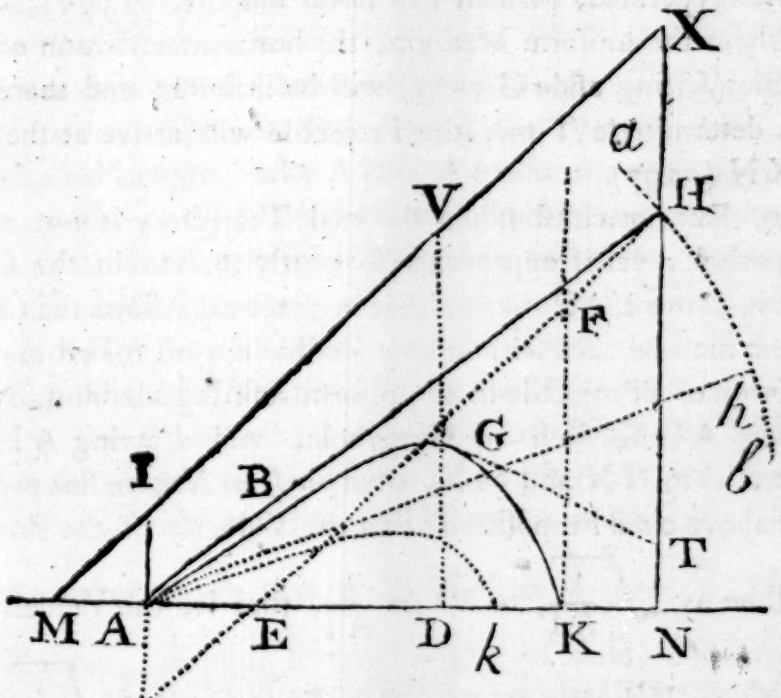
3117. BUT now that the ENGINEER may be aſſured from a genuine *Phyſico-mathematical* THEORY, that the Path which a Projectile deſcribes, when thrown from the Gun in any oblique Direction, even when the Reſiſtance of the Medium is no more than proportional to the *Square of the Velocity*, cannot poſſibly be a PARABOLA, nor any regular *geometrical Curve*, he is to conſider in the firſt Place, that ſince the Body is prevented going on in the Right-lined Direction A B by the perpendicular Action of Gravity by which it is brought down from any Point C to the Point D in the Curve of the Parabola A D F in Vacuo, ſo if we ſuppoſe the Projection to be made in a reſiſting Medium, the Reſiſtance will leſſen the Velocity of the Ball, and prolong the Time of its arriving to the Diſtance of D, and therefore as Gravity acts all the Time uniformly, it will produce in the Projectile a greater perpendicular Deſcent from the Point C, and carry it from C to ſome Point E below D; and ſince this is the Caſe every where, it follows that the Curve A E G deſcribed by the Projectile, cannot be the *Parabola* A D F, but ſome other Curve contained within it.



3118. Again, ſince the Velocity of the Projectile in the deſcending Part of the Curve H G will be always much leſs than that in the aſcending Part A H, as we have above demonſtrated

(3102).

(3102) therefore Gravity will in this Part carry the Ball through much greater perpendicular Spaces (in the ſame Time) from the parabolic Curve; and ſo the Curve in the Part HG will have a leſs Degree of Curvature, and therefore the Point G, where it falls on the Horizon, will be much nearer to the Line HI of the greateſt perpendicular Altitude, than the Point A from whence it was projected, different from what happens in the *Parabola*.



3119. And Sir ISAAC NEWTON has ſhewn, that the Curve AHG approaches nearer to the Form of an *Hyperbola* than that of a *Parabola*; for an *Hyperbola* will be truly deſcribed in a reſiſting Medium, whoſe Denſity is every where inverſely as the Tangent to the ſaid Curve, as he has proved in Prop. X. Lib. II. of the *Principia*. And therefore if AGK be the hyperbolic Trajectory of the Ball, projected from the Point A in the Direction AH, (in a Medium of a variable Denſity) whoſe Aſymptotes are MX, XN, cutting the horizontal Line MN in the Points M and N; and of which the latter XN is perpendicular thereto; then ſince in this Caſe the Denſity in any Points A and G is inverſely as the Tangents AH and

GT, it is manifeſt if the Medium be now ſuppoſed of an uniform Denſity, ſuch as is equal to the mean Denſity of the Medium in Queſtion, then in ſuch an one, on Account of the greater Denſity at A, the Velocity of the Projectile will be more diminiſhed than in the other Caſe, and conſequently the Trajectory will be continued within the Hyperbola AGK (by 3117) in the aſcending Part AG. But in the deſcending Part GK it will approach nearer to the Aſymptote XN than the Hyperbola, (which can never meet it, by 776); conſequently in the uniform Medium, the horizontal Motion of the Projectile (ſetting aſide Gravity) will be infinite; and therefore after a determinate Time, the Projectile will arrive at the ſaid Line XN (3097).

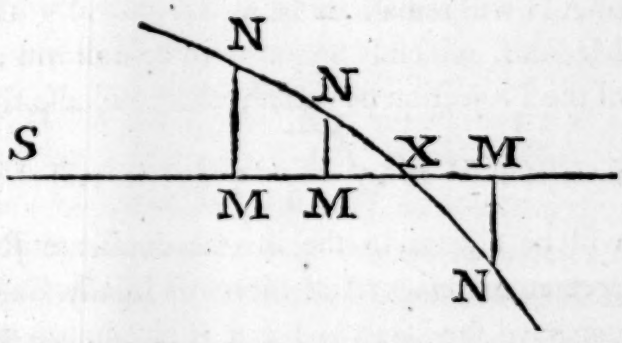
3120. But notwithſtanding the real Trajectory is not truly an Hyperbola, yet it approaches ſo nearly thereto in the ſmall Part of it above the Horizon, that in practical Affairs the *Conic-Hyperbola* may be uſed without any ſenſible Error, for eſtimating the Motion of a Projectile in an uniform reſiſting Medium. Now ſuppoſing AGK ſuch an Hyperbola, and drawing AI and HC parallel to NX and MX, then Sir *Iſaac Newton* has proved (in the above cited Propoſition) that the Velocity of the Projectile will be as $\sqrt{\frac{1}{AI}}$, or $V^2 = \frac{1}{AI}$, that is, the Velocity at A is to the Velocity at any other Point G, as $\sqrt{\frac{1}{AI}}$ to $\sqrt{\frac{1}{GV}}$. Alſo, that the Reſiſtance of the Medium is to the Force of Gravity (in any Point A) as AH to $\frac{1}{2}$AI. Moreover, from the Nature of the Curve, AM = KN; and AI = AC; and by Conſtruction IC = (2AI =) HX.

3121. Theſe Things premiſed, it is evident that if the Lines AI and AH be given in Magnitude and Poſition, the Hyperbola AGK may be deſcribed, becauſe in this Caſe we have given HX = 2AI, and therefore the Center of the Hyperbola X. Hence alſo the Aſymptote XN, and ſince the Point I is given, we have alſo the other Aſymptote XM cutting the horizontal Line in M. Hence, laſtly, we have given the Point K, by taking

taking K N = A M (776 to 780). It remains now to ſhew how the Lines A I and A H may be determined in the preſent Caſe of the Problem by Experiments, and thence an expeditious Method of deſcribing the Hyperbola A G K, or Trajectory of the Projectile; and thence a Solution of all the Caſes of the Problem.

3122. In order to this, let two equal Balls be diſcharged with the ſame Velocity on two different Angles of Elevation H A K, and *h* A *k*; and let the Points K and *k* where they fall on the Horizon be obſerved, and the Proportion of their Diſtances A K and A *k* be found by Meaſuration. And let A K : A *k* :: *d* : *e*. Then having erected the Perpendicular A I of any aſſumed Length, take A H or A *h* alſo of any Length whatſoever, and then by Scale and Compaſſes find the Lengths A K, and A *k*, as directed in the laſt Article; and if they are found to be in the ſame Ratio as *d* to *e* (by Experiment) then was the Length A H rightly aſſumed, and the Hyperbola A G K every Way ſimilar to that deſcribed by the Ball in the Air.

3123. But if not, take in the indefinite Right-line S M, the Length S M equal to the aſſumed Length A H; and erect the perpendicular M N equal to the Difference of the Ratios of A K to A *k*, and of *d* to *e*; that is, let $M N = \frac{A K}{A k} - \frac{d}{e}$. And in like Manner, aſſuming ſeveral Lengths of A H, find ſeveral other Points N, thro' all which draw a regular Curve N N X N, cutting the Line S M M M in X. Laſtly aſſume A H = S X, and thence again find the Lengths A K and A *k*; theſe ſhall have the ſame Ratio with *d* and *e*, or the ſame Lengths found by Experiment; and A I, and this laſt found A H, ſhall be ſimilar to, or have the ſame Proportion with thoſe which belong to the Hyperbola deſcribed by the Ball in the Air. For ſince $\frac{A K}{A k} - \frac{d}{e}$



$\frac{d}{e}$ = MN, where SM = SX there MN = o, and conſequently there $\frac{AK}{Ak} = \frac{d}{e}$; and ſo AK : Ak :: d : e.

3124. To illuſtrate this by an Example; let it be required to find AI and AH for the two Shots made upon the two Elevations of 19° 5′ and 9° 45′ (in Art. 3112, 3113) where the horizontal Ranges were 448 Yards = d, and 330 = e. Therefore $\frac{d}{e} = \frac{448}{330} = 1{,}36$. Then aſſuming AI = 10, and AH = 70, you find AK = 45, and Ak = 37,8, and ſo $\frac{AK}{Ak} = \frac{45}{37{,}8} = 1{,}19$ (by 3122). And then $\frac{AK}{Ak} = \frac{d}{e} = 1{,}36 - 1{,}19 = 0{,}17$ = MN. In like Manner, if you aſſume other Lengths AH = 60, 50, 45, 40, reſpectively you will find other Values of MN = 1,23. 1,28. 1,39. 1,50. Then drawing a Curve through the extreme Points of all the Ordinates MN, and it will interſect the Line SM in X, ſo as to give the Abſciſſa SX = $46\frac{1}{2}$ = AH ſought; and in this Caſe AK will be to Ak as 448 to 330; or meaſured upon the Scale by which AI and AH were laid down, they will be AK = 25,7 and Ak = 18,8.

3125. Having thus obtained the Proportion or Values of the Lines AI and AH in this one Caſe, they remain unalterably the ſame for all other Angles of Elevation, for Diſcharges of the ſame Velocity. And indeed in all Caſes whatſoever, the Value of AH will remain as being reciprocally as the Denſity of the Medium, which is ſuppoſed to be uniform; but if the Velocity of the Projection be varied, then will alſo the Value of AI. For ſince it is $V = \sqrt{\frac{1}{AI}}$ (3120) it will be $AI = \frac{1}{V^2}$, or AI will be always in the inverſe duplicate Ratio of the Velocity greater or leſs. And therefore in any Caſe of a given Velocity you have the Lines AI and AH, and ſo the Trajectory may be deſcribed for any Angle of Elevation HAN, according to this Theory.

3126.

3126. Hence likewiſe the Ratio of the Reſiſtance (R) in the Point A to Gravity (G) is given. For R : G :: AH : $\frac{4}{3}$ AI (3120) :: 4,65 : $\frac{4}{3}$ = 1,333 (3124). Therefore if the Weight of the Bullet were $\frac{1}{1\frac{1}{2}}$ *lb.* then $\frac{4}{3}$: 4,65 :: $\frac{1}{1\frac{1}{2}}$: $\frac{13,95}{48}$ = 0,288 *lb.* the ſame nearly as found by the common Method to the ſame Velocity (in 3076).

3127. Having now found the Values of AI and AH for any propoſed Velocity of Projection, we may proceed to find the Random AK generally for any given Velocity and Angle of Elevation HAN; in order to this let AH = *a*, AI = *b*, and then HX = 2 AI = 2 *b*. Alſo let AK = *x*, AN = *z*, and NH = *y*; and we ſhall have *z* — *x* = KN = AM = AE, and AC = AI = *b*. Alſo EN = AK = *x*. Then by the ſimilar Triangles EAC, ENH, we have this Analogy AE (*z* — *x*) : EN (*x*) :: AC (*b*) : HN (*y*). And thence compounding, *z* : *x* :: *b* + *y* : *y*; whence $x = \frac{zy}{b+y}$, and $z = \frac{bx + xy}{y}$; and $zz = xx \times \frac{\overline{b+y}^2}{yy}$. But becauſe of the right Angle ANH, it is $aa - yy = zz = xx \times \frac{\overline{b+y}^2}{yy}$. And ſo $\frac{a^2y^2 - y^4}{\overline{b+y}^2} = xx$; and therefore $x = AK = \sqrt{\frac{a^2y^2 - y^4}{\overline{b+y}^2}}$.

3128. Hence alſo if the Amplitude AK be given, then may the Sine HN of the Elevation neceſſary for ſtriking any Object K at the given Diſtance AK be found by the ſame Theorem. For we ſhall thereby get $b^2x^2 = \overline{a^2 - x^2} \times y^2 - y^4 - 2bx^2y$; from which Equation by a few eaſy Trials, the Value of *y* = NH may be found. Or in a Method eaſy for Practice thus. On the Center A with the Radius AH deſcribe a Circle, as (*ab*); and on the given Point K erect the Perpendicular KF; then apply a Ruler to the Point C in ſuch Manner that the Part FH intercepted between KF, and the Circle *ab* may be equal to CE. Then through the Point H, draw the Right-line AH, and it ſhall be the Direction or Elevation required. For ſuppoſing the Angle given, we have always CE = FH, becauſe (by

(by Conſtruction 3120) the Angles A C E and H E N are ſimilar, and therefore C E : A E :: E H : E N :: F H : K N; hence ſince A E (= M A) = K N, we have C E = F H. And the Point H muſt be in the Circle *a b*, becauſe of the given Length A H.

3129. By putting the above Equation (3127) into Fluxions, and making the Fluxion of x = A K equal to nothing, we ſhall get an Equation for the Sine N H of the Angle or Elevation which of all others will produce the greateſt horizontal Random A K; for then we ſhall get $2a^2yy - 4y^3\dot{y} = 2bx^2\dot{y} + 2x^2y\dot{y}$; and dividing by $2\dot{y}$, we have $a^2y - 2y^3 = bx^2 + x^2y = x^2 \times \overline{b+y}$. But $xx = \frac{a^2y^2 - y^4}{\overline{b+y}^2}$; wherefore $a^2y - 2y^3 = \overline{b+y} \times \frac{a^2y^2 - y^4}{\overline{b+y}^2} = \frac{a^2y^2 - y^4}{b+y}$. Whence by Reduction, and by dividing by y, we get $aab = 2byy + y^3$.

3130. Here it is plain the Quantity $2byy$ is leſs than aab, and therefore yy leſs than $\frac{1}{2}aa$, or H N^2 leſs than $\frac{1}{2}$ A H^2; and conſequently the Angle H A N is leſs than half a Right-angle, which it would be in an unreſiſting Medium, (ſee 1155) and in that Caſe, we have $yy = \frac{1}{2}aa$, and $y = a\sqrt{\frac{1}{2}}$, which Value of y, if it be ſubſtituted in the above Equation, in order to reduce it, it becomes $aab = 2byy + yya\sqrt{\frac{1}{2}}$; whence $yy = \frac{aab}{2b + a\sqrt{\frac{1}{2}}}$ and ſo $y = \sqrt{\frac{aab}{2b + a\sqrt{\frac{1}{2}}}}$.

3131. Let us now ſee what the Quantity of this Angle is for the Values of A I and A H as determined above (3124) where, if we put A I = b = 1, then A H = a = 4,65, and the Equation is $aa = 2yy + y^3 = 21,6225$. Now $y = \sqrt{\frac{21,6225}{2 + 4,65\sqrt{\frac{1}{2}}}} = \sqrt{\frac{21,6225}{5,255}} = 2,03$. But it is plain this Value of y is too ſmall, for it makes $2yy + y^3$ little more than 16, whereas it muſt be = 21,6255; this was occaſioned by making $y = a\sqrt{\frac{1}{2}} = 3,25$ according to the cuſtomary Method, which makes one of the Roots too large, and conſequently the whole Quantity $2yy + y^3$ too ſmall. The beſt Way, then, to approximate to the Value of y is (*Tentando*) by Trial, for by this Means we ſhall eaſily find that $y = 2,26$ nearly. Therefore ſay, as A H = 4,65 : H N = 2,26 :: Radius : Sine of 29° nearly; whereas *in Vacuo* we have ſhewn this Angle is preciſely 45 Degrees (1155).

3132.

3132. If it be required to find what this *Maximum* of Amplitude is for the above found Values of A I and A H, you have it determined by the Theorem in (3127) (and alſo for any other Degree of Velocity, as ſhewn in 3125). So according to the preſent Caſe (3124) we ſhall find the greateſt Value of $AK = x = \sqrt{\frac{a^2y^2 - y^4}{b - y^2}} = 2{,}83$ nearly; and therefore ſay, as AK = 2,57 (by Experiment) : AK = 2,83 (a *Maximum*) :: 448 Yards : 490 Yards nearly, the greateſt Amplitude.

3133. If the greateſt Ordinate or Height of the Projection be required, let the Line D N be taken in the horizontal Line A N, a Mean proportional between A M and A N, and through the Point D, draw the Ordinate G D, and it will be the *Maximum* required. This Ordinate G D is equal to the Difference between the vertical Line N X, and a 4th Proportional to the Lines D N, A N, and 2 A I = I C; or $GD = NX - \frac{2AI \times AN}{DN}$. The Demonſtration hereof may be eaſily deduced from the abovementioned Propoſition of Sir *Iſaac*'s *Principia* (3119). But it is too prolix and intricate to be here inſerted.

3134. To exemplify this in the laſt Caſe of the greateſt Amplitude AK; we have AK = x = 2,83, and y = HN = 2,26 (3131). Alſo we have $z = AN = \frac{bx + xy}{y} = 4{,}1$ nearly (3127); then AN — AK = KN = 1,27 = AM. Therefore $DN = \sqrt{AN \times AM} = 2{,}28$. Now NX = NH + HX = 2,26 + 2 = 4,26; and $\frac{2AI \times AN}{DN} = \frac{2 \times 4{,}1}{2{,}28} = 3{,}6$. Therefore GD = 4,26 — 3,6 = 0,66; which gives about 110 Yards for the greateſt Height of the Projection. It is obſervable alſo, that DK = (DN — KN =) 1,01 and AD = (AK — DK =) 1,82; and ſo the Point D is nearly $\frac{2}{3}$ of the whole Random from the Point A; whereas *in Vacuo* it is always the Middle Point between A and K.

3135. From a bare Inſpection of the Trajectory A B G K, it is evident, the Path of the Projectile will nearly coincide with the Tangent A H for a conſiderable Diſtance A B, which in a Shot of 24 *lb.* will be about 500 Yards (as the Angle there will not exceed $\frac{1}{2}$ a Degree;) and hence we ſee the Reaſon why Mr. *Anderſon*, the moſt eminent of all practical Authors on Gunnery, found himſelf obliged to ſuppoſe that the Track of Shells and Cannon Balls was much leſs incurvated,

vated, than what it ought to be on the *parabolical Hypotheſis*; and in order to reconcile this Circumſtance with the ſaid Theory, he imagined that every Shot was impelled to a certain Diſtance A B in a *ſtrait Line*, or that in ſuch a Diſtance it was not affected by the Power of Gravity. But ſo ſtrange and abſurd a Suppoſition, as the Suſpenſion of the Power of Gravity for a Moment, plainly evinced how ignorant he was of the great Diminution of the Velocity of the Shot in its Flight from the Reſiſtance of the Air. The Notion, therefore, of a *Point-blank Shot*, as they call the Diſtance A B, is entirely groundleſs, and owing to the vulgar Error of the Air's Reſiſtance being inconſiderable.

3136. We have now delivered all we can think neceſſary relating the Theory and Practice of Gunnery; and it is with ſome Regret we are obliged to conclude, that from a true and genuine Theory, the *practical Part* of this moſt neceſſary Science appears very perplext and difficult; it is for this Reaſon we deſiſt from purſuing it any further. The celebrated Mr. *J. Bernoulli* has given a moſt exquiſite Theory of the Path of a Projectile in a reſiſting Medium, but as the Practice reſulting thence depends upon the Quadrature of mechanical Curves, it can by no Means be rendered uſeful.

3137. The late learned Mr. *Simpſon* has alſo given us an admirable Theory, but the practical Rules that might be from thence deduced would prove ſo difficult, that few of our ENGINEERS, I fear, will care to be at the Trouble either to underſtand, or reduce them to uſe. Mr. *Muller* has ſupplied us with *converging Series* for the various Caſes of Projectiles in a *reſiſting Medium*; but then he has given no Demonſtration of them, and they are upon Suppoſition that the Reſiſtance is as the *Square of the Velocity* only, which is far from the Truth, as we have ſhewn; beſides *infinite Series* ſeem but little adapted to Caſes of *practicable Gunnery*.

3138. In the laſt Place, Mr. ROBINS himſelf, in a poſthumous Tract, has left us ſome of the beſt Methods of reducing the Genuine (though complicated) Theory of military Projectiles to practice, but even theſe will be attended with Trouble and Difficulty enough; and beſides, being publiſhed without any Inveſtigation, or Rationale, which he propoſed doing if he had lived, they will be found as little ſatisfactory as ready in Practice. And I think we may conclude, Sir *Iſaac Newton* would not have given us the above Method for deducing the *practical Rules of Gunnery* if he had known of any other that was better.

INSTI-

INSTITUTIONES HOROLOGICÆ,

OR, A

PHYSICO-MATHEMATICAL THEORY

OF

CLOCK-WORK.

CHAP. I.

Of the Nature *and* Deſign *of* CLOCK-WORK *in general; and the* PRINCIPLES *on which it depends.*

3139. A THEORY of CLOCK-WORK, I am inclined to think will be looked upon as a Novelty by the *Engliſh Mechanic*; as I have ſeen nothing of that Kind publiſhed in our own Language. Indeed I have met with but one Treatiſe on the Subject of Clock-work in *Engliſh*, *viz.* The *artificial Clock-maker*; but this is altogether *practical* and refers in every Particular, almoſt, for the *Rationale*, to a *Treatiſe* in *Latin*, entituled *Horologium Oſcillatorium*, by Mr. HUGENS.

3140. But this Treatiſe of Mr. HUGENS is upon the Theory of one Sort of Clock only, *viz.* that which moves with a *Weight*, and is regulated by a *ſingle Pendulum*. This great Author being the firſt that applied a PENDULUM to a Clock for this Purpoſe; though *Pendulums* were long before in Uſe as CHRONOMETERS, to meaſure Time by their equable Vibrations in aſtronomical Obſervations, and on many other Occaſions. But more of this

hereafter, when we come to treat directly of Nature, Uſe, and various Forms of the Pendulum.

3141. The Deſign of Clock-work is twofold, *viz.* (1.) To meaſure Time exactly, or to divide a given Portion of Time into very ſmall equal Parts; for Inſtance, to divide the Time of a Day into *Hours*, *Minutes*, *Seconds*, *&c.* (2.) To produce Motions, or periodical Revolutions ſimilar to any given ones, as thoſe of the heavenly Bodies, together with their Phaſes, Aſpects, Poſitions, *&c.* at a determinate Time. But Clock-work of this Sort is uſually called a *Planetarium*, *Orrery*, *&c.*

3142. A CLOCK is the principal Machine, or Capital of all mechanical Contrivances, for *meaſuring* TIME. Its Principle of Motion is derived from a two-fold Power or Force, *viz.* that of a *Weight*, or a *Spring*. For either of theſe Forces are ſufficient to actuate, or put into Motion, the Syſtem of Wheels and Pinions which compoſe the intermediate Parts or Body of the Machine; the Indexes fixed on the Axles of the Wheel point out the proper Diviſions of the integral Portion of Time on appropriated Circles upon the Face of the Clock, and a Pendulum or Balance is added, to regulate the Motion communicated to the Machine, or render it uniform.

3143. The two phyſical Principles of all *Automata*, *viz.* the WEIGHT and the SPRING are now to be conſidered. With Reſpect to the firſt, as its Force is derived from the Power of Gravity only, and this Power being always the ſame in a given Quantity of Matter (968) it follows that the Force of a given Weight is a *conſtant Quantity*, or always remains the ſame in the ſame Medium, and therefore in ſuch Caſe, this becomes abſolutely a uniform Power or Principle of Motion; ſuch as is neceſſary in perfect Clock-work.

3144. The ELASTICITY of a well-tempered Steel *Spring* is a fit Power or Principle of Motion in a Clock, for when it is bent or coiled to any given Degree, the Intenſity of its Force continues the ſame; but as it is coiled more or leſs about its Axis, the Force becomes increaſed or diminiſhed; and ſince the Action of a Spring in Clock-work is by unbending itſelf by Means of its renitent Force, the Force it exerts on the Wheels would gradually decreaſe; and therefore if it was at firſt ſufficient to keep the Clock in Motion, it could not long continue ſo, but the

the *Pendulum* would at Length ceaſe to move, and the Clock ſtand ſtill.

3145. When a SPRING, therefore, is applied to Clock-work, it becomes neceſſary to contrive a Method by which its variable Force may be rendered equable or uniformly the ſame on the Wheels of the Clock from its firſt or greateſt to its laſt or leaſt Degree of intenſive Force. And this is done by giving a requiſite Form to the Barrel on the Axis of the firſt Wheel, which it is connected with by a proper Chain, or String.

3146. The Force, therefore, of the Weight or Spring is the *Primum Mobile*, or firſt Mover, in Clock-work; and it may be proper here to obſerve that the Force they communicate to the Machine is continually diminiſhed by the Wheel-work, till at Length upon the ſerrated Teeth of the Crown-wheel it is but juſt ſufficient to keep the *Pendulum* in Motion; that is, the Force is there required to be equal to the Reſiſtance the Pendulum meets with from the Air and the Axis of its Motion, for then its Motion will be continued in the Arch of Vibration propoſed.

3147. But if the Force of the ſaid Crown-wheel on the Pallets of the Verge of the Pendulum be ſuperior to the Reſiſtance, it will only cauſe the Pendulum to vibrate farther, or in a ſomewhat larger Arch than the given one. But if the Force be leſs than the Reſiſtance of the Pendulum, then the Arches of Vibration will decreaſe gradually to nothing, or the Pendulum and Clock will ceaſe to move.

3148. The more perfectly the component Parts of the moving Syſtem are wrought and finiſhed, the leſs Force of a Weight or Spring will be required to keep it going. But when the Work is coarſe, and clogged with Duſt, inſpiſſated Oil, *&c.* the Force of the Weight or Spring will be ſo much diminiſhed thro' the Train of the Work, that unleſs it be in Proportion encreaſed it will not keep the Clock in Motion. But whatever be the Condition of the Clock, while it does move, it will meaſure Time equally; for on Suppoſition the Pendulum continues of the ſame Length, it makes all its Vibrations, in larger or ſmaller Arches, in the ſame Time, as we have formerly ſhewn (1122, 1126); and, therefore, if once a Clock be ſet right, or

has its Pendulum duly adjuſted, it muſt (as *Hugens* ſays,) always *meaſure Time truly, or not meaſure it at all.*

3149. Upon the Whole we may conclude, that ſince a *Weight* is, in its own Nature, a conſtant Principle and muſt neceſſarily act with an uniform Tenour; and on the other Hand, the Action or Force of a *Spring* is in itſelf alway variable, and can only be rendered of an equable Tenour by an Artifice, it muſt follow that the former is more eligible in Clock-Movements than the latter; and that a Spring is preferable to a Weight only on a ſingle Account, *viz.* of its bringing the Clock into a more compendious or portable Form than can be admitted where a Weight is made uſe of for giving it Motion.

CHAP. II.

The NATURE, FORM, *and* ACTION *of the* FUSEE, *explained from* Mechanical *and* Mathematical PRINCIPLES.

3150. WE are now to explain the Theory of that Invention by which the SPRING is made to act with an equable Force on the Syſtem of Wheel-work by the Mediation of a Part called the FUSEE, which for that Purpoſe is required to have a peculiar Form; every one knows how a Weight acts upon the Cylinder, and thereby communicates an equal Force and Movement to the Machine. But the Manner in which the Spring and Fuſee do the ſame Thing conjointly, is not ſo obvious, but yet will be eaſy to conceive by attending to the following Particulars.

3151. The Chain being fixed at one End to the Fuſee, and at the other to the Barrel, when the Machine is winding up, the Fuſee is turned round, and of Courſe the Barrel; on the Inſide of the Barrel is fixed one End of the Spring, the other End being fixed to an immoveable Axis in the Center. As the Barrel moves round, it coils the Spring ſeveral Times about the Axis, thereby increaſing its elaſtic Force to a proper Degree; all

all this while the Chain is drawn off the Barrel upon the Fuſee and then when the Inſtrument is wound up, the Spring by its elaſtic Force, endeavouring conſtantly to unbend itſelf, acts upon the Barrel, by carrying it round; by which Means the Chain is drawn off from the Fuſee, and thus turns it about, and conſequently the whole Machinery is put in Motion.

3152. Now as the Spring unbends by Degrees, its elaſtic Force, by which it affects the Fuſee, will gradually decreaſe; and therefore unleſs there were ſome mechanical Contrivance in the Figure of the Superficies of the Fuſee to cauſe, that as the Spring is weaker, the Chain ſhall be removed farther from the Centre of the Fuſee, ſo that what is loſt in the Spring's Elaſticity is gained in the Length of the Lever; I ſay, unleſs it were for this Contrivance, the Spring's Force would always be unequal upon the Machine, and ſo would produce an unequable Motion of the Parts thereof.

3153. The Figure of the Curve, which ſhall form the Superficies of the Fuſee by a Revolution about its Axis, may be inveſtigated as follows. (Fig. 1.) Let B C D be the Curve, A L the Axis of the Fuſee produced; let D be the Point where the End of the Chain is fixed on the Fuſee when the Watch is down, or the Spring uncoiled, and B the Point where it touches it when the Spring or Machine is wound up. From the Points B and D let fall the Perpendiculars to the Axis B A and D H; in which produced, let there be taken A E and H I proportional to the Force or Strength of the Spring, when the Chain is at B and D. Through E, I, draw the Right-line E I K interſecting the Axis ſomewhere in K; and from any Point C in the Curve, draw C F perpendicular to the Axis in G; then will F G be as the Strength of the Spring when the Chain is at G.

3154. Now ſince the Force acting on the firſt Wheel ought always to be uniformly the ſame; and this Force being always as the Strength of the Spring expreſſed by F G, and the Diſtance at which the Chain acts from the Axis of the Fuſee conjointly: Therefore the Force at any Point C will be as the Rectangle F G × G C, and ſince this is a given Quantity it may be made F G × G C = $a\,b$, and ſo we have F G = $\frac{a\,b}{G\,C}$.

3155.

3155. Therefore to determine the Equation of the Curve BCD, let KH = a, HI = b, HG = x, and GC = y. Then becauſe of the ſimilar Triangle HKI and GKF, we have HK : HI :: GK : FG = $\frac{ab}{y}$; that is, $a : b :: a + x : \frac{ab}{y}$; whence we have $aa = ay + xy$, which is the Equation of the Curve, and ſhews it to be that of the *Hyperbola* with Reſpect to the Space between the Curve and its Aſymptotes, as is evident from (779).

3156. Hence when $x = o$, then $a = y$, or HK = HD; alſo when the Point G arrives at A, then y = AB. And becauſe EA × AB = IH × HD, we have EA : IH :: HD : AB :: $a : y$ :: ſo is the greateſt Force of the Spring to its leaſt Force, on the Fuſee (3154).

3157. Becauſe the Ordinates HD, AB, *&c.* are at Right-angles to AL, (3153) the *Curve* BCD is that called an *equilateral Hyperbola* (ſee 780). By the Revolution of which about its Axis or common Aſymptote AK the true Form of the Solid or *Fuſee* is generated, as in Fig. 2.

3158. In that Figure ADE, FGH, and IKL, MNO, are oppoſite equilateral Hyperbolas deſcribed about the Aſymptotes PQ, RS, interſecting at Right-angles in the common Center C. Put CT (= TD) = 1, then CD = $\sqrt{2}$ = Radius of the Circle DKGN touching the four equal Hyperbolas in their reſpective Vertices. And CF (= CY = $\sqrt{CQ^2 + QY^2}$) = 2, is the Diſtance of the Focus of each Hyperbola from the Center. Laſtly, the Parameter ab = KN, the Diameter of the Circle. All which is evident from what we have heretofore demonſtrated of the Properties of the Hyperbola in general (765, *&c.*)

3159. It is therefore demonſtrated that the Section of any given Fuſee BDKX through its Axis ZT is determined by two equal Arches BD and KX of two equal and adjacent Hyperbolas beginning from their Vertices D and K.

3160. It is alſo evident, that ſince TD and ZB do repreſent the Force of the Spring, when it is wound up, and when quite down, therefore in every Fuſee truly made the *Diameters*

DK *and* BX *of the greateſt and leaſt Ends thereof muſt be exactly proportionate to the greateſt and leaſt Force of the Spring.*

3161. Alſo it follows, that when the Proportion of the greateſt and leaſt Force of the Spring is known, or the Ratio of TD to ZB is given, then alſo the Length of the Fuſee TZ is a given Quantity; or *there can be but one determinate Length for the Fuſee, to anſwer to the two given Forces of the Spring.*

3162. Laſtly, it is evident, that when the Length of the Fuſee, and one of the Forces, TD or ZB are given, then the other Force is given or determined, and not to be aſſumed at Pleaſure.

3163. Having thus determined the *geometrical Form of the Fuſee*, we next proceed to illuſtrate the Theory of the ſeveral Caſes by Examples. Therefore put $TD = a$, $ZB = y$, and $TZ = x$; and then the Equation $aa = ay + yx$ (3155) will appear in its uſual Form. Whence (1.) If a and y are given, to find x; we have $\frac{aa}{y} - a = x$. (2.) When x is given, or $x = 1$, we have given the Ratio of a to y; for then $aa = ay + y$, conſequently $a : y :: a + 1 : a$. (3.) When a and x are given, then $\frac{a^2}{a + y} = y$. (4.) Laſtly, when x and y are given; we have $a^2 - ay = xy$, and (compleating the Square) $a = \sqrt{xy + \frac{1}{4}y^2} + y$.

3164. The proper Numbers for expreſſing the Forces (a) and (y) of the Spring will be in *Ounces* and *Drams* Averdupois Wt. which *Ounces* may be made Tenths of an Inch in the Meaſures of the Fuſee TD, TZ, and ZB. Theſe Forces are thus determined. In Fig. 3. let ABCD be the Barrel containing the Spring, and let FBX be the Poſition of the Chain or Cord upon the Barrel and Fuſee when the Spring is wound up; then ſuppoſe the Chain diſengaged from the Fuſee, and carried under the Barrel in the Direction EH to the Pulley at H, over which it is to be hung with ſuch a Weight W appended, as will juſt counteract or balance the Force of the Spring coiled up. After the ſame Manner, if CDK be the Chain when the Clock is down, then if this be taken from the Fuſee, and paſſed under the Barrel to the Pulley K, and a Weight L hung on to the End, ſuch

as

as ſhall juſt keep the Barrel in the ſame Poſition; then this Weight L will be equal to the Force of the Spring ſo far uncoiled. Therefore $TD = a : ZB = y :: W : L$.

3165. Case I.

Suppoſe the Weight W be 63 Ounces, and the Weight L be 21; then $a = 63$ and $y = 21$; or becauſe $63 : 21 :: 3 : 1$, we have $a = 3$, and $y = 1$; and then we find $x = \frac{aa}{y} - a = 6 = 2a$, that is when the Forces are as 3 to 1, the Height or Length of the Fuſee TZ is equal to the Diameter of its Baſe or End DK. If $W : L :: 2 : 1 :: TD : ZB$; then $x = a$, or $TZ = TD$. When $W : L :: 3 : 2$; then $x = \frac{1}{2}a$; and univerſally if $W : L :: m : n :: a : y :: a + x : a$, then it will be $\frac{m - n}{n} a = x$.

3166. Case II.

Given the Length of the Fuſee $TZ = 6$, to determine the Ratio of the Forces of the Spring, or the Weights W, L, which will give the Diameters DK, and XB of the Ends of the Fuſee. Since $a^2 = ay + 6y$ (3163) we have $a : y :: a + 6 : a$; then by aſſuming the Value of a you have that of y. Thus ſuppoſe $a = 3$, then $a : y :: 3 + 6 : 3 :: 3 : 1$. And in this Caſe the Diameter $DK = 3XB$. And becauſe $a : y :: m : n$, (3165) therefore for any aſſumed Ratio, we have $\frac{n}{m - n} x = a$: thus if $m : n :: 2 : 1$, then $x = a$, or if $m : n :: 3 : 2$, then $2x = a$. Conſequently if $x = 6$, we have $a = 12$, and $y = 8$.

3167. Case III.

Given the Length of the Fuſee, and the greateſt Force of the Spring to find what the leaſt Force muſt be. Suppoſe $x = TZ = 6$, and $a = TD = 3$; then $y = \frac{a^2}{a + x} = \frac{9}{9} = 1$; ſo that if W = 63 Ounces, we have L = 21. If $x = a$, then $y = \frac{1}{2}a$, or $DK = 2XB$.

3168.

3168. CASE IV.

Given the Length of the Fuſee, and the leaſt Force of the Spring, to find the greateſt. Let $x = 6$, and $y = 1$; then $\sqrt{xy + \frac{1}{4}y^2} + \frac{1}{2}y = \sqrt{6{,}25} + 0{,}5 = 3 = a$ (3163) ſo that if $y = 21$ Ounces, the greateſt Force of the Spring will be $a = 63$. Therefore in every Caſe the Form and Dimenſions of the Fuſee are geometrically determined.

3169. It only remains now to ſhew the Method by which the Hyperbola A D E (Fig. 2.) is to be deſcribed *in Plano* in order to be made a *Gauge* for giving the true Form of the Fuſee required in any particular Caſe. Thus ſuppoſe it be found by Experiment (3164) that the greateſt and leaſt Force of the Spring to be uſed is 63 and 21 Ounces. Then having drawn two Lines P Q and R S at Right-angles in C for Aſymptotes, let the Angles P C S and R C Q be biſected by a Right-line G D continued each Way indefinitely.

3170. Then having determined the Diameter of the Baſe D K of the Fuſee, take that Extent in your Compaſſes and ſet it off each Way from the Center C in the Line D G to F and C (in Fig. 4.) then will thoſe two Points be the Focuſſes of two oppoſite Hyperbolas A D E and F G H (774).

3171. Having provided a Ruler A B C of the Form (in Fig. 4.) you fix one End of a String A B F on the End A, and the other End in the focal Point F of the intended Hyperbola. This Chord A B F muſt be juſt ſo much leſs in Length than the Ruler A B C, as is equal to the Diameter G D of the Circle; that is, A B C — G D = A B E. Then if with a ſteady Hand you move the Ruler about a Pin fixed in the oppoſite Focus C, and at the ſame Time keep the Chord nicely to the Edge of the Ruler, as at B, with a proper Pencil or drawing Point, that Point B will deſcribe the required Hyperbola A D E (Fig. 2.)

3172. For the *Ellipſis* in the Figure to (768) becomes a *Circle* in the preſent Caſe, and the tranſverſe Axis T V there becomes the Diameter D G here; but in every Caſe the Difference of two Lines C B, F B, drawn from any Point B in the Curve of the Hyperbola will be equal to the tranſverſe Axis, as is ſhewn (769) and therefore univerſally C B — B F = G D; or the

Point B is conſtantly in, and therefore deſcribes the *Hyperbolic Curve* required.

3173. When the Curve is thus drawn on Paper or Paſtboard, it will be eaſy to transfer it to a Plate of Braſs, or Steel; and thereby form a *Gauge*, for giving a true Figure to the Fuſee propoſed. In ſuch Fuſees of the larger Sort, where the Diameter of the Chord or Chain is large enough to be conſidered, it muſt be added to the Diameters D K and X B of the greateſt and leaſt Helix of the Fuſee.

3174. In order to draw the Fuſee by Scale and Compaſſes, it has been ſhewn that there is a ſtated Proportion between the Diameters, and Length of the Fuſee; and therefore in whatever Numbers one is expreſſed, the other may be expreſſed in the ſame. Thus for Inſtance; if T D be to Z B, as 63 to 21, then, becauſe, in this Caſe, $x = 2a$, or T Z = 2 T D (3165); therefore T Z = 126, of the ſame equal Parts. So that if T Z = 1,26 Inches, or 12,6 *Tenths* of an Inch, then T D = 0,63, or 6,3 *Tenths*; and Z B = 0,21, or 2,1 *Tenths* of an Inch, which are laid down from any decimal Scale, or other Scale of equal Parts.

3175. Theſe Things are, I think, all that belong eſſentially to the *Theory of the Fuſee* in Clock-work; and could the Artiſt in Practice execute this Part to the Perfection of Theory, it would then communicate a Motion to the Machine as equable as that produced by a *Weight* itſelf. And by this geometrical Conſtruction of the Fuſee, thoſe which are uſually made by Trial with the Lever, may be compared and corrected in regard to their Figure and Dimenſions.

CHAP.

CHAP. III.

The Rationale *of* CALCULATION *in* CLOCK-WORK; *the ſame applied and illuſtrated in a* DESCRIPTION *of the* original AUTOMATON *invented by* HUGENIUS.

3176. THE Nature of the *Weight*, and the *Spring* with its *Fuſee*, being, as the Origin or Principles of Motion in this Kind of Machinery, explained; it remains to ſhew the Conſtruction and Diſpoſition of the Syſtem of Wheel-work to anſwer the general Purpoſe of a Clock. But as little of a mathematical Theory is here required, or employed, it will the ſooner be diſpatched.

3177. The Communication of Motion being by Wheels and Pinions, it is in the firſt Place neceſſary to take Care, that the Diameters of the Wheel and Pinion it drives, have the exact Proportion of their Numbers of Teeth reſpectively, that the Teeth of one may exactly correſpond to the Cavities or Interſtices of the other; thus if a Wheel of 80 Teeth be propoſed to drive a Pinion of 8 Teeth, then the Diameter of the Wheel muſt be to that of the Pinion exactly as 80 to 8, or as 10 to 1. If this be done throughout the Syſtem, the Movements will be every where natural and exact.

3178. In the next Place, from the Nature of the Lever and Axis in *Peritrochio*, it appears that the Power or Force on the Pinion is to that on the Circumference of the Wheel on the ſame Axis, as the Diameter of the Wheel is to that of the Pinion; conſequently by this Means, and by the Friction of the Parts, the Force at firſt impreſſed by the Weight or Spring is conſtantly diminiſhing through the whole Compages, till at laſt it is but very ſmall on the Pendulum, *viz.* juſt enough to continue it in Motion.

3179. The Revolutions of two immediate Axis in a given Time are inverſely as the Number of Teeth in the Wheel of one to that Number of Teeth in the Pinion of the other, which it drives, thus if the Number of Teeth in the Wheel and Pinion

be 80 and 8, then the Axis of the Pinion will be turned round 10 Times to 1 Turn of the Axis of the Wheel; therefore the *Quotient of the Wheel, divided by the Pinion it drives*, is the Ratio of Turns to Unity. Thus 8) 80 (10, as before.

3180. Hence if there be any Number of Wheels A, B, C, D, &c. acting upon ſo many contiguous Pinions *a*, *b*, *c*, *d*, &c. and let the Quotients of each Wheel, by the Pinion it drives, be *m*, *e*, *f*, *g*, &c. then $\frac{A}{a} = m$; or $a : A :: 1 : m$. Again, B is the Wheel on the ſame Axis with the Pinion *a*, and drives the Pinion *b* on the next Axle; then $\frac{B}{b} = e$, is the Number of Revolutions in this third Axis to one on the Axis of the Pinion *b*; and therefore $me = \frac{mB}{b} = n$ the Number of Turns in this third Axle to one of the firſt Axle, or that of the Wheel A, ſo that $b : B :: m : n$. Let the Wheel on the Axis of the Pinion *b* drive another Pinion *c*, then $\frac{C}{c} = f$ with Reſpect to the Turns of the preceding Axis which are expreſſed by *n*; therefore $\frac{nC}{c} = nf = o$; and ſo we have $c : C :: n : o$; and for the next Axle, we have $d : D :: o : p$, and ſo on for as many Axes as are required in the Train of the Work.

3181. Then placing theſe Analogies one under another, and multiplying all the Antecedents and Conſequents together, we have ab $cd : ABCD :: mno : mnop :: 1 : p$. Wherefore $\frac{ABCD}{abcd} = p$; whence we have this Rule. $\left.\begin{array}{l} a : A :: 1 : m \\ b : B :: m : n \\ c : C :: n : o \\ d : D :: o : p \end{array}\right\}$ *Divide the Product of the Number of Teeth in the Wheels by the Product of the Teeth in the Pinions, the Quotient will be the Number of Turns of the Axle of the laſt Pinion* d *in one Turn of the firſt Axle of the Wheel* A.

3182. Since the Expreſſions $\frac{A}{a}$, $\frac{B}{b}$, $\frac{C}{c}$, &c. are but Ratios of the Numbers of the Teeth in the Wheels and Pinions, it is evident, any Numbers having the ſame Ratio will anſwer the ſame Purpoſe, or give the ſame Number of Revolutions to an Axis

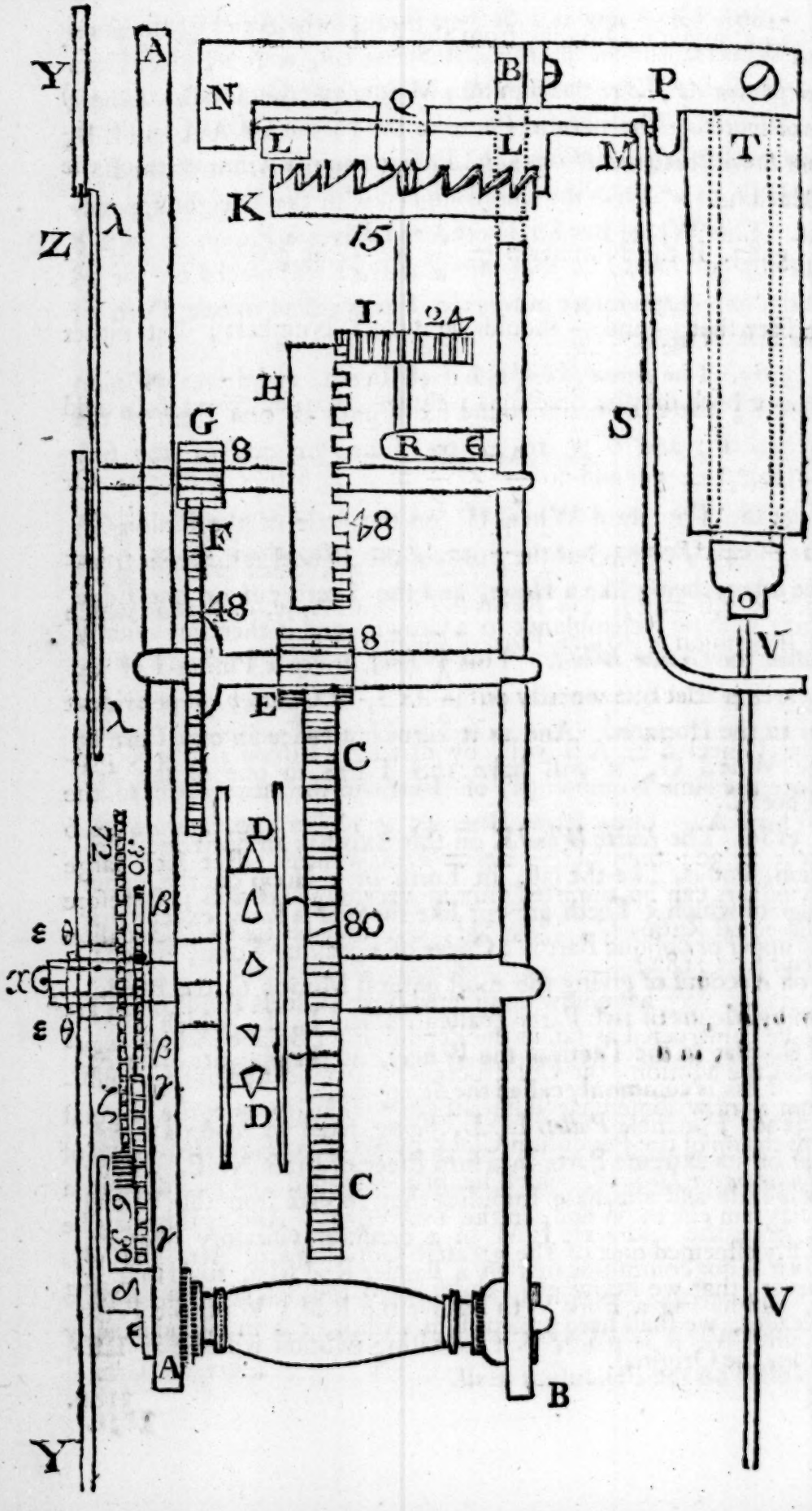
Axis at a given Diſtance from the Firſt. Thus $\frac{A}{a}$ may be $\frac{80}{8}$, or $\frac{60}{6}$, or $\frac{50}{5}$, *&c.* (as they all give the ſame Quotient, *viz.* 10) and therefore ſuch Numbers are to be choſen by the ſkilful Artiſt as will beſt ſuit the general Deſign and Circumſtances of the Machine.

3183. If the Number $n = \frac{A}{a} \times \frac{B}{b}$, be given, it is not neceſſary that $\frac{A}{a}$ and $\frac{B}{b}$ ſhould be whole Numbers; but either one or both may be Fractions; thus $\frac{A}{a}$ may be $\frac{33}{5} = 6{,}6$, and $\frac{B}{b} = \frac{40}{8} = 5$; and ſo $\frac{A}{a} \times \frac{B}{b} = n = 6{,}6 \times 5 = 33$. Or if $\frac{A}{a} = \frac{45}{6} = 7{,}5$, and $\frac{B}{b} = \frac{18}{5} = 3{,}6$; then $\frac{A}{a} \times \frac{B}{b} = 7{,}5 \times 3{,}6 = 27$, the Turns in the third Axle, or that on which is the Pinion b, to one Turn of the firſt Wheel.

3184. Becauſe we may put $\frac{A}{a} \times \frac{B}{b} = \frac{S}{s} = n$, it is evident one Wheel $S = AB$ will, by driving a Pinion $s = ab$, produce the ſame Number (n) of Turns in the next Axle to the Wheel A. Thus let $AB = 45 \times 18 = 810$, and $ab = 6 \times 5 = 30$; then $\frac{810}{30} = 27 = n$, as before. But ſuch large Numbers can be admitted only in very large Works; therefore the equal Ratios $\frac{81}{3}$, $\frac{162}{6}$, $\frac{243}{9}$, *&c.* may be taken as Occaſion requires (3182).

3185. We have now conſidered how Wheels and Pinions are to be conſtructed as far as the Pendulum or Balance which regulates the Motion; the Conſtruction for Application of the Pendulum is now ſomewhat different from what it was in the original Invention of the Pendulum Clock by Mr. *Chriſtian* HUGENIUS of *Zulichem* in *Holland*, which he firſt deſcribed and publiſhed in a Diagram cut in Wood, in the Year 1657. And as this may be juſtly eſteemed one of the greateſt Curioſities of Art, and was never (that we know of) exhibited to the View of an *Engliſh* Reader, we ſhall here preſent him with it, cut in Wood exactly from the Original.

3186.


Y
A
N
Q
B
P
T
L
L
M
K
Z
λ
15
I
24
H
G
8
R
S
F
48
48
8
E
C
D
80
V
B
A
Y

3186. The View is a Section through the Axles of the ſeveral Wheels, which, in this firſt *Automaton*, were all placed in a perpendicular Line through the Middle of the Clock, as here repreſented. The parallel Plates of the Frame are A A and B B, and the firſt or *great Wheel* C C is placed on the Arbor of the Barrel D D, to which is the Chord applied with the Weight appended. This Wheel has 80 Teeth, and drives a Pinion E with 8 Teeth, (or *Leaves* as they are uſually called) placed on the 2d Axle, which therefore moves ten Times round in one Turn of the firſt Wheel.

3187. The *ſecond Wheel* F has 48 Teeth, and drives a Pinion G of 8; therefore turns round ſix Times in one Turn of the Wheel F; and $6 \times 10 = 60$ Turns in one of the firſt Wheel.

3188. The third Wheel H (on the Axle of the Pinion G) has alſo 48 Teeth; but the Form of this Wheel is different from the other, being like a Hoop, and the Teeth cut on one Edge gives it ſome Reſemblance to a Crown, and is therefore uſually called the *Crown Wheel*. This Wheel drives a Pinion I of 24 Leaves placed horizontally on an Axis, of Courſe, perpendicular to the Horizon. And as it turns but twice in one Turn of the Wheel G, it will have 120 Turns to one of the firſt Wheel C.

3189. The *fourth Wheel* K on this Axis has an horizontal Poſition, and is, like the laſt, in Form of a Hoop on the upper Edge of which 5 Teeth are cut like thoſe of a *Saw*, except that the upper or oblique Part is a Curve of a peculiar Form, neceſſary on Account of giving the moſt natural Motion to the Pendulum by Means of two Parts (called *Pallets*) L, L, which alternately play in the Teeth of the Wheel, at the oppoſite Sides K, L. This is commonly called the *Swing-wheel*.

3190. For theſe *Pallets* L, L, being fixed to an Axis moveable on its extreme Parts in a firm Piece of Braſs N, P, will by the gentle and alternate Impulſes they receive from the Wheel K, keep the ſaid Axle L M in a conſtant vibratory Motion, which being communicated to a ſlender Rod M S, fixed to it at M, and having a Fork V to receive the Rod V V of the Pendulum T V, it is plain, that vibrating Motion will be at laſt impreſſed on the Pendulum itſelf.

3191. But this Motion thus communicated to the *Pendulum*, muſt be ſuch as nicely quadrates or coincides with the proper Motion of the Pendulum itſelf, which depends on its Length. Now as the Wheel K has 15 Teeth, and each Tooth ſtrikes each Pallet 15 Times in one Turn, both the Pallets together muſt receive 30 Impulſes, and conſequently the Pendulum muſt have 30 Swings or Vibrations in one Turn of the Wheel K; and therefore it muſt vibrate $30 \times 120 = 3600$ Times in one Turn of the great Wheel C.

3192. Now 3600 is the Number of Seconds in one Hour; therefore a Pendulum vibrating Seconds, whoſe Length is 39,2 Inches (1125) was applied to the Clock; and then the Axis of the firſt or great Wheel C turned round once in an Hour.

3193. As the Axis of this Wheel paſſed through the Plate A A, then an Index put on the End ε ε was carried round once in an Hour, and over a large Circle (on the Face of the Clock) divided into 60 equal Parts, thereby indicating the *Minutes* at its extreme Part.

3194. On the ſame Axis (near the Plate A A) was fixed a Wheel β β of 30 Teeth which drove another Wheel γ γ of the ſame Number of Teeth, and a Pinion of 6 Leaves. This Pinion drove the Wheel ζ of 72 Teeth; and therefore it moved but once round in 12 Turns of the Pinion or Axle of the Wheel C. Conſequently an Index placed on the Socket θ θ of the Wheel ζ (moveable about the common Axis) will ſhew the HOURS as it paſſes over a Circle divided into 12 equal Parts, on the Face of the Clock Y Y.

3195. Laſtly; the Axis of the Crown-wheel H has 60 Turns to one of the great Wheel C, (3187) that is, it has 60 Turns in an Hour, or it turns once round in a *Minute*. The Axis of this Wheel, therefore, paſſing through the Plate A A had a Plate λ λ fixed upon its End with a Circle divided into 60 equal Parts denoted by their proper Numbers, which ſhewed the SECONDS of Time as they paſſed by a Hole z made in the exterior Plate or Face of the Clock Y Y.

3196. This original Pendulum-clock, conſtructed with ſuch Simplicity as to have no more than *four Wheels and Pinions* in the Body of the Clock, ſhewed HOURS, MINUTES, and SECONDS of Time, with all the Exactneſs and Equability that a mecha-

nical

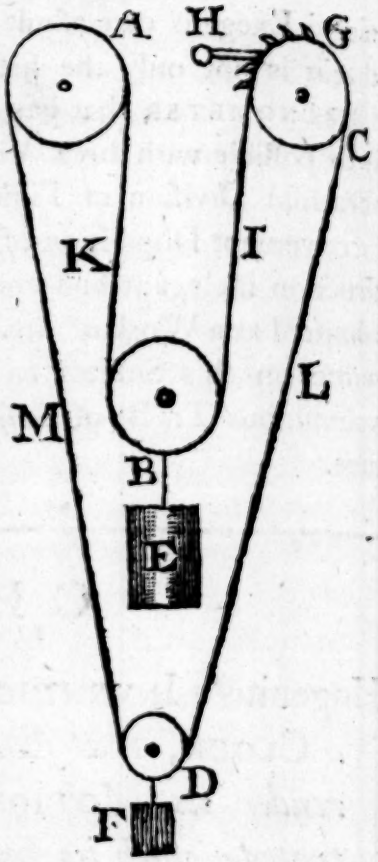
nichal Exegeſis can admit of. And I think we may conclude that it is not only the firſt but the moſt perfect Pattern of a CHRONOMETER that has been or can be propoſed; for though it be poſſible with three Wheels and Pinions only to produce this threefold Diviſion of Time; yet the great Diſproportion, and inconvenient Diſpoſition of the Parts, would render ſuch a Conſtruction inelegant and immechanical, and therefore not to be admitted as a Work of Art. They who would ſee a great deal wrote on this Subject to very little Purpoſe, may conſult the voluminous Tracts of *Caſp. Scottus*, and others of the like Genius.

CHAP. IV.

Hugenius's INVENTION *for applying a* WEIGHT *to a* CLOCK, *that ſhall act upon it inceſſantly, and render its* MOTION *conſtant, explained and illuſtrated; with its Improvement in* ROYAL PENDULUMS.

3197. THE celebrated Author of the foregoing Invention, ſo uſeful in public and private Life, did afterwards improve it by another, which no leſs demonſtrated his ſingular Genius and Sagacity for *automatical Machinery*. This ſecond Invention was to render the Motion of the Clock, or rather of the Pendulum, more equal than before; but to underſtand the Reaſon of it, the Manner in which he ſuſpended the Weight for keeping the Clock in an unintermitting Motion, is firſt to be explained.

3198. It is eaſy to underſtand that a Weight ſuſpended upon a Clock in the Manner it is upon a Jack, will keep the Clock going till it wants winding up; but during the Time of winding it up, the Action of the Weight is taken off from the great Wheel of the Clock, as it is all that while diſengaged from the Barrel on which the Weight hangs; and therefore the Clock not being impelled by the Weight, will, during that Time, ſtand ſtill, and conſequently ſo much Time will be loſt.



3199. Our Author therefore tells us, that he invented the Method which follows, of a *perpetual* or *endleſs Line* for that Purpoſe, which being of a proper Length, the two Ends were nicely ſpliced and connected together; it was then applied in the Clock as repreſented in the Figure adjoined, by the Letters A B C D A. Where A repreſents the ſulcated Wheel D D (in the Figure of the Clock) with ſeveral *Spiculæ* or ſmall pointed Pins fixed in the Surface of the evacuated Part, as is there ſhewn.

3200. The Pulley C is fixed to a *Rachet-wheel* G, but both moveable on a fixed Axis, from the Left-hand towards the Right only; for all Motion the contrary Way is prevented by the Trigger or Catch at H falling conſtantly into the ſerrated Teeth of the Wheel G. The Surface alſo of this Pulley C is ſpiculated, like that of the Wheel D D on the Arbor of the firſt Wheel of the Clock.

3201. *The endleſs Line or Cord, being put over the Barrel* A, and Pulleys B and C, will, by Reaſon of the *Spiculæ* or pointed Pins, hang firmly on A and C, ſuſpending the large Weight E affixed to the Pulley B. Now it is evident from the Nature of the Pulley (1050) that the Weight E equally ſtretches the Parts of the Chord I and K; and therefore acts with an equal Force, *viz. half its Weight*, on the Wheel A, and Pulley C.

3202. But ſince this Force upon the Pulley C tends to move it from Right to Left, and all Motion that Way is ſtopped (3200) it follows, that the Line I is to be conſidered as fixed all the Time the Clock is going; but the Parts K and M, by Vertue of the Weight E, will be conſtantly moving over the Surface of the Wheel A, and thereby communicating Motion to it, and, of Courſe, to all the Machinery of the Clock.

3203.

3203. As to the Pulley D, and the ſmall Weight F, it is plain they are deſigned only to give free Motion to the Parts of the Chord L, M; and keep them in a perpendicular Poſition.

3204. Thus it appears, that as long as the Weight E can deſcend, ſo long the Clock will keep going; and therefore as many Times as the Circumference of the Wheel or Barrel A is contained in the Diſtance through which the Weight can deſcend, ſo many Hours will the Clock go without drawing up, becauſe the ſaid Wheel A goes once round in an Hour (3192). In this original Clock, the Wheel D D is one Inch Diameter, or bout 3 Inches in Circumference, and went 30 Hours without drawing up; therefore 90 Inches, or $7\frac{1}{2}$ Feet was the perpendicular Deſcent of the Weight, or Height of the Clock.

3205. When the Weight was down, it was eaſily drawn up again by applying the Hand to the Part of the Line at L, and drawing downwards it will move the Pulley C, till the Weight E aſcends to the Top, during which Aſcent it conſtantly acts or gravitates on the Wheel A, by Means of the Rope K, with the ſame Force as when at Reſt, and therefore the Clock goes conſtantly by this Contrivance without ever looſing a Moment of Time, if it be not neglected.

3206. *Hugenius* informs us that in thoſe of his Clocks which were eſteemed the beſt, *the Weight E was ſix Pounds*; the Power therefore by which the Clock was animated, was that of 3 Pounds. The Weight of the Pendulum was alſo 3 Pounds; and of the Length, to ſwing Seconds.

3207. By an Addition of Pulleys, it would be eaſy to make the Weight deſcend more ſlowly, and, of Courſe, to make the Clock go longer before it wants to be drawn up. In the Modern Conſtruction of Clocks, a different Method of applying the Weight is uſed, and it is wound up upon a Barrel with a Handle or Winch; by this Means, the uſual Time the Clock will go, is 8 Days in thoſe called *Royal Pendulums*, in which the Swing-wheel K has a Poſition (not parallel, as here, but) perpendicular to the Horizon, or the ſame with all the other Wheels. Every Thing of this Sort is evident by Inſpection of any 8 Day Clock, and needs no other Inſtruction.

3208. I ſhall only here obſerve, that the Form of the Teeth in this Wheel ſhould be exactly circular on that Side or Part by which acts on the *Pads* or two Arms of the *Clock* of the Pendulum; in the extreme Parts of thoſe Arms there ſhould be a ſmall cylindric Pin fixed, always touching and moving in the circular Surface of the Teeth of the Wheel; the Radius of the Circle for the Teeth is the neareſt Diſtance of the Pin in the End of the Pad to the Axis or Arbor of the Pendulum on which it is fixed.

3209. Moreover it is farther neceſſary, that the Teeth on this Wheel be of ſuch a Number and Length, that the Pins upon the Ends of the Pads, may very nicely *take and eſcape* them alternately, or without the leaſt Intermiſſion of Action. Such a Conſtruction is both natural and mathematical, but it is not in Uſe, that I have ſeen; I know of no Method that can be ſubſtituted that is eaſier, or that will produce a more ſteady or equable Motion of the *Second-hand* fixed on the End of this Arbor, or (in the Clock-maker's Language) ſo accurately cauſe the Pendulum to *beat dead Seconds*.

CHAP. V.

The Rationale *of* HUGENIUS's Invention *for* continuing and regulating the MOTION *of the* PENDULUM *by a* ſingle Wheel *only*; *thereby rendering it moſt* EQUABLE *for* USE *at* SEA.

3210. HUGENIUS having conſidered (and probably found by Experience) that the Action of a Weight being propagated through a Multiplicity of Wheels and Pinions to the *Pendulum*, could not be ſo conſtant, and agitate the Pendulum with a Force ſo equable, as it would do if the Number of Wheels were leſs; and of Courſe concluded, that if the Weight could be applied to the Arbor of *that Wheel only* which actuated the Pen-

Pendulum, it would then communicate a Motion to it that would be the moſt ſimple and conſtant that the Nature of Things admits of in this Kind of Machinery.

3211. At Length he invented the Method of ſuch an Application, which was as follows. An *endleſs Line* with its proper Weights and Pulleys, (every Way ſimilar to that deſcribed in Fig. to Art. 3199, only as much leſs in Proportion as the Force of the Weight was leſs,) was applied to the Arbor of the Swing-wheel in a ſmall pinnulated Grove of a Barrel, (which we now ſuppoſe to be A) on one Part, and to a Ratchet-wheel C on the other, ſupporting the ſmall Wheel E, with the other much ſmaller at F.

3212. It is manifeſt that the Weight E depending on the Line I B A, conſidered as fixed on the Pulley C, does now ſolely act on the Arbor of the Swing-wheel by the Part A B, and therefore the Motion communicated to the Pendulum, muſt be the moſt Uniform poſſible, as we cannot conceive any Thing incident to the Weight E that can make any Alteration in its gravitating Force, and as there can be no Cauſe for any Irregularity in the Motion of the Pendulum.

3213. As the Body of the Machinery in the Clock is in this Conſtruction diſingaged from the Swing-wheel and Pendulum, ſo none of its Anomalies can affect either; the proper Office of all this Part being now only to raiſe up the Weight E as often as it is down, which the Author tells us, was every *Half-minute*, or 30 Seconds. For ſince in every Revolution of the Wheel A, the Weight E muſt deſcend through a Space equal to the Circumference of that Wheel; therefore it muſt want very frequently to be drawn up, which could not be done but by Means of the Machinery of the Clock, which by its Nature becomes a conſtant Agent for that Purpoſe.

3214. Where it not for the Frequency of drawing the Weight up, the Pendulum, with one Wheel to continue its Motion, would be the moſt perfect *Chronometer* that could be made; but as it is thus connected with the Clock-part, it becomes juſt as uſeful as if it wanted no Attendance to draw it up at all.

3215. By this Invention, therefore the Clock with all its Wheels, is, in Effect, reduced to *one ſingle Wheel*; and the Mo-

Motion just as equable and perfect as if it consisted of no more; and the Author assures us, that the Equability of Motion in Clocks of this Sort was apparently greater than in Clocks of the common Construction.

3216. As a Proof of this, he mentions a very extraordinary Circumstance, which he discovered by Accident; it was this; as two of those Clocks stood together on a Shelf, he observed the contrary Vibrations of each of them did so exactly agree and coincide, that they never in the least receded from it. And if this Concordance of their Motions was on Purpose disturbed, yet in a very little Time it would of it's one Accord return, and continue, in such perfect Coincidence, that only *one Sound* could be heard of both their Vibrations.

3217. Having wondered some Time at so unusual a Phænomenon, he resolves to enquire into its Cause with the utmost Diligence, and at Length found that it proceeded from a Motion of the Shelf or Plank on which they stood, though in itself altogether insensible. For the Reciprocations of the Pendulums communicated some Degree of Motion to the Clocks, firmly placed as they were; and this Motion being impressed on the Board, did necessarily cause that, if the *Pendulums* vibrated otherwise than by contrary Strokes very exactly, they must at Length do so, and then the Motion of the Shelf would entirely cease.

3218. For whatever Motion be communicated to the Shelf by the contrary Motion or Vibrations of the Pendulums as those Motions are impressed in contrary Directions, they will, if equal, destroy each other ( ) and if they are not impressed equally, and at the same Instant on the Board, yet as the Difference equally affects the Board and the Clock, it will, by this Alternation, constantly tend to an Equibrium both in the Clocks and the Shelf, till at Length that being affected, the Pendulums must move by contrary Strokes very nicely.

3219. But our Author concludes, that notwithstanding the Fact is evident, yet a Cause of such a tender and delicate Nature could never have Efficacy enough to produce such an Effect, unless the Motions of the Clocks had been previously, and by other Means rendered most equable and consentient between them-

themſelves; and which therefore is by this Experiment fully aſcertained and demonſtrated.

3220. It is above 100 Years ſince this Diſcovery was made and publiſhed to the World, and was intended, by the Inventor, to render his Clock more immediately uſeful at Sea; for this Purpoſe he contrived it in a particular Form with a triangular Pendulum, and a Conſtruction which gave it always an upright and ſteady Poſition on Ship-board; and thus many of them were made and uſed at Sea for the better diſcovering of the Longitude; and with conſiderable Succeſs, as we find by an Account thereof (in ſeveral Voyages) by the Author in his *Horolog. Oſcillator.* Pag. 17, *&c.* where alſo you have Iconiſms of the Pendulum, the Clock, and its Frame in which it was ſuſpended in the Ship.

3221. But after all, we find the Uſe of this moſt promiſing Invention diſcontinued at Sea, which would never have been, were it poſſible for any Thing of this Nature to ſucceed there; and if this be the Fate of a CLOCK actuated and regulated by the conſtant Power of a WEIGHT, it may tend to moderate our Expectations of the Longitude by Means of *Watches*, wherein the ſame Artifice has been attempted by Means of *Springs*. But how much Inferior all *Automata* by Springs are to thoſe which move by Weights, common Experience can ſufficient teſtify, and will be farther evident by conſidering their Nature and Conſtruction, explained in the next Chapters.

3222. I ſhall conclude this with a Query to the ingenious Artificer, *viz.* whether it is not practicable to move a Pendulum by a *ſingle Wheel*, and a Weight that ſhall not need winding up at all, by the Addition of another equal Pendulum to the Clock? —— I ſee nothing to contradict it in the Theory; if this were done, the utmoſt Equability of Motion from Clock-work would thence be derived to the Pendulum, and ſuch a *double Pendulum* Clock would become the moſt perfect *Automaton* in Nature.

CHAP.

CHAP. VI.

Concerning the Invention *of* portable AUTOMATA, *or* WATCHES; *of the* BALANCE, *and* REGULATOR; *and the* THEORY *of* Iſochronal VIBRATIONS *in* SPRINGS.

3223. IN the Invention and Application of Pendulums to *Horological Automata*, Mr. *Hugens* ſtands unrivaled; but with regard to the Invention of *Clocks in Miniature*, or of a portable and pocket Form, which we generally call WATCHES, there is (at leaſt, there has been) great Diſpute. Of theſe Watches there is not the leaſt Mention or Intimation in the *Horologium Oſcillatorium* by *Hugenius*, though printed in 1673, from whence it ſeems probable, that at that Time he knew of no ſuch Thing.

3224. It is alſo certain, that many Years before, our celebrated Countryman, Dr. HOOKE, exhibited ſeveral of thoſe Watches as his own Contrivance and Invention, and a Patent was offered him for the ſame in 1663, but not liking the Conditions, he refuſed it. Dr. *Derham* (in his *Artif. Clock-maker*) tells us, he ſaw a Watch preſented to King *Charles* II. on which was this Inſcription, *viz. Robert Hook, inven.* 1658. *T. Tompion, fecit*, 1675. Mr. *Ward*, in the Life of Dr. *Hooke*, ſays, he ſaw the ſame Inſcription on a round Braſs-plate, which formerly had been a Cover to the Balance of one of Mr. *Hooke*'s Watches; and he farther adds, that Mr. *G. Graham*, informed him, he heard Mr. *Tompion* ſay, he was employed three Months that Year by Mr. *Hooke*, in making ſome Parts of thoſe Watches before he let him know for what Uſe they were deſigned; and that Mr. *Tompion* uſe to ſay, he thought Mr. *Hooke* was the Inventor of them.

3225. From hence it appears, that theſe Watches were invented by Dr. *Hooke*, within one Year after the Clock itſelf, which (as before obſerved, 3185) was not made public till the Year 1657. Notwithſtanding this, in the Year 1674, Mr. *Hu-*

Hugens publiſhed a Pocket Watch of his own Invention, as he aſſerts, but it was in ſeveral Reſpects different from Dr. *Hooke*'s. And as it is inconſiſtent with the great Character of *Hugenius*, for Learning and Probity, to ſuppoſe him capable of *Plagiariſm*, therefore it muſt be allowed, that the Reduction of Clock-work, to the Size of a *Pocket-watch*, was ſeparately the Invention of each of theſe ingenious *Competitors*.

3226. This Invention conſiſts in applying a *Balance-wheel*, to beat the Time inſtead of a *Pendulum*; for though they are called *Pendulum-watches*, it is only becauſe of the *alternative Motion* of the Balance being ſomewhat like that of a *Pendulum*.

3227. This *Balance* has *its Motion* regulated by a *Spiral-ſpring* properly applied, an Idea of the Balance and Spring is eaſy to be formed from the View of any Watch-work. As alſo the Manner in which they are actuated by the Machinery of the Watch; for this being the ſame as in a Spring-clock, only *in ſmall*, there is nothing new till we come to the Regulation by a *Wheel and Spring* inſtead of a *Pendulum*.

3228. From barely conſidering the Size of a Watch, it is plain no Pendulum can be admitted into its Conſtruction; for in the firſt Place, a Pendulum cannot be made of ſo ſhort a Length as is requiſite, with any Degree of Exactneſs; and ſecondly, the Pendulum always muſt have a perpendicular Poſition, which cannot be allowed in the Uſe of a Watch, which is incident to all Kinds of Poſitions.

3229. As the Balance-wheel has in itſelf no Principle of, or Diſpoſition to Motion, but is actuated ſolely by the Machinery of the Watch, it is to be conſidered alone as no other Thing than a Check upon the Motion of that Syſtem, or a Reſervoir in which it is ultimate received and abſorbed, like the *Fly of a Jack* in a great Meaſure, only the Fly has a circular Motion, whereas that of the Balance is an Oſcillatory one.

3230. The Motion of the *Balance* and the *Fly* is therefore equally ſubject to the Inequalities of the Motion of the Wheel-work of the Watch; but this can be regulated in the Vibrations of the Balance by Means of a Spring, whereas a *circular Motion* of a Wheel admits of no ſuch Regulation or Correc-

CHAP. VI.

Concerning the Invention *of* portable AUTOMATA, *or* WATCHES; *of the* BALANCE, *and* REGULATOR; *and the* THEORY *of* Iſochronal VIBRATIONS *in* SPRINGS.

3223. IN the Invention and Application of Pendulums to *Horological Automata*, Mr. *Hugens* ſtands unrivaled; but with regard to the Invention of *Clocks in Miniature*, or of a portable and pocket Form, which we generally call WATCHES, there is (at leaſt, there has been) great Diſpute. Of theſe Watches there is not the leaſt Mention or Intimation in the *Horologium Oſcillatorium* by *Hugenius*, though printed in 1673, from whence it ſeems probable, that at that Time he knew of no ſuch Thing.

3224. It is alſo certain, that many Years before, our celebrated Countryman, Dr. HOOKE, exhibited ſeveral of thoſe Watches as his own Contrivance and Invention, and a Patent was offered him for the ſame in 1663, but not liking the Conditions, he refuſed it. Dr. *Derham* (in his *Artif. Clock-maker*) tells us, he ſaw a Watch preſented to King *Charles* II. on which was this Inſcription, *viz. Robert Hook, inven.* 1658. *T. Tompion, fecit*, 1675. Mr. *Ward*, in the Life of Dr. *Hooke*, ſays, he ſaw the ſame Inſcription on a round Braſs-plate, which formerly had been a Cover to the Balance of one of Mr. *Hooke*'s Watches; and he farther adds, that Mr. *G. Graham*, informed him, he heard Mr. *Tompion* ſay, he was employed three Months that Year by Mr. *Hooke*, in making ſome Parts of thoſe Watches before he let him know for what Uſe they were deſigned; and that Mr. *Tompion* uſe to ſay, he thought Mr. *Hooke* was the Inventor of them.

3225. From hence it appears, that theſe Watches were invented by Dr. *Hooke*, within one Year after the Clock itſelf, which (as before obſerved, 3185) was not made public till the Year 1657. Notwithſtanding this, in the Year 1674, Mr. *Hu-*

Hugens publiſhed a Pocket Watch of his own Invention, as he aſſerts, but it was in ſeveral Reſpects different from Dr. *Hooke*'s. And as it is inconſiſtent with the great Character of *Hugenius*, for Learning and Probity, to ſuppoſe him capable of *Plagiariſm*, therefore it muſt be allowed, that the Reduction of Clock-work, to the Size of a *Pocket-watch*, was ſeparately the Invention of each of theſe ingenious *Competitors*.

3226. This Invention conſiſts in applying a *Balance-wheel*, to beat the Time inſtead of a *Pendulum*; for though they are called *Pendulum-watches*, it is only becauſe of the *alternative Motion* of the Balance being ſomewhat like that of a *Pendulum*.

3227. This *Balance* has *its Motion* regulated by a *Spiral-ſpring* properly applied, an Idea of the Balance and Spring is eaſy to be formed from the View of any Watch-work. As alſo the Manner in which they are actuated by the Machinery of the Watch; for this being the ſame as in a Spring-clock, only *in ſmall*, there is nothing new till we come to the Regulation by a *Wheel and Spring* inſtead of a *Pendulum*.

3228. From barely conſidering the Size of a Watch, it is plain no Pendulum can be admitted into its Conſtruction; for in the firſt Place, a Pendulum cannot be made of ſo ſhort a Length as is requiſite, with any Degree of Exactneſs; and ſecondly, the Pendulum always muſt have a perpendicular Poſition, which cannot be allowed in the Uſe of a Watch, which is incident to all Kinds of Poſitions.

3229. As the Balance-wheel has in itſelf no Principle of, or Diſpoſition to Motion, but is actuated ſolely by the Machinery of the Watch, it is to be conſidered alone as no other Thing than a Check upon the Motion of that Syſtem, or a Reſervoir in which it is ultimate received and abſorbed, like the *Fly of a Jack* in a great Meaſure, only the Fly has a circular Motion, whereas that of the Balance is an Oſcillatory one.

3230. The Motion of the *Balance* and the *Fly* is therefore equally ſubject to the Inequalities of the Motion of the Wheel-work of the Watch; but this can be regulated in the Vibrations of the Balance by Means of a Spring, whereas a *circular Motion* of a Wheel admits of no ſuch Regulation or Correc-

tion, and therefore can have no Place in this Sort of Mechaniſm.

3231. Hence then it is evident, that the whole Artifice of a Watch conſiſts in giving Motion by the firſt Spring and Fuſee to the Balance, or oſcillating Wheel, as nearly equable as poſſible; and then to correct the Irregularities of thoſe Oſcillations, and render them iſochonal as far as may be, by a fine Spring properly applied to the ſaid Balance, though not by its *vibrating Property*, as many have ſuppoſed.

3232. But as this Doctrine of the *iſochronal Vibrations* of a Spring is a curious Point, I ſhall be a little more particular and explicit in deriving it from its firſt Principles, and then illuſtrating it by a Figure. Therefore let G be any Force (whether *Gravity*, *Elaſticity*, *&c.*) which conſtantly acts on a Body and produces an accelerated Velocity; and let S, T, V, be the Space, Time, and Velocity of ſuch Motion; then we have ſhewn (991) that it is $S : TV$, univerſally; and alſo, that it is $GT : V$ (*ib.*) therefore we have $S \times V : GT^2 V$, or $S : G \times T^2$; and conſequently we have $S : G :: T^2 : 1$; therefore when the Ratio of S to G is given, the Time T will be a given Quantity or always the ſame. So that if G be the Force of Elaſticity, and S the Space through which it vibrates by that Force, then, however thoſe Quantities vary in Magnitude if they keep the ſame Proportion, the Vibrations of the Spring will all be iſochronal, or performed in equal Times.

3233. For a farther Illuſtration of the Similarity of Motion in a Pendulum and Spring; let A *a* (Fig. 1.) be a Spring, or a ſtrait elaſtic Wire fixed at the End A, and at the other End ſuppoſe it drawn out of its natural or perpendicular Situation, by a Line paſſing over a Pulley B, with a Scale C at the End, and Weights put into it for trying the Experiments. Then admit we put a Dram into the Scale C which draws the Wire or Spring from A *a* to the Site A *b*; and then we add another Dram, and it draws it into the Poſition A *c*; and a third Dram being put into the Scale, draws the Wire into the Situation A *d*; and ſo on, as long as the Wire or Spring can retain its rectilineal Form; then will the Spaces *a* A *b*, *a* A *c*, *a* A *d*, be as 1, 2, 3; that is, as the Weights applied which retain the Spring in thoſe Poſitions reſpectively; therefore the elaſtic Force of the Spring

at

at *a*, *b*, *c*, being alſo as thoſe Weights, and conſequently as the Spaces deſcribed by the Wire, *viz.* the ſmall right-lined Triangles *a* A *b*, *a* A *c*, *a* A *d*, it follows, that the Wire will deſcribe each of them in the ſame Time; and therefore all its Vibrations, as long as it continues rectilineal, will be performed in equal Time.

3234. But if upon adding 3 Drams more, we obſerve the Wire drawn into the Poſition A *e*, and there appears incurvated, it is evident then, becauſe the Space *a* A *e* is not equal to twice the Space *a* A *d*, the Time in which it moves from *e* to *a* cannot be equal to the Time in which it moves from *d* to *a*. If 3 more Drams be added, ſuppoſe all the Nine draw the Spring to the Poſition A *f*, which is there more curved than before; and therefore *a* A *f* is ſtill more deficient from 3 Times the Space *a* A *d*, and conſequently the Time of deſcribing it will differ ſtill more from the Time of deſcribing the Space *d* A *a*. Upon the Whole, then, it appears, *that after the ſtrait Spring begins to be bent into a Curve, the Times of Oſcillation are no longer iſochronal.*

3235. In like Manner, with regard to a Pendulum *A a* (Fig. 2.) we have formerly demonſtrated (1105) that the Times of Deſcent through any Number of Chords (*a d*, *a e*, *a f*,) of a Circle are all equal. Therefore if we take very ſmall Arches *a b*, *a c*, *a d*, they will, as to Senſe, coincide with their Chords, and ſo of Courſe the Time of deſcribing each of them will be equal; but if the Arch be ſo large that it can no longer be eſteemed rectilineal, as *a e*, or *a f*, then the Times of deſcribing them ceaſe to be iſochronal. So that very ſmall Diſtances only on each Side the Perpendicular *a* A, both in the Spring and Pendulum, will admit of iſochronal Vibrations.

3236. But what has been hitherto ſaid of the Spring and its Vibrations has been with a View rather to ſhew its Nature and Congruity with the Pendulum, than its Application to Uſe in automatical Machinery on that Account; for though the Pendulum by Means of its oſcillatory Property, becomes ſuch an excellent Regulator in all Kinds of Time-pieces, yet the Spring poſſeſſed of the ſame Property, will anſwer no ſuch Purpoſe on that Account; becauſe the Time of a Vibration is ſo exceeding ſmall. Now the Vibrations of a Spring (ſuch as we have conſider-

ed) ought to coincide with thoſe of the Balance, if its Regulation were to be produced by them; but the Vibrations of the Balance, or *Beats* of a Watch, are not more than about 16000 in an Hour, that is, about $4\frac{1}{2}$ *per* Second; whereas a ſtrait Spring, of the Length adapted to a Watch, will vibrate many hundred Times in a Second; and, indeed, a Spring of any Size, or Kind, can have no Uſe in that Reſpect, all its Effect, as a *Regulator*, being derived from its elaſtic re-active Force, by Way of *conſtant Preſſure* (and not *Oſcillation*) which therefore muſt be more particularly explained, and is the Subſtance of the following Chapter,

---

CHAP. VII.

The THEORY *of* SPRINGS *conſidered as* REGULATORS *of the* BALANCE *of* CLOCKS *and* WATCHES.

3237. AS a ſtrait Spring, in Length, cannot be more than about $\frac{2}{3}$ of the Diameter of the Watch-plate, it will be found too ſhort to admit of a proper Tenour of Action as a Regulator of Motion in ſuch ſmall Balances; for the Action of a Spring ought to be very free and eaſy, but yet at the ſame Time ſtrong enough to govern the Motion of the Balance, and controul the Irregularities of its Vibrations derived from the Syſtem of Machinery. Therefore a Spring of a Spiral, undulated, or ſome other Form, wherein there is a conſiderable or ſufficient Length in a conciſe Space for anſwering its Purpoſe in a WATCH, muſt be choſen.

3238. As the Action of the Spring under all thoſe different Forms will be the ſame, I ſhall chuſe that of (Fig. 3.) to explain it by. Thus let CL be the Spring in its natural Situation, or ſuch as it has when left to its itſelf; at the End L ſuppoſe it to reſt againſt, or be fixed to, an immoveable Support K, then by its elaſtic Force it will make Reſiſtance to any Power or Weight by which it is preſſed inwards, or drawn outwards. Thus let

p be

The THEORY of SPRINGS & PENDULUMS.

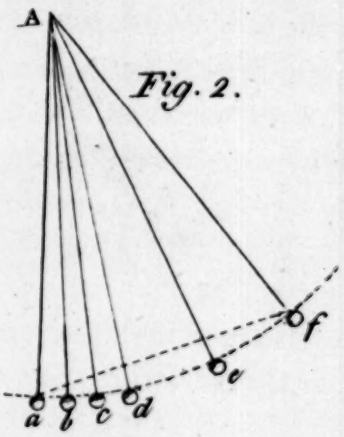


Fig. 2.

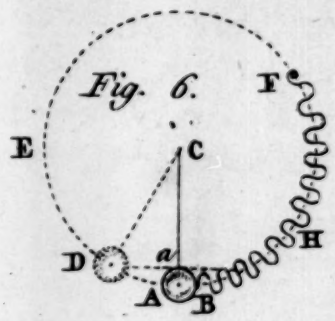


Fig. 6.

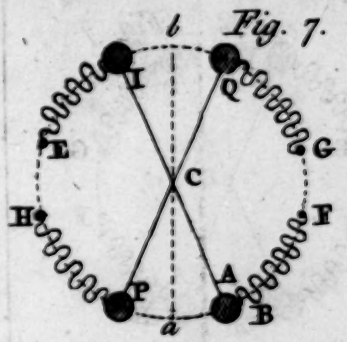


Fig. 7.

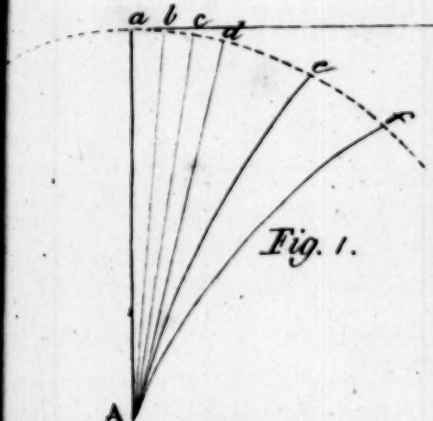


Fig. 1.

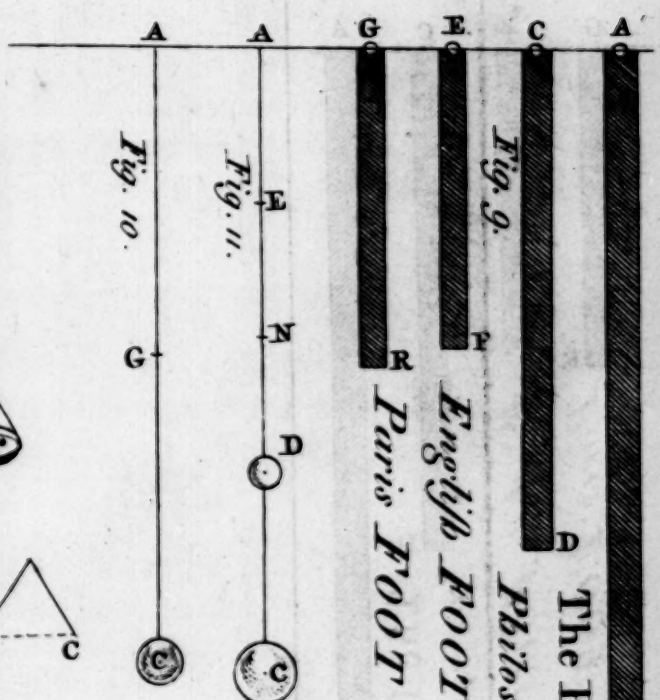


Fig. 10. Fig. 11. Fig. 9. Paris FOOT. English FOOT. Philosophical FOOT. The Philosophical YARD or STANDARD.

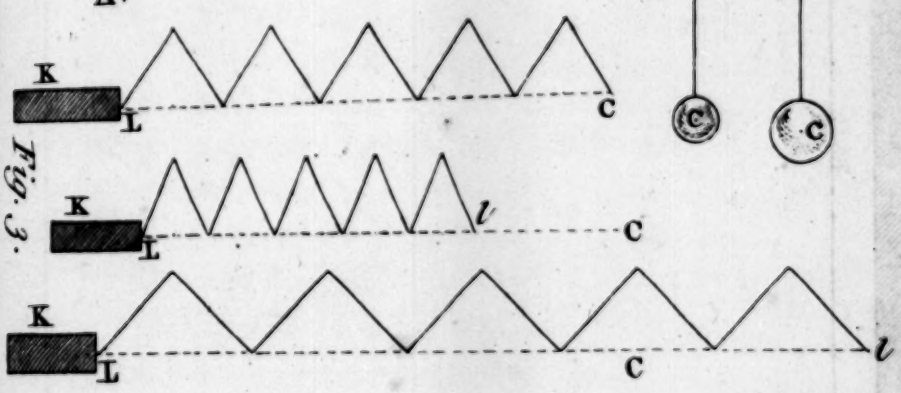


Fig. 3.

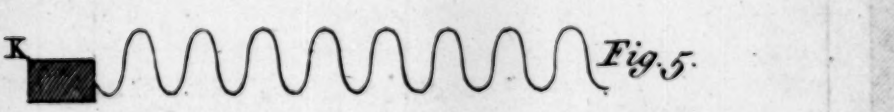


Fig. 5.

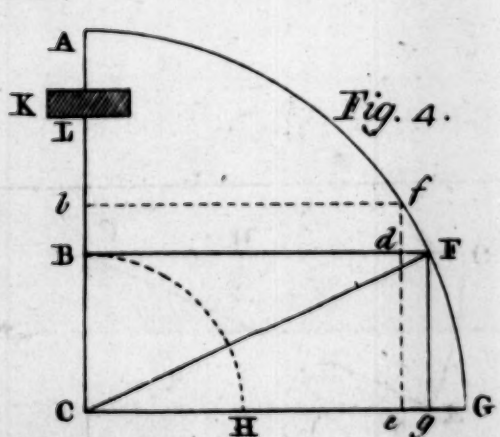


Fig. 4.

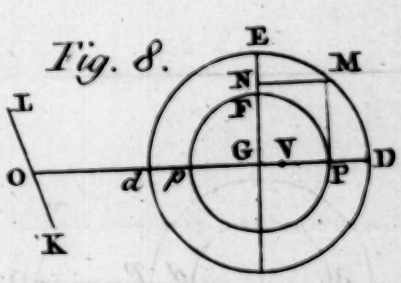


Fig. 8.

p be a Force applied which bends it through any Space or Length $Cl = s$; then from the Nature of the Spring (3233) $2p$ will bend it through a Space equal to $2s$, and $3p$ will bend it thro' $3s$; and ſo on. So that s will be every where proportional to p.

3239. Then if P denote the Power by which the Spring is wholly compreſſed, and S = Space of total Compreſſion; then it will be $S : s :: P : p$; and the ſame Analogy will reſult from a partial and total Extenſion. This Principle is not only deduced from a phyſical Theory as before-mentioned, but is confirmed moſt accurately by Experiments on Springs on every Form.

3240. And here I may obſerve, that uſeful Inſtrument called the *Spring* (or *Cylindric*) BALANCE for weighing Bodies, is nothing more than the Practice of this Principle; for the Number of pound Weights in the Body appended is indicated by the Number of *equal Diviſions* on the Bar, which is the Space thro' which the Spring is compreſſed by the Weight of the Body, as each ſingle Diviſion correſponds to that of a pound Weight.

3241. This is the Effect of a Spring in regard to *Preſſure*; we ſhall next inſtitute a Compariſon between this *elaſtic Force* and a *percuſſive Force*, or that of a *ſtriking* Body moving with a certain Degree of Velocity. For if a Body whoſe Weight or Maſs of Matter is M, and moving with an uniform Velocity V, ſtrike upon the End C of the Spring, it will bend the ſame through a Part or the whole Space S; and the Quantity of Motion by which it affects the Spring at firſt will be MV (970); but by the Reaction of the Spring, the Velocity V will be gradually diminiſhed; ſo that when the Spring is bent through any Space s, the original Velocity V will be reduced to v, and the Quantity of Motion or Momentum of the Body will there be only Mv, ſo that the Loſs of Motion in bending the Spring through the Space s, will be $MV - Mv$.

3242. Let A = Space through which a Body deſcends *in Vacuo* in one Second, by the Power of Gravity; and C = Celerity acquired in that Deſcent. Then we have $C^2 : A :: V^2 : a$ = Space of the Body M muſt deſcend thro' by Gravity to acquire the Velocity V. (991) Alſo $\sqrt{A} : 1'' :: \sqrt{a} : T$ = Time of the Deſcent through a.

3243.

3243. Suppoſe, then, by the Stroke of the Body M moving with the Velocity V, the Spring be bent through a Space s in the Time t. In this Caſe we have $S : s :: P : p = \frac{sP}{S}$ = to the *elaſtic Force* of the Spring at l, or at the End of the Space s.

3244. Now let the momentary Decrement of Velocity in the percutient Body be $-\dot{v}$, then will the inſtantaneous or fluxionary Momentum produced by the elaſtic Force of the Spring be $-M\dot{v}$ in the Moment of Time i; and we have ſhewn (1000) that in Caſe of Forces (M and $\frac{sP}{S}$) acting uniformly, the Quantities of Motion (MV, $-M\dot{v}$,) generated are proportional to the generating Forces, and the Times (T, i,) conjointly; and therefore $MV : -M\dot{v} :: M \times T : \frac{sP}{S} \times i$; which gives $-\dot{v} = \frac{VPsi}{MST}$.

3245. Again, in the ſame Caſe of Forces acting uniformly, the Spaces are as the Velocities and Times conjointly (971) therefore $2a : VT :: i : vi$; whence $i = \frac{TVi}{2av}$.

3246. Therefore $-\dot{v} = \frac{VPs}{MST} \times \frac{TVi}{2av}$; which gives $2v\dot{v} = -\frac{V^2Psi}{MSa}$; the Fluents of which are v^2 and $-\frac{V^2Ps^2}{2MSa}$; but when the former of theſe was V^2, the latter was $= 0$, becauſe of $s = 0$; therefore $v^2 = V^2 - \frac{V^2Ps^2}{2MSa}$.

3247. To reduce this Equation to more ſimple Terms by a Conſtruction, let $\frac{2aMS}{P} = R^2$, then will $v^2 = V^2 - \frac{V^2s^2}{R^2}$, or $v^2 = V^2 \times \frac{R^2 - s^2}{R^2}$. Now if R = Radius CG of a Circle, (Fig. 4.) and $s = CB = gF$, the right Sine of the Arch GF; then by the Property of the Circle, it is $R^2 - s^2 = BF^2$; therefore $v^2 = V^2 \times \frac{BF^2}{R^2}$; and ſo $v = V \times \frac{BF}{R}$, that is, *the*

Ve-

Velocity v *is to the original Velocity* V, *as the Cosine* BF *to the Radius* CG.

3248. Because $i = \frac{TV\,s}{2av}$, and $v = V \times \frac{\sqrt{R^2 - s^2}}{R}$, therefore $i = \frac{TV\,s}{2a} \times \frac{R}{V \times \sqrt{R^2 - s^2}} = \frac{T}{2a} \times \frac{R\,s}{\sqrt{R^2 - s^2}}$.

But drawing bf indefinitely near to BF, and fd perpendicular to the same, we have from the similar Triangles BCF and dfF, as $BF : df :: CF : fF = \frac{CF \times df}{BF} = \frac{R\,s}{\sqrt{R^2 - s^2}}$.

Therefore $i = \frac{T}{2a} \times fF$; and the Fluents of this Equation are $t = \frac{T}{2a} \times GF$; which gives this Analogy $t : T :: GF : 2a$.

3249. Thus the *Times*, *Velocities*, and *Spaces*, respecting the Motion of Springs, are easily assignable by a *trigonometrical Calculus* in the Parts of a Circle. But in most practical Cases, particularly *Watch-work*, the Space through which the Spring is bent by the Stroke is but a Part of the whole Length, and therefore the Quantity of Motion MV, or Force of the Stroke being consumed in bending the Spring, through the Space s in the Time t, the Velocity will there cease, or $v = 0$; in which Case the Time may be expressed independent of any particular Arch of a Circle, as follows.

3250. It is evident from the Theorem in (2247) that $R^2 = s^2$, when $v = 0$; therefore s being now equal to Radius CH = CB (Fig. 4.) and also the Sine of the Arch GF, that Arch now becomes the Quadrant of a Circle HB. Then if m = Periphery of a Circle whose Diameter is 1, we have $1 : m :: 2s : 2sm$, therefore $\frac{2sm}{4}$ = Quadrant HB. Wherefore in this Case, $t = \frac{T}{2a} \times GF = \frac{T}{2a} \times \frac{2sm}{4}$. But $\frac{T}{2a} = \frac{T}{\sqrt{a} \times 2\sqrt{a}}$; also $1'' : T :: \sqrt{A} : \sqrt{a}$; therefore $\frac{T}{\sqrt{a}} = \frac{1''}{\sqrt{A}}$; whence $t = \frac{1}{\sqrt{A}}$

$\frac{1}{\sqrt{A}} \times \frac{2sm}{4 \times 2\sqrt{a}}$. But it being $s = R = \sqrt{\frac{2SMa}{P}}$ (3247) we have $\frac{s}{\sqrt{a}} = \sqrt{\frac{2MS}{P}}$. Consequently $t = \frac{m}{2} \times \sqrt{\frac{MS}{2PA}}$ $= \frac{m}{2} \times \sqrt{\frac{Ms}{2pA}}$ in Seconds of Time.

3251. Because $v^2 = V^2 \times \frac{BF^2}{R^2}$ (3247) we have $V^2 - v^2 = V^2 - V^2 \times \frac{BF^2}{R^2} = V^2 \times \frac{R^2 - BF^2}{R^2} = V^2 \times \frac{CB^2}{R^2} = \frac{V^2 s^2}{R^2}$; then because $R^2 = \frac{2MSa}{P}$ (2247) we have $V^2 - v^2 = V^2 s^2 \times \frac{P}{2MSa}$.

3252. But $\frac{V^2}{a} = \frac{C^2}{A}$ (2242) therefore $V^2 - v^2 = \frac{C^2 P s^2}{2MSA}$. And therefore $MV^2 - Mv^2 = \frac{C^2 P s^2}{2SA} = \frac{C^2 ps}{2A}$. This Theorem may be of Use in that Part of *Mechanics* concerned in the Doctrine of the *Vires Vivæ & Mortuæ*.

3253. Hence when $v = 0$, we have the initial Velocity $V = Cs\sqrt{\frac{P}{2MSA}} = C\sqrt{\frac{ps}{2MA}}$. And because $t = \frac{m}{2} \times \sqrt{\frac{MS}{2PA}}$; therefore the Product of the Velocity and Time will be $Vt = \frac{mCs}{4A} \times \sqrt{\frac{MSP}{MSP}} = \frac{mCs}{4A}$.

3254. And because $\frac{mC}{4A}$ is a constant Quantity, we have $Vt : s$; or the *Space through which the Spring is bent always as the Velocity and Time conjointly*, the same as in the Case of Bodies descending by *Gravity* (971).

3255. Wherefore in the same, or different Springs, *the Spaces through which they are bent in a given Time will be as the Velocities*; and with a given Velocity, *they will be as the Times employed in bending them.*

3256.

3256. Becauſe in the Expreſſion of the Time, $t = \frac{m}{2}\sqrt{\frac{MS}{2PA}}$, the Quantities m and $2A$ are conſtant; therefore it will always be $t : \sqrt{\frac{MS}{P}}$; or $t^2 P : MS$. But we had alſo $t : \frac{s}{V}$ (2254) therefore $t : \frac{MS}{tP} : \frac{s}{V}$. Conſequently $MSV : ts P$. Hence we obtain a Compariſon between theſe ſix principal Quantities in a *Spring*, and a Body bending it with a *percuſſive Force*.

3257. Thus if the Length S and Force P of a Spring be given we have $MV : ts$; but in this Caſe $t^2 : M$ (2256); therefore $tV : s$, or *the Space is as the Rectangle of the Time and Velocity*, as in *Falling-bodies* (991).

3258. Likewiſe in the ſame Spring, ſince $t^2 : M$, therefore when M, or the Body is given, *the Time* (t) *of bending the Spring will be the ſame*, whatever be the Degree of Velocity V, or the Space s through which it is bent.

3259. But in a given Spring S, P, and Body M, ſince t will be conſtant, we have $V : s$; that is, the *Space through which the Spring is bent, will always be proportional to the Velocity of the Body*; which is another Caſe ſimilar to deſcending Bodies urged by different accelerating Forces.

3260. It was proper on ſome Accounts to give the foregoing Theorems expreſſing the *Velocity*, *Time*, *&c.* in the Form they there have; but in the particular Caſe of the Space deſcribed, or Deſcent in *one Second*, we have $2A = C$; thoſe Theorems, therefore, will be expreſſed more ſimply for Calculation, thus (putting $m = 3{,}1416$) $t = \frac{m}{2} \times \sqrt{\frac{MS}{2AP}} = 1{,}57\sqrt{\frac{MS}{CP}}$ $= 1{,}57\sqrt{\frac{Ms}{Cp}}$; and $V = C \times \sqrt{\frac{ps}{2AM}} = C \times \sqrt{\frac{ps}{CM}}$ $= \sqrt{\frac{Cps}{M}}$; and $MV^2 = Cps$; and laſtly, $Vt = \frac{mCs}{4A}$ $= 1{,}57s$;

$= 1{,}57\,s$; or $t = 1{,}57\,\frac{s}{V}$, in Seconds. And in all these Cases, $C = 32\,\frac{1}{6}$ Feet, or 386 Inches.

CHAP. VIII.

The foregoing THEORY *of* SPRINGS *farther considered, and exemplified by* CALCULATIONS *in their various* APPLICATION *to* WATCH-WORK.

3261. IT has been already shewn, that an *Automaton* put into Motion by a Weight, and having that Motion regulated by a Pendulum, is the most perfect that can be in Nature; and that in the next Degree of Perfection is the *Spring-clock*; but when we consider the Structure of a WATCH, left wholly to the Power and Action of Springs, we shall find it naturally destitute of any absolute Principle of equable Motion.

3262. The Action of a *Weight* is constant, and the Oscillations of a Pendulum are equable, from an *innate Principle*; but we cannot consider the Action of a Spring either as a *First-mover* or *Regulator* of Motion, as constant and equable in its artificial Application in Watches; this we have also shewn in regard to the First, in the *Theory* of the *Spring* and its *Fusee*; and in respect to the Second, the Office of the Spring as a *Regulator*, is not derived from the *Natural*, but what we may properly call, the *artificial Vibrations* thereof. The natural Vibrations of all Springs are in themselves as perfectly isochronal as those of Pendulums (3232); but the Case is quite otherwise with respect to the *artificial Vibrations*, whose Equability of Motion, being the Result of the compound Action of two Strings together, cannot be supposed so perfectly constant, and regular, as no *mechanical Combination of Causes* can act with the Simplicity and Uniformity of *Nature itself*.

3263.

3263. If the Watch be required to beat *Quarter-ſeconds*, or to oſcillate 14400 Times in an Hour, theſe Oſcillations by a Pendulum are rendered equable without any Trouble; but to effect this by a Spring, or Number of Springs, will require much more mechanical Skill and Contrivance, and at laſt be attended with ſome Degree of Inconſtancy or Irregularity.

3264. Indeed, the fine Spring connected with the Balance is to be conſidered as ſomewhat more than a *Regulator*; for it does, as it were, *form* or *modulate* thoſe Oſcillations, as well as *regulate* them; without ſuch a Spring the Balance would oſcillate, it is true, but ſince the Motion produced by the Action of the Crown-wheel, on one Pallet, muſt be in a Moment ſtopped, deſtroyed, and generated anew in a contrary Direction by its Action on the other Pallet, and this in ſo quick and conſtant an Alternation, the Effect would be too violent and ſhocking to be ſufferable, and too irregular to be of any Uſe.

3265. But by the Application and Co-operation of the Spring, the ſaid contrary Motions are gradually generated and deſtroyed; and the Times of the Oſcillations of the Balance-wheel rendered as equable and uniform as the niceſt Mechaniſm can admit of. To effect this, many different Methods have been invented, principally by Dr. *Hooke*, Mr. *Hugens*, Mr. *Leibnitz*, *&c.* Some of theſe were by ſingle Balances, others by double Ones; ſome had only one regulating Spring; others had two, or more. The Attempt to regulate a Watch by a *Loadſtone* inſtead of a *Spring*, is not worth mentioning.

3266. There were two Ways in which the double Balance was applied; in one of them each ſingle Balance had a Verge with one Pallet only, placed on each Side of the *Crown-wheel* diametrically oppoſite to each other, for that Wheel had the ſame Poſition with the *contrate Wheel*, or as it now has in the preſent *horizontal Watches*. On the Verge of each Balance was fixed a ſmall Wheel; theſe Wheels were proportioned to the Diameter of the Crown-wheel; having the ſame Number of fine Teeth, they played in each other, and ſo gave an equal Motion to the larger Balance-wheels juſt above them.

3267. The other Way, with two Balances, had the two ſmall Wheels, by which they moved each other, and more-

over to each Balance-verge, there was added a ſpiral Spring as a *Regulator*. In this Method, one Balance only had a Verge on which both the Pallets were, and it was moved by the Crown-wheel, placed in the ſame perpendicular Poſition it now has in common Watches. With reſpect to theſe two Inventions, Dr. *Derham* ſuppoſes the firſt was never proſecuted ſo far as it deſerves; and the Second has this Excellency, that no Jerk or the moſt confuſed Shake can in the leaſt alter its Vibrations. And that the Reaſon why this Method of conſtructing Watches came into Diſuſe, he judges was the great Trouble and vaſt Niceneſs required in it. If this was the Caſe, it reflects no great Honour on the Reputation of Artiſts in this Way.

3268. The Spring, as a *Regulator*, is applied in Watches, with one End fixed to the Verge of the Balance, and being coiled ſeveral Times round, to give it a ſufficient Length, it has its other End faſtened to a Part towards the Extremity of the Watch-plate, which has lateral Teeth, and is moved by a ſmall Pinion on an Axis in the Center of a ſmall ſilvered Plate on one Side of the Cock or Balance, divided into 30 ſmall equal Parts. This Plate is alſo fixed on the ſaid Axis, and by the Key is moved againſt an Index to the Right or Left, thereby increaſing or remitting the Force of the Spring when the Watch goes too ſlow or too faſt; and this Apparatus for correcting the Regulator, and thereby the Time of the Watch, we find by common Experience, is but too frequently neceſſary in the moſt excellent Pieces of this Sort of Machinery.

3269. Mr. *De la* HIRE, in a *Memoir of the Academy of Sciences*, objects very much to the common ſpiral Form of the Regulator, and eſpecially to the fixing one End thereof to, or near the Verge or Axis of the Balance. He thinks, if the End of the common ſpiral Regulator were faſtened to a Part of the Radius of the Balance inſtead of the Axis, it would have more Force to govern the unequal Movements thereof, and at the ſame Time be leſs ſubject to its Irregularities; and farther adds, that the Spring bent into an *undulatory* or *wave-like Form* (as in Fig. 54) is much better than the Spiral, in that a greater Length may be had in a leſs Space, and the Spring thereby ſuſtain itſelf with greater Eaſe, and act without that Compound and diſtorted Mo-

Motion which the Spires of the common Regulator are ſubject to, in their lengthening and ſhortening. And what he has aſſerted, he tells us, he has fully evinced by Experiments with Watches of this new Conſtruction.

3270. But I have conſtantly obſerved, that theſe Inventions, Alterations, and Innovations concerning the Balance and its Regulator, have all proceeded (*rudi Minerva*) from the natural Force of mechanical Genius, with very little, if any, *Rationale* from Principles of a *philoſophical* or *mathematical* THEORY of the *Nature and Action* of SPRINGS; which certainly muſt reflect very great Light on a Subject that has always been looked upon as overwhelmed in Difficulty and Obſcurity. For, (if I judge right in theſe Matters) there ſeems to be a much better and more natural Method of governing and regulating Watch-work pointed out by the foregoing Theory, than that of the Balance-wheel and its uſual Regulator.

3271. For by this Theory, it is evident, that with Reſpect to the *Power* of the Spring, the *Momentum* of a *Body* which ſtrikes it, the *Space* through which it is bent, and the *Time* of bending it, if any of theſe Quantities are known or given, the reſt may be found; becauſe their Relations are all determined by the Theorems $t = 1{,}57\sqrt{\frac{\overline{M}s}{C\,p}}$, and $V = \sqrt{\frac{\overline{C}ps}{M}}$ (2260) as alſo the real Quantities of each reſpectively.

3272. Therefore ſuppoſe the Train of a Watch be 14400, or 4 Beats *per* Second; then let A be a Body ſuſpended by a Thread A C from the Center C, (Fig. 6.) and being raiſed to the Point D let it deſcend through the Arch D A and ſtrike the Spring B F with the Velocity acquired in that Deſcent; and then A *a* is the verſed Sine, or perpendicular Deſcent of the Body to acquire the ſame Velocity. Laſtly, the Momentum of the Body A will by its Impulſe on the Spring bend it through a certain Space B H. Now for the given *Length* and *Strength* of the Spring, and the given Space and Time of bending it, the other Quantities may be found by Calculation, as follows.

3273. Conſidering this Spring B F as a *Regulator*, it muſt be very fine and tender; and therefore we will ſuppoſe it ſuch that a ſingle *Scruple* only, or 20 Grains, ſhall be juſt equal to its whole

whole Force, or be able to keep it bent through its *whole Length*, which may be one Inch, and let the Time of bending it through half its Length be $\frac{1}{4}$ of a Second; then there is given B F $=$ S $=$ 1, P $=$ 20, s $=$ 0,5 $=$ A H, p $=$ 10, and t $=$ 0″,25; to find M, the Weight of the Body A; V, the uniform Velocity of its Motion; and a $=$ A a, the Height it muſt deſcend perpendicularly to acquire that Velocity.

3274. To find the *Weight of the Body* A, we have from the Theorem (3260) $t = 1{,}57\sqrt{\frac{MS}{386\,p}}$, this Equation $386\,p\,t^2 = 2{,}4649\,M\,s$; and therefore this Analogy, $p : M :: s : 156{,}6\,t^2$; which will give $M = \frac{156{,}6\,t^2\,p}{s} = 196$ Grains, in the preſent Caſe; ſo that the ſtriking Body A is near 20 Times the Weight of p.

3275. The uniform Velocity of its Motion is $V = \sqrt{\frac{C\,p\,s}{M}}$ $= 3{,}138$ Inches *per* Second, this is equal to the Circumference of a Circle A E F whoſe Diameter is juſt *one Inch*.

3276. Laſtly, we find A a $=$ a $= \frac{A\,V^2}{C^2}$ (3242) $= \frac{V^2}{4\,A} = \frac{V^2}{772} = 0{,}0127$, or about $\frac{13}{1000}$ of an Inch; and therefore the Arch A D ſo exceeding ſmall as to be altogether inconſiderable.

3277. If therefore the Pallets of the Verge be ſo proportioned to the Teeth of the Crown-wheel as to move the Body A thro' $\frac{3}{4}$ of an Inch each Stroke without the Spring; it will, when the Spring is added, be moved through only half an Inch, or bend the Spring from B to H in $\frac{1}{4}$ of a Second as required.

3278. If the Maſs of Matter in the Body or Globe A were diſpoſed into the Form of a Circle A E F, or ſo as to make the Perimeter of a fine Wheel, it would then become the Balance-wheel of a common Watch; and being connected with the End of the Spring B F, that Spring would alſo become the Regulator; and the Watch thus conſtructed, would beat Quarter-ſeconds.

3279.

3279. Hence in Watches of a large Size, and eſpecially Table-clocks, where more Springs than one may conveniently be applied, this Regulation of Time might be moſt commodiouſly performed by a ſingle Balance-lever with two Springs, or a double One with four; the Reaſon of which will appear from (Fig. 7.) where A I and P Q are two Levers, whoſe Weights A, I, P, Q, are equal to each other, and to the Weight A in Fig. 6th. Alſo the Springs B F, I E, P H, G Q, are all ſeverally equal to B H in that Figure. Theſe Levers croſs each other in the common Center of Gravity C, where they are fixed at Right-angles to the Verge.

3280. As the Weights and Springs are increaſed in the Balance, it is neceſſary the Force of the Spring at the Fuſee ſhould be encreaſed in Proportion. And it is further to be obſerved, that in the quieſcent State of the Levers, the four Springs are in a State of Compreſſion; the two Springs fixed at F and H being compreſſed on each Side from (*a*) and the other two from the Point (*b*). If *a* B be one Half of the Arch through which the Levers oſcillate, then it is plain, that in each Oſcillation, while two Springs are compreſſed by one Lever, the other two Antagoniſt Springs are relaxing; and therefore as one Pair retards, the other equally accelerates the Motion of the Levers; and ſo no Inequality of Motion can ariſe from the joint Action of Springs; but on the contrary, as perfect a Correction of the Irregularities of the Watch-work is obtained, as can be produced by the Agency of *Springs*.

CHAP.

CHAP. IX.

The METHOD *of finding the* CENTER *of* OSCILLATION *in all Kinds of* PENDULUMS, *deduced from a* NEW THEORY. *Alſo the* NATURE *of a* univerſal MEASURE of LENGTH, *or* philoſophical FOOT, *explained and exemplified.*

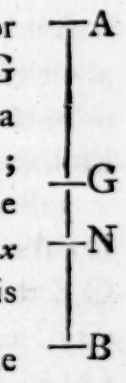
3281. WE now quit the Subject of *Watch-work*, where nothing valuable with regard to the Equation and Regulation of Time can be hoped for, or expected; and return again to the farther Conſideration of thoſe *Automata* which are regulated more by *Nature* than *Art*, *viz.* by the Power of Gravity governing the Oſcillations of *Pendulums*. But here the Artiſt muſt follow pretty cloſely the Dictates of the omniſcient Mechanic, and work, if he propoſes to merit Applauſe, by the Rules of divine Geometry.

3282. Now as the PENDULUM is the Principle of Truth and Perfection in Clock-work, all Circumſtances relative to it ſhould be conſidered with the greateſt Attention, and principally that which concerns the *Center of Oſcillation*; and here will ariſe the following Queſtions, *viz.* what this *Center* of *Oſcillation* is in the BALL or BOB of a common Pendulum? What is the Diſtance thereof from the Point of Suſpenſion? And how that Diſtance is to be preſerved unaltered?

3283. I believe very few Mechanics in this Way know ſo little of Art or Nature, as to ſuppoſe that the *Center of the Bob*, is the *Center of Oſcillation*. But fewer ſtill know where it is, or how to find it in the Pendulum. Yet the Knowledge thereof is of the laſt Conſequence in very large Balls, and ſome Mr. GRAHAM had which exceeded 60 Pounds; alſo in ſhort Pendulums in Table Clocks, this Center of Oſcillation ſhould be nicely aſcertained. Mr. *Hugens* lays the greateſt Streſs on this Point; and all his Folio-treatiſe is wrote profeſſedly on this important Subject.

3284.

3284. But the Method he purſued to diſcover it, will give us too much Trouble and Fatigue; a nearer and eaſier Way has, ſince his Time, been found out, and which we ſhall now elucidate in that Caſe firſt, where the Weight of the Ball or Bob is ſo great, that the Weight of the Rod by which it is ſuſpended is inconſiderable in Compariſon of it. In order to this, a Retroſpect to a few Things already demonſtrated in theſe Inſtitutions will be neceſſary, and tend greatly to ſhorten the Operation.



3285. Let A B be a Line oſcillating about a Point or Center A; the Diſtance of the Center of Gravity A G $= g$, and of the Center of Oſcillation A N $= n$; $\dot{x} =$ a ſmall Particle or Weight; $x =$ its Diſtance from A; and let the Sum of all the Particles or Weight of the whole Line be S. Then $\dot{x}x$ is the *Moment*, and $\dot{x}xx$ the *Force* of the Weight $\dot{x}$; and the Sum of all the $\dot{x}x^2$ is $\frac{x^3}{3}$, and when $x =$ S, we have $\frac{S^3}{3}$ for the Force F of the whole Line. All which is evident from (1086 to 1098).

3286. But $\frac{S^3}{3} = S \times \frac{1}{2} S \times \frac{2}{3} S = Sgn = F$, agreeable to what was ſhewn in (1095). Alſo when $x = \frac{1}{2}$ S, then the Sum of all the $\dot{x}x^2$ will be $\frac{x^3}{3} = \frac{S^3}{3 \times 8} = \frac{S^3}{24}$; and twice that Sum will be $\frac{2}{24} S^3 = \frac{1}{12} S^3$; which therefore will be as the *Sum of the Products of all the Particles* ($\dot{x}$) *each multiplied by the Square of its Diſtance from the Center of Gravity* G.

3287. Therefore put $\frac{1}{12} S^3 = q$; and we ſhall have $Sgn = q + z$; and $Sgn - q = z = \frac{1}{4} S^3 = S \times \frac{S^2}{4} = Sg^2$. Therefore $Sgn - Sg^2 = q = Sg \times \overline{n - g}$. And conſequently $\frac{q}{Sg} = n - g = GN$; which gives this general Rule for finding the Diſtance of the Center of Oſcillation N from the Center of Gravity G, *viz. divide the Sum of the Products of the Particles or Weights, ſeverally multiplied by the Squares of their Diſtances from the Center of Gravity, by the Solidity or Maſs multiplied by the Diſtance of the Center of Gravity from the Axis of Oſcillation, and the*

Quotient will be the Diſtance of the Center of Oſcillation from the Center of Gravity.

3288. If we put G N = d; then becauſe q = S g d it will be $\frac{q}{d}$ = S g = $\frac{F}{n}$ (2286) therefore $q\,n$ = F d, and we have F : q :: n : d :: A N : G N. The reaſoning is the ſame in regard to Planes and Solids, and conſequently the Rule is the ſame in all; but I have choſen the eaſieſt and moſt ſimple Method by which it may be demonſtrated. As a Proof of its Univerſality I ſhall apply it to find the Center of Oſcillation in a Globe, and ſhew it to be the ſame as *Hugenius* found it with ſo much Labour and Prolixity.

3289. Let D E d be the Section of a Globe through its Axis; D d the Diameter perpendicular to the Axis K L of Oſcillation; G E the Radius at Right-angles to D d; G the Center of Gravity, and V the Center of Oſcillation in the Line of Suſpenſion O G; and let P F p be any concentric Circle. Let the Ordinate P M = y, G P = x, and n : 1 :: Circumference of a Circle : to Radius; and draw N M parallel G D. Then ſuppoſe a cylindric Surface to ſtand on the Circumference P F p perpendicular to the Plane D E d, and terminated by the Surface of the Sphere, then becauſe b : 1 :: p : x, we have $n\,x$ = p = Circumference P N p P, and therefore $2\,y\,n\,x$ = to the ſaid cylindric Surface (832).

3290. Now G P = x, is the Diſtance of the Particles in each Section of this Surface (perpendicular to the Axis of Oſcillation) from the Center of Gravity of the Section; therefore $2\,n\dot{y}\,x \times \dot{x}^2 = 2\,n\,y\,x\,\dot{x}$, is the Fluxion of the Surface or Weight, which multiplied by (the Square of the Diſtance from the Center of Gravity) G P^2 = x^2, gives the Fluxion $2\,n\,y\,x^3\,\dot{x}$, whoſe Fluent is $\frac{1}{2}\,n\,y\,x^4$.

3291. The Sine and Co-Sine of an Arch are inverſely as their Fluxions; for (in Fig. 4.) the Triangles $f\,d$ F and F C g are ſimilar, therefore $f\,d$: dF :: C g : gF; that is, $\dot{x} : \dot{y} :: y : x$; whence we have $\dot{x} = \frac{y\,\dot{y}}{x}$; and putting the Radius of the Globe G D = a, we have $x = \sqrt{a^2 - y^2}$; therefore $2\,n\,y\,x^3\,\dot{x} = 2\,n\,y \times \frac{y\,\dot{y}}{\sqrt{a^2 - y^2}} \times x^3 = 2\,n\,y^2\,\dot{y} \times \overline{a^2 - y^2} = 2\,n\,a^2\,y^2\,\dot{y} - 2\,n\,y^4\,\dot{y}$, whoſe

whoſe Fluent is $\frac{2}{3} n a^2 y^3 - \frac{2}{5} n y^5$; and when $y = a$, this Fluent is $\frac{4}{15} n a^5$, analogous to the Quantity (q) in the general Expreſſion (2287).

3292. Again, the Solidity of the Sphere is $\frac{1}{6} p d^2$ (836) but $p = na$, and $d = 2a$, therefore $S = \frac{1}{6} \times na \times 4a^2 = \frac{2}{3} na^3$, and putting $OG = z$, we have $\frac{2}{3} na^3 \times z$ analogous to Sg in (2287); therefore $\frac{2}{3} na^3 z$) $\frac{4}{15} na^5$ ($= \frac{2}{5} \frac{a^2}{z} = GV$, the Diſtance of the Center of Oſcillation from the Center of the Sphere, as required. And is the very ſame with that found by *Hugenius*, *Bernouilli*, *Varignion*, *Mac Laurin*, *Simpſon*, *Emerſon*, &c. by tedious and obſcure Methods.

3293. If $Od = d$, then $OG = d + a = z$; and then alſo $\frac{2a^2}{5z} = \frac{2a^2}{5d + 5a}$; and when $d = 0$, or the Sphere oſcillates about a Point d in its Surface, we have $dV = \frac{2a^2}{5a} = \frac{2}{5} a$, or $\frac{7}{10}$ of the Diameter dD, and not $\frac{6}{10}$, as it has been determined by Mr. *Carrè*, *Hayes*, *Stone*, and others.

3294. Put $\frac{2a^2}{5z} = (GV =) v$, then $2a^2 = 5zv$, and $a^2 = \frac{5}{2} vz$; therefore $z : a :: a : \frac{5}{2} v$. Now the Length of a Pendulum vibrating Seconds is 392 *Tenths* of an Inch. Then ſuppoſe the Diameter of the Globe or Ball = 4 Inches, we have $a = 20$, and $z + v = OV = 392$. And then $392 + v : 20 :: 20 : 1$, very nearly; ſo $\frac{5}{2} v = 1$, or $5v = 2$, and $v = \frac{2}{5}$ of the Tenth of an Inch, or $\frac{1}{25}$ of an Inch.

3295. In a whole Day, or 24 Hours, there are ($24 \times 60 =$) $1440'$; and if we put $OV = 1440$, it will be eaſy to find the Length of a Pendulum that ſhall gain juſt one Minute *per* Day. For ſince the Times of a Vibration in the two Pendulums will be inverſely as the Number of Vibrations performed in a given Time (or *one Day*,) thoſe Times will be as the Numbers $1440'$ and $1439'$. Again, the Lengths of Pendulums are as the Squares of the Times of their Vibrations, or as 1440^2 to 1439^2; but $1440^2 : 1439^2 :: 1440 : 1438$, very nearly; then $1440 : 1438 :: 392 : 391{,}46$, then $392 - 391{,}46 = 0{,}54$, or a little more than $\frac{1}{20}$ of an Inch; therefore $\frac{1}{20} : \frac{1}{25} :: 25 : 20 ::$

60″ : 48″. Whence it appears, that the Diſtance G V, ſmall as it is, in a *Second Pendulum*, is enough to make the Clock err 48 Seconds *per* Day, if not regarded.

3296. A Cubic Inch of Braſs weighs 4,4 Oz. Troy, and the Cube being to its inſcribed Sphere nearly as 2 to 1 (846). A Sphere or ſolid Braſs Globe of one Inch Diameter, will weigh 2,2 Oz. Troy; therefore as $1^3 : 4^3 :: 1 : 64 :: 2{,}2 : 141$ Oz. or 11 $\frac{3}{4}$ *lb.* Troy, (841) the Weight of the Ball of the above-mentioned Pendulum. But if it were 5 Times heavier, the Quantity G V would ſtill be greater in the Ratio of the Squares of the Diameters of the Globes; and hence it appears, of how great Conſequence it is, in the Science of meaſuring Time, to be able to aſcertain this Point or *Center of Oſcillation in ſimple Pendulums.*

3297. Before we quit this Theory, it will be very material to obſerve, that the Doctrine of a *perpetual and univerſal* MEASURE is founded in it; and that what is called a phyſical or *philoſophical* YARD, is nothing more than the Length of a ſimple Pendulum vibrating in a *Second* of Time. That is, the *univerſal* YARD is equal to 392 *Engliſh Lines*, or *Tenths* of an Inch. And the *univerſal* FOOT is a third Part of that Number, or 130 $\frac{2}{3}$ of ſuch Lines. But this Notion of a HORARY YARD, and FOOT, and the Manner of aſcertaining it by the Pendulum of a Clock well adjuſted to equal Time by the Revolution of the Stars, as deſcribed by HUGENIUS, is to tedious here to inſiſt on.

3298. I ſhall offer the following Method as the moſt conciſe and eaſy for this Purpoſe. Let any Meaſure propoſed be conſidered as a uniform ſolid Body, whoſe Length is the *general Standard.* If ſuch a Meaſure or Solid were ſuſpended on an Axis to vibrate freely on one End (paſſing through the central Line of the Axis) it has been ſhewn, that the Center of Oſcillation in ſuch a Body will be juſt $\frac{2}{3}$ of its Length from the Axis of Suſpenſion (1097). And therefore any ſuch Body equal in Length to $392 + 196 = 588$ *Tenths* of an Inch, will vibrate exactly in *one Second of Time*; and is of Courſe, the *univerſal* or *philoſophical* YARD.

3299. This YARD muſt be divided into 10000 equal Parts, in which the Length of other Standard Meaſures are to be expreſſed

preſſed by comparing the Times of their Vibrations with that of the univerſal Yard. For this Purpoſe, let its Length be L = 10000, and N = 3600, the Number of Vibrations in an Hour; and let *l* and *n* denote the ſame Quantities in any other Meaſure propoſed; then we have T : *t* :: $\sqrt{L}$: $\sqrt{l}$ :: *n* : N, the Times of a ſingle Vibration T, *t*, being inverſely as the Numbers N and *n* performed in a given Time. Whence this Theorem $\frac{\sqrt{L} \times N}{n} = \sqrt{l}$.

3300. Then as the *Engliſh* Standard Foot is = 120, ſay, as 588 : 10000 :: 120 : 2040; ſo that the *Engliſh* Foot is 204 of the ſame Parts of which the *univerſal* YARD contains 1000. Now this is diſcoverable from its Number of Vibrations in an Hour, which ſuppoſe were found, by a well adjuſted Clock to be 7969; for by the Theorem $\frac{\sqrt{10000} \times 3600}{7969} = \sqrt{l}$, whence l = 2040, as before.

3301. After the ſame Manner the *Paris Royal* Foot would be found 2179,6 Parts of the *univerſal* YARD; and therefore as 2040 : 2179,6 :: 1000 : 1068 :: 12 Inches : 12,8 Inches; or more accurately the *Paris Foot* is 12 $\frac{816}{1000}$ Inches. The *Philoſophical*, *Engliſh*, and *Paris* Feet are therefore as 333,3, 204, 218; and after this Manner, by counting the Vibrations made in a Minute or an Hour by any other Standard Meaſure, may its Length be aſcertained, and Ratio expreſſed, in Parts of the *univerſal* YARD.

3302. And as a farther Illuſtration of this Matter, I have repreſented theſe Meaſures ſeparately in the Copper-plate (Fig. 9.) where A B is the *philoſophical* YARD; C D the *philoſophical* FOOT; E F the *Engliſh Foot*; and G H the *Pariſian Foot*; all in their due Proportion of Length. From what has been ſaid, it is evident of how much Importance the Theory of a ſimple Pendulum is, not only as the moſt perfect CHRONOMETER, but as a STANDARD for Meaſures of LENGTH of every Kind, and to all Ages.

CHAP.

CHAP. X.

The THEORY *of a* Compound PENDULUM *explained, and its* CENTER *of* OSCILLATION *inveſtigated.*

3303. THE Conſideration of a *Compound Pendulum* comes next in Courſe, nor is the Theory thereof leſs curious or neceſſary than that of a ſimple One, ſince in the common Conſtruction of Clocks, the Weight of the Rod or Wire is too conſiderable to be neglected; and then the Rod and Ball together are to be eſteemed as a Pendulum conſiſting of two different Weights, whoſe compound Center of Gravity, and Center of Oſcillation are to be determined.

3304. Again, after all the Efforts of human Art or Skill, the *Pendulum* can never be perfectly equable in its Motion, but will, from the very Condition of Nature itſelf, be liable to have the Times of its Vibration altered, by an inſenſible Acceleration or Retardation from the variable State of natural Cauſes. Therefore, for common Uſe, HUGENIUS recommended a moveable Ball, or Weight for correcting the Irregularities of Vibration, as by its different Poſition on the Rod, it will cauſe a *ſmall Alteration of the Diſtance of the Center of Oſcillation*, by which Means the Clock may be adjuſted to true Time.

3305. In a Pendulum of this Sort there are three different Weights to be conſidered, *viz.* that of the large *Ball or Bob*, that of the *Rod*, and that of the *Corrector*, moveable upon it. The Center of Oſcillation of all which muſt be found, which will not be difficult after we have found that for two Weights. Therefore let A C (Fig. 10.) be a Pendulum, Biſect its Length in G; and put A G $= g$, and A C $= a$. Alſo call the Weight of the Bob (c) and that of the Rod or Wire (b).

3306. Now the *Momentum* and *Force* of each Part are ſeparately to be eſtimated. The *Momentum* of the Rod is univerſally $\frac{1}{2}ab$ (1072); the *Moment* of the Ball C is (ac), being as the Weight multiplied by its Diſtance (1089). The *Force* of the Rod is $b \times \frac{1}{2}a \times \frac{2}{3}a$ (1095) $= \frac{1}{3}ba^2$; and the *Force* of the Ball C is ca^2 (1089); therefore the *Sum of the Forces* divided by the Sum of the Moments, gives the Diſtance (n) of the Center of Oſcillation, that is, $\frac{\frac{1}{3}ba^2 + ca^2}{\frac{1}{2}ba + ca} = n = \frac{\frac{1}{3}ba + ca}{\frac{1}{2}b + c}$ as in (1094). And from hence the Length of the Pendulum $AC = a = \frac{\frac{1}{2}bn + nc}{\frac{1}{3}b + c}$; which is therefore determined for a *Second-pendulum*, by putting $n = 39{,}2$ Inches; or for *Half-ſeconds*, if $n = 9{,}8$ Inches. *Note*, in all theſe Caſes, the Length A C is the Diſtance between the central Line in the Axis of Suſpenſion, and the Center C of the Bob.

3307. Now let AC (Fig. 11.) be the Pendulum with the Addition of the *Corrector* D, or ſmall moveable Weight $= d$. And let its Diſtance be $AD = f$. Then will its *Momentum* be fd, and its *Force* dff; and now the Sum of the Forces of each Part divided by the Sum of the Moments will be $\frac{\frac{1}{3}a^2b + ca^2 + df^2}{\frac{1}{2}ab + ac + df}$ *the Diſtance of the Center of Oſcillation or Length of an iſochronal Pendulum*, which for the Future we will call p.

3308. Then if it be required to find (f) or the Poſition of the Corrector D, we have $\frac{\frac{1}{3}a^2b + ca^2 + df^2}{\frac{1}{2}ab + ac + df} = p$, which gives this Equation $f^2 = fp + \frac{\frac{1}{2}abp + cap - \frac{1}{3}a^2b - a^2c}{d}$, then by compleating the Square, and extracting the Root, we have $f = \frac{1}{2}p \pm \sqrt{\frac{\frac{1}{4}p^2 + \frac{1}{2}abp + cap - \frac{1}{3}a^2b - a^2c}{d}}$.

3309. Hence it is obvious, there will always be two real Roots or Values of (f) while $\frac{1}{2}abp + cap$ is leſs than $\frac{1}{3}aab + aac$, or while the Length p is leſs than $\frac{\frac{1}{3}a^2b + a^2c}{\frac{1}{2}ab + ca}$, which

is

is the Length of the iſochronal Pendulum (π) conſiſting of the Rod A C, and the Weight C only (3306).

3310. When, therefore, we would accelerate the Motion of the Pendulum by the Application of the Weight D, we have the Choice of two Places for it between A and C, *viz.* D or E, which Places are equally diſtant from N, the Point which Biſects the iſochronal Pendulum p. For $f = \frac{1}{2}p +$ or $-$ the radical Quantity as above (3308).

3311. Therefore, when $f = \frac{1}{2}p =$ A N, it will accelerate the Motion of the Pendulum the moſt of all; in which Caſe we have $\sqrt{\frac{\frac{1}{4}p^2 + \frac{1}{2}abp + cap - \frac{1}{3}a^2b - a^2c}{d}} = 0$, which will give the Equation $\frac{1}{4}p^2 + \frac{\frac{1}{2}abp + cap}{d} = \frac{\frac{1}{3}a^2b + a^2c}{d}$; from which, by Reduction, compleating the Square, &c. we ſhall find $p = \frac{a}{2d}\sqrt{\frac{4}{3}bd + 4cd + bb + 4bc + 4cc} - \frac{ab - 2ac}{2d}$; and from hence the Value of $f = \frac{1}{2}p$, is known when its Effect is a *Maximum*.

3312. For accommodating this Theorem to practice it will be moſt convenient to aſſume $b = d = 1$; then we ſhall have the Equation f or A D $= f = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + \frac{1}{2}ap + acp - \frac{1}{3}a^2 - a^2c}$; and when $f = \frac{1}{2}p$; then $p = \frac{a}{2}\sqrt{2\frac{1}{3} + 8c + 4cc} - a \times \overline{1 - 2c}$.

3313. To give an Example of the Uſe of theſe Theorems; let the Pendulum of the Clock be required to beat *Seconds* preciſely, and ſuppoſe the Weight C $= 50lb.$ and $b = d = 1lb.$ Then we have $p = n = 1440$ (2295), $c = 50$, and the Theorem in (2306) will become $\frac{\frac{1}{2}n + nc}{\frac{1}{3} + c} = a = 1444{,}8$. And by ſubſtituting theſe Values of a, b, and d, in the Equation for (f), we get $f = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + 72962p - 105061210}$.

3314. Therefore when the Accelleration is greateſt of all, we have $f = \frac{1}{2}p$; and $\frac{1}{4}p^2 + 72962p = 105061210$, or $p^2 + 291848p = 420244840$; whence we find $p = 1436$, nearly; but

but 1440 — 1436 = 4′, or the greateſt Acceleration is 4 Minutes *per* Day, by placing the Weight D at N, or making f = AN = 718 = $\frac{1}{2}p$, when a *Minimum*.

3315. Since by ſhortening the iſochronal Pendulum (p) only 4 Parts of the 1440, the Length of f = AD, or the Space through which the Weight D moved upwards is 718; it is evident that a ſmall Alteration in the Length of AC, will allow of a very large *Scale of Correction* on the Rod of the Pendulum both in regard to *retarding* as well as *accelerating* its Motion.

3316. For, to accelerate the Motion of the Pendulum one Minute *per* Day, the Center of Oſcillation moves through but 2 Parts of the 1440, or the Length of the Pendulum is then p = 1438 as we have ſhewed (3295). Therefore putting n = 1438, we ſhall have by Theorem (3306) $\frac{\frac{1}{2}bn + nc}{\frac{1}{3}b + c} = a =$ 1441,8 which is juſt 3 leſs than 1444,8, the Length for ſwinging in a Second preciſely (3313). Hence the Motion of the Center of the Weight C is to that of the Center of Oſcillation as 3 to 2. And it is univerſally $a : n :: \frac{1}{2}b + c : \frac{1}{3}b + c$; or, in the preſent Caſe, $a : n$ :: 50,5 : 50,3 :: 1444,8 : 1438 :: 392 : 391,46, as we have before obſerved (3295).

3317. From all which it appears, that the Center of Oſcillation is but 5 Parts of 1440 diſtant from C the Center of the Ball, and alſo, that when the Corrector D has its Poſition, ſuch as to produce no Acceleration, it muſt be either upon the Center of Oſcillation, or at the Axis of Suſpenſion A; or $f = \frac{1}{2}p = \frac{1}{2}a = \frac{1}{2}$AC, very nearly. But in the common Conſtruction of the Pendulum this Diſpoſition of the *Corrector* D can have no Place, becauſe it cannot be brought nearer to the Center of the Weight C than the Length of its Semidiameter, which, in a Ball or Globe of 12*lb.* only, is 2 Inches (3296); but in a Globe of 50*lb.* it is more than 3 Inches. Now this is more than the Diſtance to which the Weight D muſt be removed to accelerate

N. B. The Reader is deſired to correct the Expreſſion of the Radical Symbols in Inſtitutions 3308 and 3311, thus; $\sqrt{\frac{\frac{1}{4}p^2 + \frac{\frac{1}{2}abp + cap - \frac{1}{3}a^2b - a^2c}{d}}}$.

the Motion 1 Minute *per* Day; for by the Theorem (3313) if we put $p = 1438$, we ſhall get $f = 1331{,}5$; if then we ſay, as $1440 : 1331{,}5 :: 392 : 362{,}5$, we have the Diſtance of the Weight D from the Center C, but 29,5, or not quite 3 Inches.

3318. Therefore in order to have a Scale of a ſufficient Variation for the Poſition of the Weight D, we muſt give to the Rod A C a greater Length in Compariſon of the Length (p) of a Pendulum vibrating Seconds, and conſider the Weight C or Bob of the Pendulum as *moveable upon* it in the ſame Manner as the Weight D is. Thus let A B $= a$, be the Length of the Rod or Wire; and A C $= e$, the new Diſtance of the Weight C upon it; then will the Theorem (3306) become $\frac{\frac{1}{3}a^2b + e^2c}{\frac{1}{2}ab + ec} = p$; and therefore $\frac{1}{3}a^2b + e^2c = \frac{1}{2}abp + ecp$, whence $e^2c + ecp = \frac{1}{2}pab - \frac{1}{3}a^2b$; and $e^2 - ep = \frac{\frac{1}{2}pab - \frac{1}{3}a^2b}{c} = \frac{\frac{1}{2}pa - \frac{1}{3}a^2}{c}$, by putting $b = 1$, as before. Then compleating the Square and extracting the Root, we get $e = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + \frac{\frac{1}{2}pa - \frac{1}{3}a^2}{c}}$. And the Theorem in (3312) for A D will now become $f = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + \frac{1}{2}pa + pec - \frac{1}{3}a^2 - e^2c}$. Theſe Theorems for A D and A C afford a new Conſtruction of the Pendulum whoſe Nature and extenſive Uſe will be fully declared in the enſuing Chapter. (See the Plate of the *Univerſal Pendulum*, Fig. 1, 2.)

CHAP.

CHAP. XI.

The THEORY *and* CONSTRUCTION *of an* Univerſal PENDULUM *for meaſuring the Time of* Solar, Lunar, *and* Planetary DAYS *very correctly.*

3319. IN order to ſhew the Univerſality of the new Structure of the Pendulum premiſed in the foregoing Chapter, it will be neceſſary to conſider that the Deſign of a Clock is to meaſure conſtantly, that equal Portion of Time which is called a *Mean Day*, and is divided into 24 equal Parts or Hours. As Time in itſelf flows equally, the Clock, if it could be kept to an equable Motion, would be an adequate Meaſure of it; and this would be the greateſt Excellency and Uſe of ſuch a Machine.

3320. But this, however, has, in my Opinion, been very indifferently provided for in the governing Principle of a Clock's Motion, *viz.* a *Pendulum of iſochronal Vibrations*; and it muſt be obſerved, that after all the Precautions and Inventions for rendering this *Iſochroniſm* of Vibration permanent, yet ſuch a Diſcovery ſtill remains too obviouſly the great *Deſideratum* of Clock-work, and therefore Artiſts are obliged to have recourſe to the beſt Methods of remedying ſuch Irregularities, and of checking them even in their *Naſcent* State.

3321. How groſs and indirect the Methods of rectifying the Pendulum in general are, if compared with that of the ſecondary or moveable Weight D, invented by *Hugenius*, will be very evident on mature Conſideration. And as it is here improved, I flatter myſelf it will be allowed the moſt eaſy and extenſive that can be deſired, when it is conſidered, that the *Scale of Variation* for the Corrector D is by this Means made ſo large as to render any Clock capable of meaſuring not only the *Mean Time* or *Day*, but alſo that meaſured by the Motion of any of the cele-

ſtial Bodies, whether *Moon*, *Planet*, or *Star*; as we now proceed to ſhew.

3322. The Time that intervenes the departing of any celeſtial Body from the Meridian and its Appulſe to it again, is called the *Day* peculiar to that Luminary; but ſince none of the Planets deſcribe *circular Orbits*, but *Ellipſes*, their Motions will be variable, and the Times of their *meridional Revolutions*, or Days, will be unequal; therefore the *Mean Revolution*, in all, muſt be taken for the *Mean Planetary Day*; and theſe collected from the aſtronomical Tables are for the ſeveral heavenly Bodies, as in the Table here ſubjoined.

	H.	′	″	‴		′	″	‴	′
3323. A fixed STAR	23	56	3	28					1436
SATURN	23	56	11	00		0	8	0	$1436\frac{1}{5}$
JUPITER	23	56	23	00		0	20	0	$1436\frac{1}{3}$
MARS	23	58	08	58		2	5	30	$1438\frac{1}{7}$
SUN	24	00	00	00		3	56	32	1440
VENUS	24	02	28	00		6	24	32	$1442\frac{1}{2}$
MERCURY	24	12	25	28		16	22	0	$1452\frac{2}{5}$
MOON Leaſt	24	43	08	28		47	5	0	$1483\frac{1}{7}$
MOON Mean	24	48	43	28		52	40	0	$1488\frac{7}{10}$
MOON Greateſt	24	57	07	28		61	4	0	$1497\frac{1}{8}$

3324. The firſt Column in this Table ſhews the Hours, *Minutes*, *Seconds*, and *Thirds* in each planetary Day, the *Mean Solar Day* of 24 Hours being the Standard. In the ſecond Column are contained the *Minutes*, *Seconds*, and *Thirds*, by which each following Day exceeds the firſt or *Sidereal Day*; thus the *Mercurial Day* is longer by 16 Minutes, and 22 Seconds; and the greateſt *Lunar Day* by 61 Minutes, and 4″. In the third Column, the Quantity of each Day is expreſs in the Minutes of a *Mean Solar Day*.

3325. If therefore the Length of the iſochronal Pendulum (p) in the foregoing Equations be found for the Numbers in the 3d Column reſpectively, then will the Diſtances A D, and A C, or the Poſitions of the Weights D and C be found by the Theorems in (2318), ſo that the Clock ſhall keep Time with the Pla-

Planet, and Hour-hand point to the Hours when the Planet ſhall be on the correſpondent Hour-circles. So that by this Means any Clock may readily be adjuſted to ſhew the Time of any *Planetary Day*, or the Poſition of the Planet in any Part of its Revolution.

3326. But for this Purpoſe the Length of the Rod A B muſt be ſufficient to allow of a Scale of Poſition to the Corrector D, while the iſochronal Pendulum (p) encreaſes from its leaſt Length for the *Sidereal Day* of 1436′ to its greateſt Length for the *longeſt Lunar Day* of 1505′, in which Caſe the Weight or Bob C will poſſeſs the Extremity of the Rod, or A C $= c$ will become A B $= a$. And becauſe as the *Days* encreaſe, the *Hours*, *Minutes*, and *Seconds* increaſe in the ſame Proportion (as there is a conſtant Diviſion of each Day into the ſame Number, *viz.* 24 Parts), therefore, alſo, the Time of the Pendulum's Vibration will encreaſe with the Length of the Day. And conſequently the Numbers in the 3d Column of the Table will be as the Times of Vibration in the Pendulums appropriated to the Clocks for ſhewing the Time of the Days reſpectively.

3327. Therefore it will be neceſſary in the firſt Place to determine the Length of P the iſochronal Pendulum whoſe Time of Vibration is as 1500, by ſaying, as $1440^2 : 1500^2 :: 1440 : 1562\frac{1}{2} = P$. And becauſe the larger the Weight C is, the leſs will be the *Scale of Acceleration*, therefore we muſt take the ſaid Weight, ſuch as will allow this Scale to be ſufficiently large to comprehend all the Variation of the Moon's Motion; and by Trial, it will be found that 10*lb.* will be the greateſt Weight that can be allowed to the Bob; therefore $c = 10$; b and d being $= 1$, as before.

3328. Hence we determine A B, the Length of the Rod, by the Theorem $\frac{\frac{1}{2}P + cP}{\frac{1}{3} + c} = a = 1587{,}8$ as in (3306). Alſo by the Theorem in (2311) we find the Length of an iſochronal Pendulum, when a *Minimum*, to be $p = 1526{,}2$; therefore $1562{,}5 - 1526{,}2 = 36{,}3$; and half this Number, *viz.* 18 ſubducted from 1500 leaves 1482 for the Expreſſion of the Time of a Vibration of the Pendulum (p) compared to the Time 1500 of the Pendulum P.

3329.

3329. That the Reaſon of this Aſſertion may appear, we ſhall demonſtrate this *Lemma*, viz. *when Numbers are very large, and their Differences very ſmall, then any three or four of them that are in* geometrical Proportion *are alſo* in arithmetical Proportion. Thus let a, and $a - d$, be two large Numbers, for Inſtance 1440, and 1439, then $d = \frac{1}{1440}$ Part of a; and let it be $a : a - d : a - d : z$; then $z = \frac{a^2 - 2ad + dd}{a} = a - 2d$, becauſe (dd) vaniſhes in Compariſon of the reſt; therefore a, $a - d$, $a - 2d$, are in *geometrical Proportion*; they are alſo evidently in *arithmetical Proportion*; and hence 1440, 1439, 1438 are Numbers that have the ſame Property; for ſince $1440^2 : 1439^2 :: 1440 : 1438$, therefore $\frac{1439^2}{1440} = 1438$, and conſequently $1440 : 1439 :: 1439 : 1438$, and ſo they are geometrical and arithmetical Proportionals at the ſame Time.

3330. Since the Number 1483 anſwers to the ſhorteſt Lunar Day in the Table (3323) and the Difference between that and the longeſt Day is but about 14 Minutes; it is evident, this Scale of 18′ will ſerve for all Lunar Days. And the Values of P and f may be found for every Number betwixt 1483 and 1497 by the Theorems in (3307, 3308). And the Scale on the Rod of the Pendulum may be graduated for Uſe and the adjuſting Weight D put to its proper Place for the due Rectification of the Clock to the Length of the reſpective *Lunar* Day propoſed, which is always known from an *Ephemeris*.

3331. From the Table (3323) it appears, that the ſhorteſt *Lunar Day* 1483 exceeds the longeſt *Planetary Day* 1452; and conſequently the *Lunar Scale* on the Pendulum will be of no Uſe for the Planets. A planetary Scale therefore muſt be conſtructed that ſhall have an Extent ſufficient for the Movement of the Corrector D from the *Sidereal Day* of 1436′, to that of the Planet *Mercury* of 1452; whoſe Difference is 16 Minutes. And ſince the *Mercurial Day* of 1452 exceeds the *Solar Day* 1440 by 12′, if we allow 4′ more for the Diameter of the Weight C, we ſhall have 1456′ for the longeſt Day in the *Planetary Scale*, for which (having the Length of the Rod A B $= a = 1587{,}8$) we can find P and e, by the Theorems already premiſed.

3332. Thus *per* Theorem (3329) we have 1440 : 1456 :: 1456 : 1472 = P, the Length of the ſimple Pendulum proper for that Length of Day.

3333. Again, having given AB = 1587,8 and P = 1472, we have $e = \frac{1}{2}P \pm \sqrt{\frac{1}{4}P^2 + \frac{\frac{1}{2}Pa - \frac{1}{3}a^2}{c}} = 1494 = AE$, by the Theorem (3318). Therefore, AB — AE = EB = 94, which is a little more than an Inch, through which the Weight C is to be raiſed on the Rod of the Pendulum AB, from C to E.

3334. We are next to enquire the Length of an iſochronal Pendulum p correſponding to the greateſt Acceleration, when $f = \frac{1}{2}p$. In this Caſe we ſhall have $\frac{1}{2}p^2 + \frac{1}{2}pa + ecp = \frac{1}{3}a^2 + e^2c$, by the Theorem ſo often quoted in (3318). If we put $\frac{1}{3}a^2 + e^2c = s$, and $\frac{1}{2}a + ec = t$, we ſhall have $p = 2\sqrt{s + tt} - 2t = 1433\frac{1}{2}$, and of Courſe AN = AD = $f = \frac{1}{2}p = 716{,}7$.

3335. But the Time of Vibration of this Pendulum being a Mean proportional between 1440 and 1433,5 by the *Lemma* in (3329), that is, 1440 : x :: x : 1433,5, whence $x = \sqrt{1440 \times 1433{,}5} = 1436{,}75$, which ſubducted from 1456 the ſloweſt Vibration (3331) will leave 19′,3 which will be juſt large enough to take in the *Planetary Days* as far as they are ſenſibly different from each other: For the *Sidereal*, *Saturnian*, and *Jovian Day* differ ſo little, as not to be diſcernable in the Scale, as is evident from the Numbers in the third Column of the Table (3323).

3336. In this Scale the Length of the iſochronal Pendulum for the *Mean Day* of the SUN being 1440 = p, we ſhall find $f = AD = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + tp - s} = 929$, wherefore the Place of the Corrector D is given for regulating the Clock for common Uſe, or to ſhew *Mean Time*.

3337. In like Manner, by having the Numbers

1438 for *Mars*, you find $p = 1436$.
1442½ for *Venus* —— $p = 1445$.
1452 for *Mercury* —— $p = 1465$.

And

And from thence, by the above Theorem, the Poſition of D, or AD = *f*, will be found for each *Planet* reſpectively. And thus the *Lunar and Planetary Scales* of Acceleration are compleated; and the Clock fitted to ſhew Time *univerſally*.

N. B. For the better Illuſtration of what we have delivered concerning this new conſtructed Pendulum, we have (in Fig. 2. of the following Plate) added ſo much of the lower Part of the *Half-ſecond* Pendulum, as contains both the *Lunar and Planetary Scales*; but theſe Scales will be four Times as long in a Pendulum that beats *Seconds*.

CHAP. XII.

Concerning the BOB *or* WEIGHT *of the* PENDULUM; *and the* THEORY *of ſuch a* FORM *or* SHAPE *thereof, as ſhall meet with very ſmall* RESISTANCE *from the* AIR.

3338. IN a Treatiſe on the *Rationale* of CLOCK-WORK, I judge it will be expected that ſomething ſhould be ſaid concerning the FORM of the BOB, or WEIGHT of the *Pendulum*, with regard to the Reſiſtance of the Air in which it moves, and the conſequent Irregularity in its Motion, which will be thereby unavoidably produced; and the rather, becauſe ſo great a Judge in theſe Matters as HUGENIUS, has aſſerted it to be a very intereſting Point, *nam plurimum refert*, are his Words.

3339. But as we have ſhewn, the only Deſign of the Contrivance and Mechaniſm of a Clock is to annihilate the Reſiſtance which the Pendulum meets with from the Air, and every other Cauſe, it is plain, if it were poſſible to conſtruct a Clock with ſo much Accuracy as to do that, the Motion of the Pendulum would be as equable as if it moved without any Reſiſtance at all; for the THEORY of *Clock-work* would be a very imperfect Thing, if it could not provide againſt the Effects of Re-

Reſiſtance of every Kind, and cauſe the Clock to go with an *equable Motion*, whether the Pendulum moved *in Vacuo*, in *Air*, or in *Water*, and even in *Quicksilver* itſelf.

3340. But it may nevertheleſs be uſeful in ſome Caſes of a Pendulum, to know what *Form of the Ball* will be moſt convenient and leaſt reſiſted in its Motion through the Air, which though inſenſible in a few Vibrations, or in a ſhort Time, may yet, by a conſtant Accumulation, produce Effects too ſenſible and moleſting in long Run, maugre all the Ingenuity and Diligence of the Artiſt. This Subject therefore we ſhall now proceed to conſider more particularly.

3341. It is well known that the Reſiſtance of a Body in Motion regards the *Quantity and Figure of its Surface*, and not the Quantity of Matter moved; for a CUBIC INCH *of Matter* may be diſpoſed into the Form of a SPHERE, whoſe Diameter will then be 1,247; and its Superficies = 4,836 *Cubic Inches* (836) whereas the ſame Matter in the Form of a CUBE had a Quantity of Surface = 6 Cubic Inches. And the SPHERE has the leaſt Quantity of Surface that it can poſſibly be contained under; and therefore *has leſs Reſiſtance in Proportion to its Weight*, than any other Body, and conſequently is the *beſt Form*, for the Ball of a Pendulum in general. And becauſe the Surfaces of Spheres are proportioned to the Squares of their Diameters (842), larger Spheres will have much leſs Surface than ſmall Ones, in Proportion to their Weight, and will therefore meet with leſs Reſiſtance.

3342. The SOLID of *leaſt* RESISTANCE (whoſe Theory you will find in (2042, *&c.*) would undoubtedly merit the firſt Conſideration, were it not, that its Conſtruction will be much too difficult for Practice; and this Solid is in itſelf but little preferable to the *Fruſtum* of a *Cone* of *leaſt Reſiſtance*, which is not only made with the utmoſt Eaſe, but is at the ſame Time beſt fitted for the Purpoſes of the planetary Pendulum and Clock before deſcribed. The Theory of which we ſhall, therefore, here deliver.

3343. Let A B C (Fig. 3.) be the iſoſceles Triangle generating by its Revolution a Cone, whoſe *Fruſtum* C G F B has the leaſt Reſiſtance for a given Baſe C B, and Altitude D E. Let the *Fruſtum* be ſuppoſed to move in the *Medium* in a Direction parallel to its Axis; let S V be drawn parallel to A E, to repreſent the Direction in which the Particles of the Fluid ſtrike the

Cone. Then V I being perpendicular to the Baſe C B will repreſent the Particles ſtriking the Baſe with their whole Force, ſuppoſing the Fruſtum moved from D toward E; but in moving from E towards D, the Particles S F will come obliquely on the Side A B, and therefore cannot ſtrike any Particle F with their full Force.

3344. Let any Line L F repreſent the whole Force of the Stroke againſt the Baſe B C, and through the Point F draw M H perpendicular to the Side A B; let M F = F H; and compleat the Parallellogram A L F M; then the whole Force L F will be reſolved into two Forces L M and M F, of which one, *viz.* L M being parallel to the Side A B of the Cone, cannot affect its Motion; the other Part M F is, therefore, all the Force of the Fluid on that Point, that can cauſe any Reſiſtance.

3345. We are next to enquire what the Effect of this Force M F or F H is, in the Direction D E; to this End let F H be reſolved into two Forces F D and D H, of which the former F D is equal to, and its Effect wholly deſtroyed by the antagoniſt Force G D, ariſing from the Action of the Fluid T G on the other Side; the Force D H, being all exerted in the Direction D E, is therefore all the Force of the oblique Stroke that directly oppoſes the Motion of the Body, and conſequently that can cauſe any Reſiſtance to the Point F in the Side A B.

3346. Therefore the Effect or Reſiſtance of a Particle of the Medium at the Point F in the Surface, is to that at the Point I in the Baſe, by a direct Stroke, as D H to L F or A H, and therefore as A H × D H to A H^2. But by ſimilar Triangles H F D, H A F, it is H D : H F :: H F : H A, whence A H × D H = F H^2, therefore A H^2 is to F H^2, as the Reſiſtance at I to that at F: But A H : F H :: A B : B F; therefore the Reſiſtance at I is to that of F, as A B^2 : B E^2. And ſince what has been ſaid of the Points F and I hold equally true of all other Points in the Surface and Baſe of the Cone, *therefore the Reſiſtance to the Baſe will be to that upon the Surface of the Cone as* A B^2 *to* B E^2.

3347. Hence then the Reſiſtance to the Surface F A G of the Cone cut off, is to that againſt its Baſe F G, as F D^2 to A F^2; therefore B E^2 — F D^2 is as the Reſiſtance to the Surface of the *Fruſtum*; and by adding A F^2, (which is as the Reſiſtance upon

the Baſe or End F G) we have $BE^2 + AF^2 - FD^2$ for the Expreſſion of the Reſiſtance to the whole *Fruſtum*, when moving in the Direction E A. But $AF^2 - FD^2 = AD^2 = AE^2 - 2AE \times ED + ED^2$ (637). Therefore the Reſiſtance of the Fruſtum will be $BE^2 + AE^2 - 2AE \times ED + ED^2 = AB^2 - 2AE \times ED + ED^2$.

3348. But ſince AB^2 expreſſes the Reſiſtance to the Baſe BC (3346) and is a *conſtant Quantity*, it may be repreſented by Unity, or $AB^2 = 1$; and as any Quantity may be conſidered as divided by Unity, the Reſiſtance of the *Fruſtum* may be alſo thus expreſſed $\frac{AB^2 - 2AE \times ED + ED^2}{AB^2}$, or thus, $1 + \frac{DE^2 - 2AE \times DE}{AB^2}$.

3349. Therefore putting $DE = a$, $BE = b$, and $AE = x$, we have the laſt Expreſſion for the Reſiſtance in Symbols thus $1 + \frac{a^2 - 2ax}{b^2 + x^2}$, whoſe Fluxion (when a *Maximum* or *Minimum*) is $\frac{2ax^2\dot{x} - 2a^2x\dot{x} - 2ab^2\dot{x}}{\overline{xx + bb}^2} = 0$; whence we have $x^2 - ax - bb = 0$; and by compleating the Square and extracting the Root, we have $x = \frac{1}{2}a \pm \sqrt{\frac{a^2}{4} + b^2}$. But x being the Height of the whole Cone muſt be greater than a, which is but a Part of that Height, therefore $x = \frac{1}{2}a + \sqrt{\frac{a^2}{4} + bb}$.

3350. Hence we have the following eaſy Conſtruction; biſect DE in K, draw KC; and produce ED to A, ſo that KA may be equal to KC. Then will ABC be a Cone, whoſe *Fruſtum* BFGC ſhall meet with leſs Reſiſtance, in an uniform Fluid, than any other of the ſame Baſe and Altitude. For becauſe $AK = KC = x - \frac{1}{2}a$, and $KE = \frac{1}{2}a$, and $EC = b$, we have $KC^2 = \frac{1}{4}a^2 + bb = x^2 - ax + \frac{1}{4}a^2$, which gives $x^2 - ax - bb = 0$, as required in the Property of ſuch a *Fruſtum* (3349).

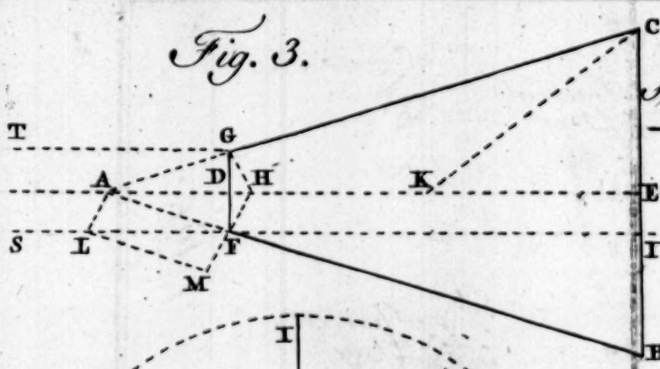
3351. In order to be certain that this Reſiſtance is a *Minimum*, and not a *Maximum*, we are to conſider this Principle, *that while a Quantity is increaſing, its Fluxion will be poſitive; but negative while it is decreaſing.* Now the Quantity $xx - ax - bb = 0$, is analogous to the Fluxion from whence it proceeds, and therefore when $x = a$, this Quantity becomes $-bb$, which is plainly negative.

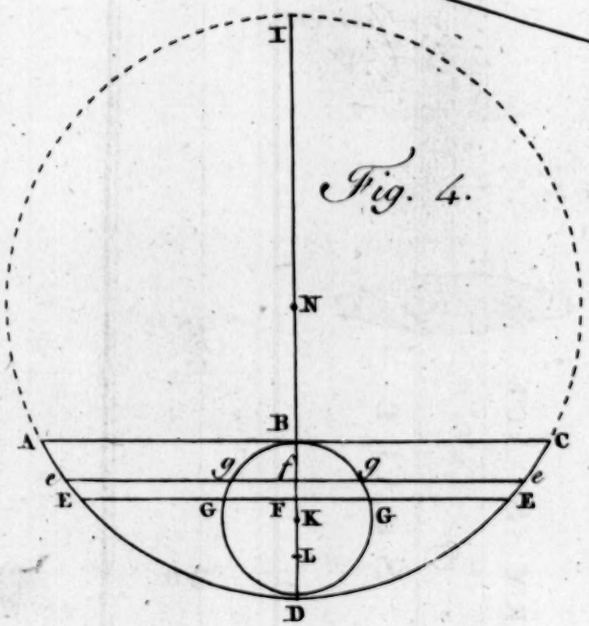
3352. Since then the Reſiſtance decreaſes from the Value of $x = a$, to $x = \frac{1}{2}a + \sqrt{\frac{a^2}{4} + bb}$, where it is a *Minimum*, it is plain that the *Fruſtum* B C G F is leſs reſiſted than a *Cone* of equal Baſe and Height, or than any other *Fruſtum* of a larger Cone, whoſe Baſe and Height are the ſame with theſe, *viz.* C B and D E.

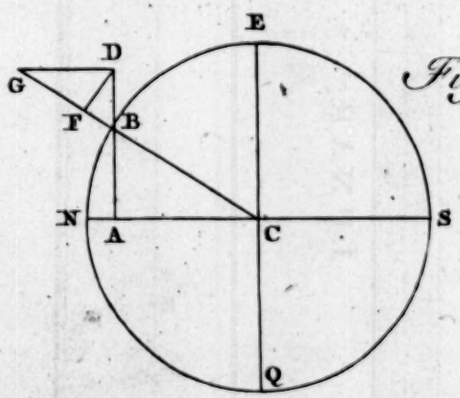
3353. The above Demonſtrations will hold for any Proportion of C E to D E; and the leſs C E is in regard to D E, the leſs will A D be in reſpect of A E; ſo that at Length the Fruſtum of a Cone, of leaſt Reſiſtance, will come near to the Form of the *Solid of leaſt Reſiſtance*, and ſerve almoſt equally as well for the Purpoſe of the *Bob* of a Pendulum, and is excellently well fitted for that of the *planetary Pendulum* deſcribed in the preceeding Chapter, which conſiſts of two ſuch Fruſtums, whoſe Center is that of their common Baſe. And further, as ſuch a *double Fruſtum* of a Cone is leſs reſiſted than a Globe of the ſame Weight, ſo the middle Segment or Proportion of ſuch a *Fruſtum* cut longitudinally, will be better and more artificial, than a Bob of the common *lenticular* Form. ——— It is ſomewhat remarkable, that the *French* and *Spaniſh* ACADEMICIANS in the Department to PERU, for meaſuring the Length of a Degree of the Meridian at the Equator, had in their Clock a Pendulum of *two truncated Cones*, but placed juſt in a contrary Direction to that which the Theory above requires, and is here ſhewn in Fig. 2.

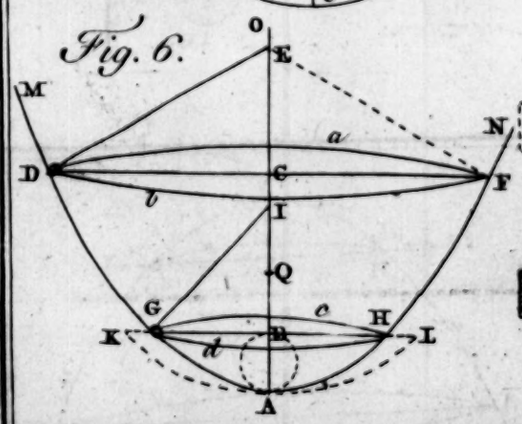
CHAP.

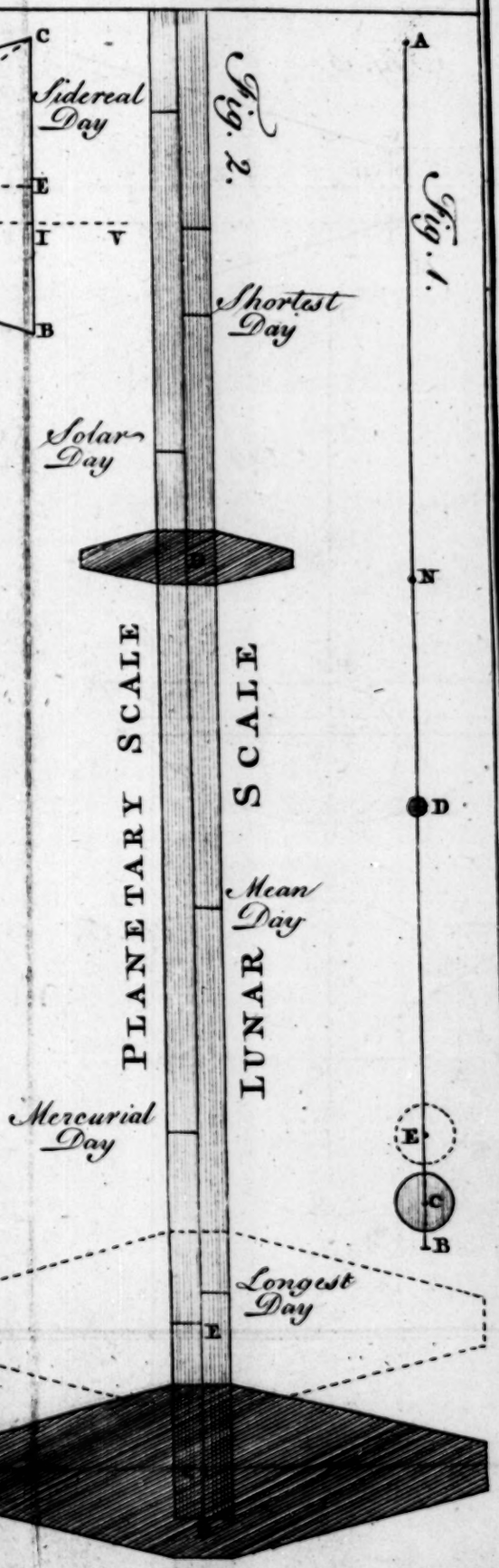
The Universal PENDULUM.











The THEORY of the Circulating PENDULUM, Oscillating ROTULA &c.

Fig: 1.

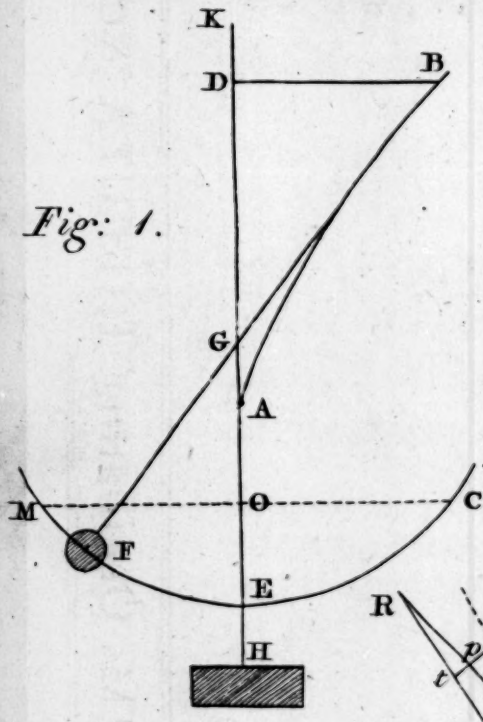


Fig: 2.

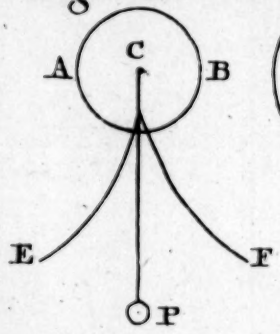


Fig: 3.

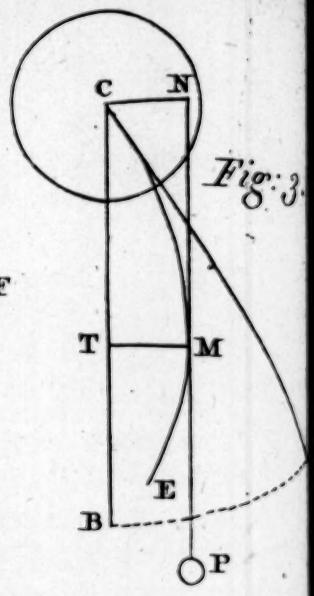


Fig: 4.

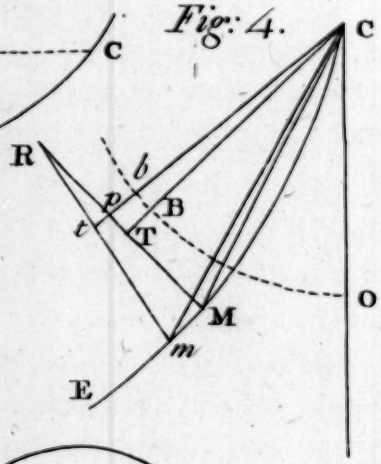


Fig: 5.

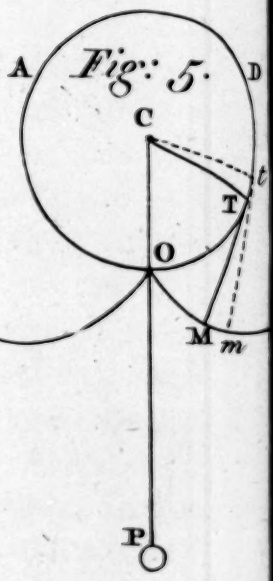


Fig: 6.

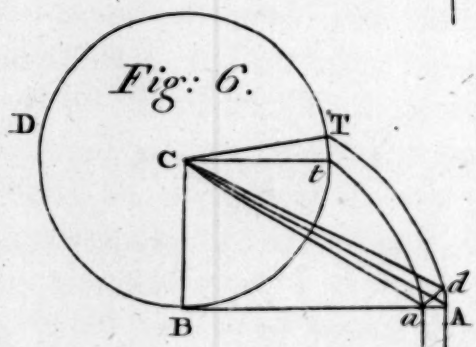


Fig: 7.

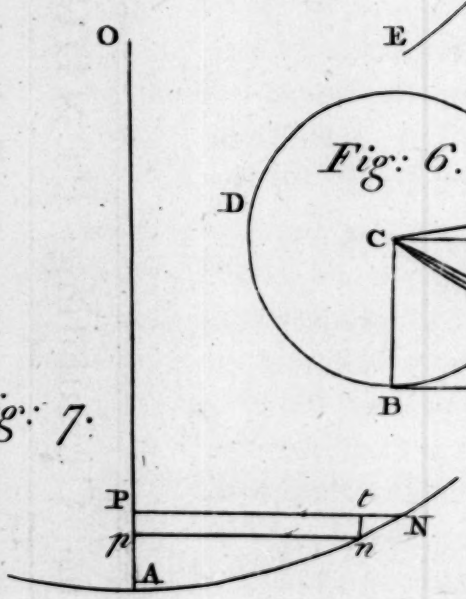
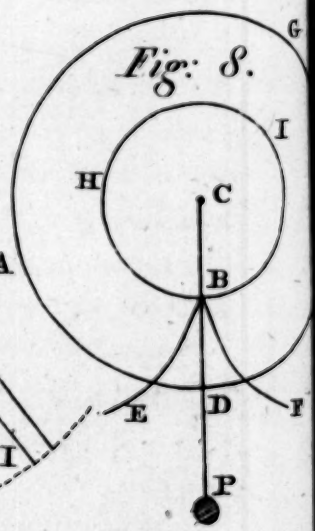


Fig: 8.



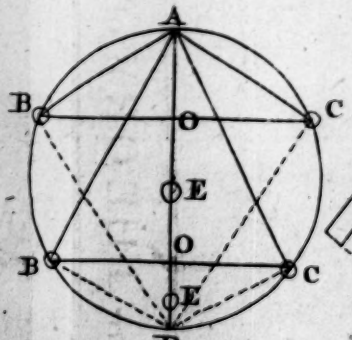


Fig: 10.

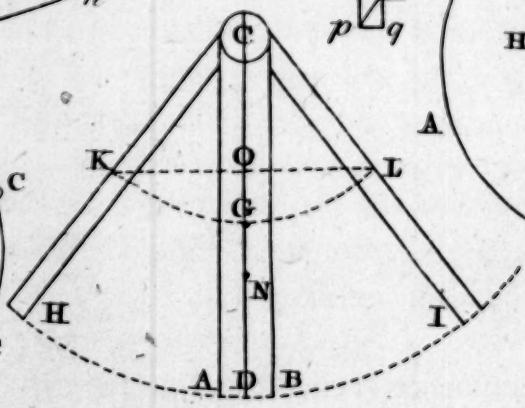


Fig: 9.

CHAP. XIII.

The THEORY *for finding the* TIME *of a* leaſt OSCILLATION *of a given* PENDULUM *ſwinging in the* ARCH *of a* CIRCLE; *and alſo the Time of any other: with proper* CANONS *for* CALCULATION.

3354. WE have formerly ſhewn (1119) that the *Time* of the *leaſt Vibration* of a given Pendulum is equal to the Time ſuch a Pendulum would take to vibrate in the Arch of a *Cycloid*. Alſo, that when the Arch of a Circle, in which ſuch a Pendulum vibrates, is ſo ſmall, that it differs not ſenſibly from the Cycloid, the Time of a Vibration will not be affected thereby; and this Conſideration makes *cycloidal Cheeks* unneceſſary in long Pendulums, or ſuch as ſwing *Seconds*, where the Arch of Vibration is not more than 3 or 4 Degrees of a Circle.

3355. But in *Table-clocks*, whoſe Pendulum is not more than about 6 or 8 Inches long, the Caſe is different; for here the Space can be but ſmall, in which there can be any ſenſible Coincidence of the *Circle and Cycloid*, and yet the Arch of Vibration in theſe ſhort Pendulums is generally much longer in Proportion to their Lengths than in the long Ones; whereas they ought to be proportionably ſhorter. Theſe Clocks muſt therefore be ſubject to an Error that cannot be avoided but by *cycloidal Plates*, ſuch as *Hugenius* invented and preſcribes for this Purpoſe; the Nature of which ſee already largely explained (1120), (or by any of the *new Species of Pendulums* hereafter to be deſcribed) for the common Method of moving the Bob up and down upon the Rod, by a Screw at the Bottom, is very fallacious and inartificial, and can never procure an equable Motion to the Clock, unleſs the Arch of Vibration be very ſmall indeed.

3356. When the circular Arch deſcribed by the Pendulum begins to deviate from the *Cycloid*, the Times of Vibration will begin to encreaſe or exceed the Time of a *leaſt Vibration*, and therefore it will be neceſſary to ſhew how this Error ariſes, and in what Proportion it encreaſes, that the ingenious *Tyro*, in Clock-

work,

work, may not be left in the Dark in that Part of the Theory which is moſt eſſential to his Art, or conducive to the Truth of his Work. And this we ſhall be able to do in the plaineſt Manner, by following the Footſteps of the late learned *Profeſſor* SAUNDERSON.

3357. The general Problem is, *to find (independent of the Cycloid) the Time preciſely of a leaſt Oſcillation of a Pendulum of a given Length ſwinging in an Arch of a Circle; and alſo to find, without any ſenſible Error, the Time of any other.* Let the Curve A D C be the Arch of a Circle, whoſe Diameter is D I, and in which the Pendulum N D is ſuppoſed to vibrate. Then in deſcending from C to any Point E, it will acquire a Velocity which will be as $\sqrt{E M} = \sqrt{B F}$ (having drawn the Chords A C, E E interſecting the Diameter I D in B and F). Now $\frac{1}{2}\sqrt{F D}$ is as the Velocity acquired in deſcending through $\frac{1}{4}$ I D, and by that Velocity it would deſcribe uniformly a Space equal to $\frac{1}{2}$ I D in the Time of the ſaid Deſcent through $\frac{1}{4}$ I D (993). Then the Space $\frac{1}{2}$ I D divided by the Velocity $\frac{1}{2}\sqrt{I D}$ will give the Time $\sqrt{I D}$ of the Deſcent through $\frac{1}{4}$ I D, *i. e.* through *half the Length of the Pendulum* (971.)

3358. Draw *e e* very near to E E; then may E *e* be conſidered as the Fluxion of the Arch D E; and $\frac{E e}{\sqrt{B F}}$ will be the Time wherein the ſmall Arch E *e* is deſcribed by the Pendulum, or the *Fluxion of the Time of a Vibration.* But $E e = \frac{\frac{1}{2} I D \times F f}{\sqrt{I F \times F D}}$ $= \frac{\sqrt{I D}}{\sqrt{I F}} \times \sqrt{I D} \times \frac{1}{2} \frac{F f}{\sqrt{F D}}$; therefore we have the Fluxion of the Time $\frac{E e}{\sqrt{B F}} = \frac{\sqrt{I D}}{\sqrt{I F}} \times \sqrt{I D} \times \frac{1}{2} \frac{F f}{\sqrt{B F \times F D}}$.

3359. Biſect B D in K, and K D in L; and when the Arch A D C is ſmall, the Quantity I F cannot differ ſenſibly from I K,

IK, nor $\frac{\sqrt{ID}}{\sqrt{IF}}$ from $\frac{IL}{IK} = \frac{IL}{IF}$ (becauſe ID : IL :: IL : IK or IF, therefore ID : IF :: IL^2 : IK^2 (by 672). Therefore $\frac{Ee}{\sqrt{BF}}$ is very nearly equal to $\frac{IL}{IK} \times \sqrt{ID} \times \frac{\frac{1}{2}Ff}{\sqrt{BF \times FD}}$.

3360. Upon the Diameter BD deſcribe a Circle cutting the Chords EE and *ee* in G and *g* reſpectively, then will the Fluxion of the Arch DG be $G = \frac{\frac{1}{2}BD \times Ff}{\sqrt{FB \times FD}}$ (875) conſequently $\frac{Gg}{BD} = \frac{\frac{1}{2}Ff}{\sqrt{FB \times FD}}$; and therefore the Fluxion of the Time of Vibration through DE will be $\frac{Ee}{\sqrt{BF}} = \frac{IL}{IK} \times \sqrt{ID} \times \frac{Gg}{BD}$.

3361. But the Fluent of this laſt Fluxion is $\frac{IL}{IK} \times \sqrt{ID} \times \frac{DGB}{BD}$, which is therefore the Time of *half a Vibration* or Motion of the Pendulum from D to C. The Time of a whole Vibration therefore through ADC will be $\frac{IL}{IK} \times \sqrt{ID} \times \frac{BGDGB}{BD}$.

3362. Let the Arch ADC be indefinitely ſmall, then the Quantity $\frac{IL}{IK} = 1$, and the Time (T) of a leaſt Vibration will become $T = \sqrt{ID} \times \frac{BGDGB}{BD}$. And therefore BD : BGDGB :: $\sqrt{ID}$: T; that is, *as the Diameter is to the Circumference of a Circle, ſo is the Time* ($\sqrt{ID}$) *of the Deſcent through half the Length of the Pendulum to the Time of a leaſt Oſcillation of that Pendulum* which is the very ſame Analogy as we found from the

the *Cycloid* (1124). Wherefore the Time of Oſcillation in a *Cycloid*, and in an *indefinitely ſmall Arch of a Circle* is the ſame; *viz.* T = 1 *Second*, when the Length of the Pendulum is 39,2 Inches.

3363. Therefore the Time of an Oſcillation or Vibration in a circular Arch in general, is $T \times \frac{IL}{IK}$, or becauſe IL = IK + KL, therefore the general Expreſſion of the Time of Vibration through any Arch A D C will be $T + T \times \frac{KL}{IK}$.

3364. But the Exceſs $T \times \frac{KL}{IK}$ of the Time of Vibration in a circular Arch above the Time in the Arch of a *Cycloid*, or the *leaſt of all*, is next to be transformed and fitted for Computation. In order to this put the *verſed Sine* of the Arch of Vibration BD = x; then will $KL = \frac{1}{4}x$ (3359) $2L - \frac{1}{2}x = IK$ (L = ND, being the Length of the Pendulum). And $T \times \frac{KL}{IK} = \frac{\frac{1}{4}Tx}{2L - \frac{1}{2}x} = \frac{Tx}{8L - 2x}$, the Exceſs or Error above T, the Time of Oſcillation in the *Cycloid*.

3365, Let the Time be expreſſed in *Seconds*; and let S = 3600″ the Seconds in an *Hour*, or 86400″ the Seconds in a Day or 24 Hours. Then $\frac{TSx}{8L - 2x} = r$, the Seconds loſt in an Hour or a Day, by the Pendulum vibrating in a *circular Arch* of a given verſed Sine x. For it is plain, that as $\frac{Tx}{8L - 2x}$ is the Increaſe of T the Time of Vibration in a Cycloid, ſo $\frac{TSx}{8L - 2x}$ will repreſent the Dimunition of the Number of Seconds S in a Day or an Hour; or $S - \frac{TSx}{8L - 2x}$ = Number of Vibrations in the ſame Time in the *Arch of a Circle*.

3366. If the Quantity (r) be given, we have the verſed Sine of the Arch of Vibration $x = \frac{8Lr}{TS - 2r}$. But in all theſe Theorems, the Value of x is found in Parts of L; and ſo muſt be reduced to the *tabular verſed Sine* by this Analogy as L : x :: R :

R : $\frac{Rx}{L}$; therefore putting B = 1, we have $\frac{Rx}{L} = \frac{\frac{1}{2}TS - r}{4r}$ = tabular verſed Sine of the Angle required.

3367. That the young *Automatiſt* may ſee the Uſe of this Doctrine, I ſhall illuſtrate this Theorem by an Example. Therefore let it be required to find how many Vibrations are made in a *circular Arch* of 120 Degrees, by a Pendulum whoſe Length L = 39,2, and which vibrates 86400 Times in one Day, in the Arch of a Cycloid. Here L = 39,2, T = 1″, S = 86400″, and $x = \frac{1}{2}L = 19,6$, being the verſed Sine of 60 Degrees, then $\frac{TSx}{8L - 2x} = 6278''$, which deducted from 86400, leaves 80122, the Number of Swings required. Whence ſuch a Clock will looſe daily 6278″ = 1 Hour : 44′ : 24″.

3368. Again let it be required to find in what Arch of a Circle a *ſecond Pendulum* muſt vibrate to looſe one Minue, or 60″ *per* Day. Then r = 60″, T = 1″, and $\frac{1}{2}TS = 43200$. And therefore $\frac{4r}{\frac{1}{2}TS - r} = 0,0028367$, the tabular *verſed Sine* of 4° : 19′; the Arch of Vibration therefore is 8° : 38′ required. In like Manner it is found, that the *verſed Sine* of half the Arch in which ſuch a Pendulum will looſe but *one Second per* Day, or in 24 Hours, is 0,0000896, correſponding to 46′ of a Degree, the Arch therefore is 1° : 32′; which is equal to *one Inch* in Length.

3369. Hence it appears, that in *Second-pendulum* Clocks, if the Arch of Vibration does not exceed two or three Inches, the Error will be very ſmall, and may be eaſily corrected, either in the ſimple or compound Form, by the common Methods. But it is otherwiſe with *Table-clocks*, whoſe Pendulums are ſhort, and the Arch of Vibration long. Thus for Example, let the Pendulum of ſuch a Clock be 9,8 Inches, to ſwing *Half-ſeconds*. And let $z = \frac{4r}{\frac{1}{2}TS - r}$, then $\frac{\frac{1}{2}TSz}{4 + z} = r$, the Error of the Clock in $\frac{1}{2}$ Seconds *per* Day. Therefore T = 1, S = 172800, and z = 0,060307, the tabular verſed Sine of 20 Degrees; and $\frac{\frac{1}{2}TSz}{4 + z} = 1283$ Half-ſeconds, or 10′ : 40″ is the

Error of ſuch a Clock *per Day*; which is ſtill much greater when the Pendulum is but 6 or 7 Inches long, as in moſt of thoſe Clocks, and therefore no great Exactneſs in them can be expected.

N. B. The Arch of Vibration in Table-clocks is conſiderably more than 20 Degrees, and of Courſe the Error greater on that Account.

CHAP. XIV.

The phyſical THEORY *of the* FIGURE *of the* EARTH; *the ſame demonſtrated to be a* SPHEROID; *and from thence a* SOURCE *of* ERROR *in the* MOTION *of* PENDULUMS.

3370. BEſides the Oſcillation of Pendulums in *circular Arches* inſtead of *cycloidal Ones*, there are other Sources of Error, which are unavoidable, as they are founded in the very Conſtitution of the Earth itſelf, and its Appendages. That which ariſes from the Earth itſelf is *the different Force of Gravity on different Parts of the Earth's Surface*. For ſince the Velocity in a given Time is always as the accelerating Force (998); therefore, in ſuch Latitudes, where the Power of Gravity is ſtrongeſt, there the Velocity of the Pendulum will be quickeſt, and *vice verſa*; conſequently the ſame Length of a Pendulum will not perform its Vibrations in the ſame Time in different Regions of the Earth.

3371. This variable Power of Gravity ariſes from two Cauſes: (1.) The ſpheroidical Form of the Earth, by which Means a Body on different Parts of the Surface is not equidiſtant from the Center. (2.) The *centrifugal Force* ariſing from the Rotation of the Earth upon it Axis, by which the Force of Gravity is very unequally diminiſhed from the Equator to the Poles. A Pendulum, therefore, upon different Parts of the Earth will be variouſly

ouſly influenced by this Power, and conſequently can never vibrate equally but in the ſame Place.

3372. We ſhall lay open theſe Sources of Error in the Motion of Pendulums by tracing them from their firſt Principles, and thereby convince the young *Automatiſt* that they neceſſarily ariſe from the natural Conſtitution and Diſpoſition of the Earth, taking it for granted, that the Earth in its firſt Formation, and at the Commencement of its Motion, was in a fluid State, or at leaſt ſo far, that its Parts could yield to the Force impreſſed upon them by that Motion, according to the *Laws of Nature* (964, *&c.*)

3373. Therefore let N S (Fig. 5.) be the Earth's Axis, and E Q the Diameter of the Equator, in the Earth conſidered as an *Ellipſoid.* And let B C be a Column of fluid or yielding Particles gravitating to the Center C, which (becauſe of an *Equilibrium* between all the Parts) muſt be of equal Weight with any other Column of Particles C N or C E. Put C E $= a$, C N $= b$, C B $= x$, and A B $= s =$ Sine of the Angle B C N; and laſtly, let the Power of Gravity (g) be every where as the (n) Power of the Diſtance from the Center C.

3374. Therefore the Gravity at E will be to that at B, as CE^n to CB^n; and therefore $\frac{g \times CB^n}{CE^n} = \frac{g x^n}{a^n} =$ the Power of Gravity at B.

3375. Again, the centrifugal Force (f) at E is to that at B, as C E to A B (1177). But A B : C B :: s : 1 = Radius, whence A B $= s \times$ C B $= sx$; therefore C E : A B :: $a : sx$ :: $f : \frac{fsx}{a} =$ the *centrifugal Force* at B.

3376. But ſince this Force acts in the Direction A D, let its abſolute Quantity at B be denoted by B D $= \frac{fsx}{a}$; this is reſolvable into two other Forces F D and F B, of which the latter is all that Part which *oppoſes Gravity* in the Direction B C; whence, ſince B D : B F :: 1 : s :: $\frac{fsx}{a} : \frac{fs^2x}{a} =$ the centrifugal Force at B, by which Gravity is there diminiſhed.

3377. Hence the Power of Gravity on a Particle of Matter ($\dot{x}$) at B will be $\frac{g x^n}{a^n} - \frac{f s^2 x}{a}$, impelling it towards the Center C; therefore its *Momentum* or Weight will be $\frac{g x^n \dot{x}}{a^n} - \frac{f s^2 x \dot{x}}{a}$ (1072); but this is the Fluxion of the Weight of the whole Column of Particles B C, and therefore the Fluent $\frac{g x^{n+1}}{n+1 \times a^n} - \frac{f s^2 x^2}{2a}$ will be the Weight of the said Column of Particles B C.

3378. But this is likewise the Weight of every other Column of Particles (3373), and therefore of the Column C E; and because in this Case the Angle N C E is a right One, we have $x = a$, and $s = 1$; wherefore the Weight of the Column C E will be $\frac{g a^{n+1}}{n+1 \times a^n} - \frac{fa}{2} = \frac{2g - nf - f}{2 \times n+1} \times a = \frac{g x^{n+1}}{n+1 \times a^n} - \frac{f s^2 x^2}{2a}$; this Equation by proper Reduction, and putting $s = 0$, will give $\overline{2g - nf - f} \times a^{n+1} = 2g x^{n+1}$; and thence we get $a : x :: 2g^{\frac{1}{n+1}} : \overline{2g - nf - f}^{\frac{1}{n+1}} ::$ C E : C N :: *so is the* DIAMETER *of the* EQUATOR *to the Earth's* AXIS.

3379. If Gravity be supposed *uniform*, then $n = 0$, and then $2g : 2g - f ::$ C E : C N. But $g : f :: 294 : 1$, (1198); therefore C E : C N :: 588 : 587, in such a Case.

3380. If Gravity were *proportional to the Distance from the Center* C, then $n = 1$, and we should have $g^{\frac{1}{2}} : g - f^{\frac{1}{2}} ::$ C E : C N.

3381. But in the present Constitution of Nature, it is well known that $n = -2$ (1230); therefore $2g^{\frac{1}{-1}} : \overline{2g + f}^{\frac{1}{-1}} :: 2g + f : 2g :: 589 : 588 ::$ C E : C N.

3382. What has been hitherto said, shews the Figure of the Earth is not *spherical*, but must be that of a SPHEROID, flatted at the Poles. And when it is said, Gravity is inversely as the Square of the Distance from the Center (or as x^{-2}) it is to be understood that (x) is the Distance from the *Center of Curvature*, which is the true *Center of Attraction* to a Particle on the Surface

of

of the Spheriod, and not the *Center of the Spheriod* itſelf, as is the Caſe of a Globe or Sphere.

3383. Now the *Radius of Curvature* at the Equator E is $\frac{2\,CN}{CE^2}$, and at the Pole N is $\frac{2\,CE}{CN^2}$ (933,935.) but the inverſe Ratio of the Squares of theſe Quantities is CN^6 to CE^6; and therefore Gravity at E and N is not inverſely as the *Square*, as the ſixth Power of CE and CN.

3384. Let D = Length of a Degree at the *Pole*, and d = a Degree at the *Equator*; now it is plain thoſe Degrees will be as the Radii (R, r,) of Curvature in thoſe Places, *viz.* $D : d :: R : r :: \frac{2CE}{CN^2} : \frac{2\,CN}{CE^2} :: CE^3\ CN^3$. Conſequently $\sqrt[3]{D} : \sqrt[3]{d} :: CE : CN$.

3385. Therefore if a Degree could be accurately meaſured at the Equator E, and at the Pole N, the Ratio of the *Diameter* of the Earth at the Equator to that of its *Axis* would be known. Indeed, if two Degrees be meaſured in any two diſtant Parts, that Ratio is equally known from them. But a Degree has been meaſured at the Equator, and found to contain $56767\frac{3}{4}$ Toiſes or *French* Fathoms.* Alſo, a Degree has been meaſured under the arctic Circle, and was found to be 57438; laſtly, a Degree in Latitude 45 was meaſured, and found 57050 Toiſes. From any two of theſe, the Ratio of CE to CN, may be found; and from a Mean of them all, it appears that $CE : CN :: 266 : 264\frac{7}{10}$.

3386. From a View of this Theory, it is plain, that there is a Diminution of Gravity ariſing from two Cauſes, *viz.* the *centrifugal Force*, and the *ſpheroidical Figure* of the Earth. Let CB be produced to G, ſo that BG may repreſent the centrifugal Force at the Equator, and BD that in the Latitude B. And draw GD parallel to CN, then it is $GB : BD :: CE : AB$. And $BD : BF :: (CB : AB ::)\ r : s$. Whence $BD = \frac{GB \times AB}{CE} = \frac{BF \times r}{s}$; whence $GB : BF :: r \times CE : AB \times s$ :: ſo is the *whole Diminution of Gravity under the Equator at* E *to the Diminution at the Latitude* B.

3387.

* See my new PRINCIPLES of GEOGRAPHY and NAVIGATION.

3387. As the *Diminution of Gravity* decreaſes from the Equator to the Pole, ſo there muſt be an *Increment of Gravity* conſtantly attending the ſame, which will be repreſented by BG — BF = GF, becauſe GB is the whole Diminution of Gravity at the Equator E. But becauſe BG : GD :: GD : GF (660) we have BG : GF BG^2 : GD^2 (672) :: BC^2 : AC^2; that is, *the Decrement of Gravity at the Equator is to the Increment thereof at the Latitude* B, *as the Square of Radius to the Square of the Sine of the Latitude.*

3388. And therefore, becauſe BG and BC, are conſtant Quantities, the Increment of Gravity will ever be proportioned to the *Square of the Sine of the Latitude in the Sphere*; and in the *Spheroid* of the Earth, it will be very nearly the ſame, as AC will differ inſenſibly from the Sine of the Latitude. Therefore ſince by Menſuration, it appears that CE is to CN, as 266 to 264,7 (3385) which is very near the Ratio of 230 to 229, as determined by Sir ISAAC NEWTON; it will follow, that the Ratio of the centrifugal Force at the Equator is to Gravity as 1 to 230, or that the Decrement of Gravity at the Equator is $\frac{1}{230}$ Part of the Whole.

3389. Therefore if we put this Diminution of Gravity = 10000; then will the Gravities at the *Equator* E, *London* B, and the *Pole* N, be as 2290000, 2296124, and 2300000. And conſequently the Lengths of Pendulums, vibrating in *equal Times* at thoſe Places, will be reſpectively proportional to thoſe Numbers. For the Times of Vibration in Pendulums are in a given Ratio to the Times of Deſcent through half their Length (1124) and, of Courſe, to the Times of Deſcent through their whole Length. But if a Body deſcends through different Spaces in the *ſame Time*, the Forces impelling it will be as thoſe Spaces (999); conſequently, *the Lengths of Pendulums will increaſe with the accelerating Forces of Gravity* from the Equator to the Pole, for the Times of Oſcillation to be *equal.*

3390. For Example; it will be 2296124 : 2290000 :: 39,2 :: 39,1 = the Length of a Pendulum vibrating *Seconds* at the Equator; again, 2290000 : 2300000 :: 39,1 : 39,27 = the Length of a ſecond Pendulum at the Poles. And this Proportion in the increaſing Lengths of Pendulums has been verified by Experiments made in all Parts *from the Equator to the polar Circle.*

3391.

3391. Mr. MAUPERTUIS (in his TREATISE of the FIGURE of the EARTH) has given us a Table of the Length of an iſochronal Pendulum for every 5th Degree of Latitude, as deduced from their Meaſures of a Degree in different Latitudes, and from a great Number of Experiments on Pendulums in all Parts from the *Equator* to the *polar Circle*; theſe I have reduced to *Engliſh Meaſure* for the Benefit of ſuch who do not underſtand thoſe of the *French*. If the Pendulum, which, at the Equator, is exactly adjuſted by the Motion of a fixed Star, be carried to any other Latitude, it will be accelerated, or gain upon the mean Time (3389). And the Quantity of this Acceleration upon one Revolution of the Stars, is here alſo ſhewn to every 5th Degree of Latitude.

3392. TABLE

Of the ACCELERATIONS *of a* CLOCK, *and of the Lengthenings of a* PENDULUM *from the Equator to the Pole.*

LATITUDE of the Place.	ACCELERATION in one Revolution of the fixed Stars.	Length of the PENDULUM.
0°	0″	39,0754
5	1,6	39,0768
10	6,4	39,0811
15	14,3	39,0873
20	24,9	39,0980
25	38,1	39,1097
30	53,3	39,1236
35	70,2	39,1389
40	88,1	39,1551
45	106,6	39,1718
50	125,1	39,1903
51 : 30		39,2
55	143,1	39,2049
60	159,9	39,2202
65	175,1	39,2338
70	188,3	39,2459
75	198,9	39,2554
80	206,8	39,2626
85	211,6	39,2669
90	213,2	39,2684

CHAP. XV.

The EFFECTS *of* HEAT *and* COLD *in varying the* DIMENSIONS *of* BODIES; *and thereby proving another natural* SOURCE *of Error in* PENDULUMS.

3393. THE other *natural Source* or Cauſe of Error in the Time of a Pendulum's Vibration, is the *different Temperature of the Air or Climate*; for it is found by Experience that all Bodies are ſubject to be *expanded* by *Heat*, and contracted by *Cold*, in all their Dimenſions. And therefore if the Pendulum by theſe natural Cauſes be conſtantly varying its Length, the Time of Vibration will be always inconſtant; and ſo, on this Account, can never (without a proper Rectification) be an equable Meaſure of Time.

3394. For ſince the Lengths of Pendulums are inverſely as the Squares of the Numbers of Vibrations in a given Time, (a *Day*, for Inſtance), if we expreſs the Length of a *Second-pendulum* in 100ths of an Inch, it will be 3920; and then ſuppoſe it contracted by Cold $\frac{1}{100}$ of an Inch, its Length will be 3919; and ſince in one Day there are 86400″; we ſhall have, as $\sqrt{3919} : \sqrt{3920} :: 86400'' : 86411''$. Whence it appears, that a Clock will gain or loſe every Day 11 Seconds for every 100th of an Inch it is contracted or lenthened by Cold and Heat. (See Table 3392).

3395. This is by far the moſt general Cauſe of Error in Time-pieces; for the different Power of Gravity affects the Pendulum in this Reſpect only in *different Latitudes or Climates*, but this Effect of *Heat and Cold* equally affects it in every Region, and in any particular Place. And as ſuch minute Variations in the Length of Pendulums are productive of ſuch ſenſible Errors in the Account of Time, they ought to be at all Times accurately known that the Clock might be properly rectified to every variable Degree of Heat and Cold.

3396.

3396. Hence it appears, that a proper Regulation of *Time-Pieces* with Pendulums is closely connected with the Nature of THERMOMETERS, whose sole Use is to indicate, at all Times, the comparative Degrees of Heat and Cold by a proper Scale of Variation. And if by Experiments we can find how much a given Rod of *Iron* or *Brass* extends or contracts with given Differences of Heat and Cold, denoted by the *Thermometer*, then such an Instrument being fixed by the Clock, will always shew what Degree of Correction is necessary at all Times of the Year for keeping the Clock to true Time.

3397. Now by many Experiments made on purpose to determine this Affair,* it appears, that a flat Rod of Brass $\frac{1}{2}$ an Inch wide, and $\frac{1}{12}$ of an Inch thick, and in Length $38\frac{1}{2}$ Inches, will expand or extend in Length just $\frac{1}{50}$ of an Inch with such a Difference of Heat as will cause the *Mercury* in *Farenheit*'s Thermometer to move through $22\frac{1}{2}$ Divisions; therefore $\frac{1}{100}$ of an Inch will correspond to $11\frac{1}{4}$ Divisions; but the Extension of $\frac{1}{100}$ of an Inch will occasion the Clock to lose $11''$ *per* Day. (3394.) Consequently each Division of *Farenheit*'s Termometer corresponds to an Extension of $\frac{1}{1000}$ Part of an Inch, in the Rod of the Pendulum, and to *one Second of Time* which the Clock, by that Means, loses *per Day*.

3398. Therefore, if the Clock be adjusted to equal Time, when the *Mercury* stands at 55, (the *Temperate* Degree) in the Thermometer; it will be seen in the said Instrument for any other Time, how much the Clock has lost or gained upon the equal Time, by the Height of the *Mercury* at that Time. For Instance; suppose the *Mercury* stands in a warm Summer's Day at 70, that is 15 Divisions above Temperate; therefore the Pendulum is lengthened $\frac{15}{1000}$ of an Inch, and consequently the Clock will lose $15''$ *per Day*, if not properly rectified. On the other Hand, if the *Mercury* stands at 40, (as far below *Temperate*) then it will shew the Pendulum is contracted as much, and the Clock now gains $15''$ *per Day*.

3399. Hence it appears that as the greatest Summer Heat in *England* never raises the *Mercury* higher than about 80 De-

* See a Treatise entituled, OBSERVATIONS *Astronomiques & Physiques* par G. JUAN & ANT. de ULLOA. Page 86.

grees, our *Second Pendulum Clocks*, of *braſs Rods*, can looſe at moſt but 25″ *per* Day by Extenſion from Heat. But they may gain 30″ or 40″ in a Day by extreme Cold, contracting the ſame.

3400. If the Rod of a Second-pendulum be made of *Iron*, it will vary leſs in Length by *Heat*, and *Cold*, for by the ſame Experiments it was found, that *(cæteris paribus)* the Extenſion of *Braſs* was to that of *Iron* as 20 to $13\frac{1}{4}$, which is extremely near the Proportion of 3 to 2. And the Extenſion of *Iron* to that of *Steel* was found to be as $13\frac{1}{4}$ to $12\frac{1}{3}$. It was alſo found that the Expanſion of *Braſs* to that of *Copper* was as 20 to $19\frac{1}{4}$.

3401. From what has been ſaid it appears, there is no ſuch Thing in Nature as an *equable Time-piece*; that a Pendulum-Clock is the only Machine which, by Art, can be made to approach near it; and laſtly, that ſuch a Clock will ſtand in conſtant Need of a proper Adjuſtment or Rectification by the fixed Stars, or other aſtronomical Means. To contrive to correct theſe Errors, as faſt as they ariſe, by ſome artificial Conſtruction of the Pendulum has been the Endeavour of many Artiſts, and with ſome Degree of Succeſs; but none have come to my Knowlege that appears to anſwer this Purpoſe ſo well as the following Method, which I am informed is put in Practice by an ingenious Artiſt in the North of *England*; and is as follows.

3402. A Bar of the ſame Metal with the Rod of the Pendulum, and of the ſame Thickneſs and Length, is placed againſt the Back-part of the Clock-caſe; from the Top of this a Part projects to which the upper Part of the Pendulum is connected by two *fine pliable Chains*, or *ſilken Strings*, (as in the Cut at Page 374.) which juſt below paſs between two Plates of Braſs, whoſe lower Edges will always terminate the Length of the Pendulum at the upper End. Theſe Plates are ſupported on a Foot fixed to the Back of the Caſe. This Bar reſts upon an immoveable Baſe on the lower Part of the Caſe, and is braced into a proper Groove, which admits of no Motion any Ways but that of Extenſion and Contraction in Length by Heat and Cold.

3403. Now 'tis evident, that the Extenſion or Contraction of this Bar and the Rod of the Pendulum, will be equal, and in contrary Directions; and therefore, ſuppoſe by Heat the Pendulum is increaſed $\frac{1}{100}$ of an Inch in Length, below the Edge of the braſs Plates or Cheeks, then becauſe the Bar is lengthened

juſt

juſt ſo much upwards it will raiſe or draw up the Pendulum juſt $\frac{1}{100}$ of an Inch, and thereby make its Length below the Plates ſtill the ſame as before. The caſe is the ſame in regard to Contraction by Cold; for as the Pendulum is thereby ſhortened gradually, it is as gradually lengthened by being let down between the Plates, by the equal Contraction of the Bar behind. Whence it ſhould ſeem that this is a conſtant and adequate Rectification of the Pendulum by which it will always keep true Time.*

3404. If the Clock be of the common Conſtruction, *viz.* with a Pendulum conſiſting only of a Rod and Bob; then we muſt be content with a Method of correcting it as ſoon as its Quantity is diſcovered. Thus, if the Bob be made to reſt or depend on a *Nutt* and *Screw* upon the loweſt Part of the Rod; then if there be 25 Threads to an Inch, and the Nutt be of a circular Form, and its Perimeter divided into 45 equal Parts, it is evident from (3397,) that each of thoſe Parts will correſpond to a ſecond of Time in the Clock, and to a ſingle Diviſion of the Scale of *Farenheit's Thermometer*; and therefore by moving the Nutt one Way or the other, and ſo many Diviſions, as the Thermometer directs, we ſhall be able to correct the Error ſhewn thereby to a great Degree of Exactneſs.

3405. If the Pendulum be of the compound Sort, *viz.* with a Ball and a Corrector, and conſtructed as directed (in 2334, 2335, 2336,) then it will be eaſy for the Artiſt to divide the Rod of the Pendulum into *Minutes* and *Seconds* by the Theorems there given, and conſequently, if by the Revolution of a Star, he finds how much his Clock looſes or gains upon the Mean Time in a Day or ſeveral Days; or if by equal Altitudes of the Sun, he knows at any Time how much the Clock is too faſt or too ſlow, he can very readily adjuſt it to the Mean or true Time. It is not worth while here to take Notice of the Uſe of the BAROMETER in eſtimating the Errors of the Pendulum ariſing from the *different Weight or Denſity of the Air*, becauſe the Difference of Reſiſtance to the *Bob* is on that Account ſo exceeding ſmall as ſcarcely to come under the niceſt Obſervation, and which we have before provided againſt, (3342, &c.)

* The Reader may meet with a larger Account of this Invention in a *French Treatiſe* on Clock-work by Mr. THIOUT, with a Print thereof; but as I have had no Opportunity of peruſing that Book, I do not know who was the Author of this Contrivance.

CHAP. XVI.

The THEORY *of the* CIRCULAR PENDULUM *demonſtrated, with its peculiar* ADVANTAGES *in* CLOCK-WORK.

3406. WE have now delivered all we think neceſſary concerning the Structure of Clocks and Watches of the ordinary Sort, whoſe Motions are governed and regulated by Springs and Pendulums in the uſual Way. But here we are debarred from the Uſe of a *Second-pendulum Clock* of a portable Form; though from what we have ſaid, it appears, no other Sort of *Automata* can be regarded as *equable Time-keepers*, in any tolerable Degree. But Nature has ſupplied a Method, and Art has diſcovered it, by which a Pendulum may be made very eaſily to *circulate* Seconds in the ſame Length that a common Pendulum *vibrates* Half-ſeconds; which is as follows.

3407. We have formerly ſhewn, that a Weight at the End of a String might be made to move in ſuch a Manner, that the String ſhould deſcribe the *Surface of a Cone*, while the Weight moved through the Circumference of its *Baſe*, the *Vertex* of the Cone being the Point to which the String is fixed [See Fig. to 1193.] Or thus, let D [Fig. 6.] be a Weight at the End of a String D E fixed to the Point E; then if the ſaid Weight be moved in the Circle D *b* F *a* D, the String will deſcribe the Surface of the Cone D E F; and this Motion being continued would conſtitute a *Circular*, or rather *Conical Pendulum*.

3408. Now a Pendulum of this Form that ſhall perform its *Circulations* in a *Second of Time* is required to be but $\frac{1}{4}$ of the Length of a common Pendulum to *vibrate* in a Second, as we ſhall bye-and-bye demonſtrate. The Invention of this Form of a Pendulum the celebrated HUGENIUS claims alſo as his own, and tells us he found it nearly at the ſame Time he invented the common long Pendulum. His Words are, "*Unde aliud quoque horologii commentum deduximus, eodem fere tempore quo prius illud.* ——

3409. I have been the more particular in mentioning this Gentleman's Claim to this Invention, becauſe I find it is alſo claimed by Dr. HOOK, who we have before obſerved was his *Competitor* for the Invention of *Watches*. (3324.) Dr. *Hooke* has not, I think, a very clear Title to the Invention, nor has ſaid any Thing relative to the Theory thereof. Mr. *Hugens* has only given the Theorems on which it depends, but without any Demonſtration. So that the *Rationale* of a Clock with a *Circular Pendulum* is yet a Novelty in our Tongue; nor has any foreign Author wrote expreſsly on this Subject, that I know of. I ſhall therefore proceed to explain the *Rationale* of ſuch a Clock in as plain and conciſe a Manner as I can.

3410. Let N A M [Fig. 6.] be a *Parabola* inverted, its Vertex A, and Axis A O. Then ſuppoſe a Veſſel or Bowl was excavated to the Figure of ſuch a *Parabola*, ſo that its concave Surface might be that of a *Paraboloid*; if ſuch a Bowl were properly agitated, a heavy globular Body within it might be made to circulate round it in any Part of the Surface as at D or G, and there to deſcribe the Circles D *a* F *b*, or G *c* H *d*, whoſe Diameters are D F and G H.

3411. The Ball D is ſupported or ſuſtained in the Circle by an Equilibrium of three different Forces, *viz.* (1.) The Force of *Gravity*, or that of its own *Weight*. (2.) A *Centrifugal Force* which is always a neceſſary Conſequence of circular Motion: And, (3.) The *Re-action* of the Side of the Veſſel in any Point D, which is always equal to the *Force* of its *Preſſure* againſt it, in a perpendicular Direction, ariſing from the other two Forces.

3412. The Circle which the Ball deſcribes being ſuppoſed parallel to the Horizon, O C will be perpendicular to its Plane in the Center C; and let D E be drawn perpendicular to the Side of the Veſſel in D. Then becauſe *Gravity* acts perpendicularly to the Hooizon, let it be repreſented by the right Line E C. Again, ſince the *Centrifugal Force* is in a horizontal Direction always from the Center C, it will be properly repreſented by the right Line C D; then, laſtly, becauſe the Body D is ſuſtained *in Equilibrio* at D, its *Preſſure* or *Re-action* directly againſt the Side at D will be denoted by the right Line

D E.

D E. All this is very evident from the *Mechanical Doctrine* of *Compound Forces*. (1027, &c.)

3413. In like Manner, if the Body circulates in any other Part of the paraboloid Veſſel, as at G, the three Powers will there alſo be repreſented by the three Sides of the Triangle I G B; that is, *Gravity* by the Side I B; the *Centrifugal Force* by the Side B G; and the *Reſiſtance* of the Veſſel by the Line G I, perpendicular to the Tangent at G.

3414. As by theſe three Forces the Body is kept in a conſtant *Equilibrium* in every Point of the Circumference it deſcribes, it will be ſuſceptible of any Degree of projectile Force impreſſed upon it by an horizontal Agitation of the Veſſel, and thereby be put into *Circular Motion*, every Way ſimilar to that by a central Attraction (1170,) or to that of a Body annexed to the End of a String, (1192,) and will therefore be ſubject to all the ſame Laws of Motion.

3415. Conſequently, ſince the Body in deſcribing the Circles D *a* F *b*, and G *c* H *d*, has Centrifugal Forces proportional to the Radii C D, and B G, it will deſcribe thoſe Circles in *equal Times*, (1177.) Therefore, *all circular Revolutions in every Part of ſuch a Veſſel are iſochronal, or made in equal Times.*

3416. If a Plane be freely moveable about a Center, and a heavy Body lies upon it, connected by a String to the Center, then as the Weight of the Body is ſuſtained by the Plane it may be conſidered as without Weight, and in Equilibrio. Therefore if the Plane be moved horizontally, it will impreſs a projectile Force on the Body, and it will begin and continue to move round the Center, with a circular Motion. Now this projectile Force being in the Direction of a *Tangent*, the Body will endeavour to proceed in ſuch a Line, but it is conſtantly checked and drawn therefrom into the Periphery of a Circle by the String, which String will therefore be ſtretched with a certain Force by which the Body endeavours to receed directly from the Center; and which is therefore the *Centrifugal Force* generated by the Motion of the Plane.

3417. As the Velocity of the Plane or circular Motion is greater, the Centrifugal Force will encreaſe and that in Proportion to the *Square of the Velocity*, (1175.) So that the Centrifugal Force will begin from Nothing and encreaſe to any Degree; and

and conſequently will, in one Degree of Velocity, be equal to the *Weight of the Body*, or the String will, in that Caſe, be ſtretched juſt as much as it would be by the Weight of the Body hanging freely to it at reſt.

3418. If, therefore, in this particular Caſe, a Power of Attraction to the Center, equal to the Weight of the Body, was ſubſtituted inſtead of the String, it would make no Alteration in the Motion of the Body; and hence it appears, *that when a Body is moved in a circular Orbit by a projectile and gravitating Force, there will be a Centrifugal Force produced juſt equal to the Power of Gravity.*

3419. But, in ſuch a Caſe, the Velocity of Motion is ſuch as the Body would acquire by deſcending through half the Radius of the Circle (1187.) Therefore in the Time of that Deſcent, the Body (with the ſaid Velocity uniformly continued) would deſcribe twice that Space, or a Space equal to the Radius, (993.) *Therefore the Time* (*t*) *of Deſcent through* $\frac{1}{2}$ *of the Radius* ($\frac{1}{2}$ R) *is to the periodical Time* (T) *in the Circle, as Radius* (R) *is to the Periphery* (P) *of the Circle, or as the Diameter* (D) *is to twice the Periphery* (2 P). That is, $t : T :: D : 2P$.

3420. Now in the paraboloid Veſſel, ſince CE, or BI, is a conſtant Quantity, being ever equal to half the *Latus Rectum* of the Parabola (747,) there will be one Part where the Centrifugal Force or Radius of the Circle BG will be equal to Gravity BI. And it is evident this muſt be when the Diameter of the Circle GH paſſes through the *Focus* B of the Parabola, becauſe then $BG = BI = \frac{1}{2} GH$, the *Latus Rectum*.

3421. By the Nature of the *Parabola* we have AB equal $\frac{1}{2} BI = \frac{1}{2} BG$ (742.) Therefore, the Time of deſcribing the Circle G*c*H*d* (or any other) will be to the Time of Deſcent through $\frac{1}{2} BI = AB$, as 2P is to D. (3419.)

3422. Biſect BI in Q, then $AB = BQ$; and let AB be the Diameter of the generating Circle of the Cycloid KAI. Now we have ſhewn that the Time (T) in which a Pendulum QA vibrates through the Cycloid KL is to the Time (*t*) of Deſcent through half its Length AB as the Periphery of a Circle (P) to its Diameter (D,) (1124.) Whence we have $t = \frac{TD}{P} = \frac{TD}{2P}$, which gives $2PT = TP$, or $T : T :: 2P : P ::$

2 : 1;

2 : 1; conſequently, *the Time of revolving in any Circle in the* Paraboloid *is double the Time of Vibration in the* Cycloid, *in a Pendulum whoſe Length* A Q *is* $\frac{1}{2}$ *the Latus Rectum of the* Parabola.

3423. It remains now to ſhew how a Pendulum may be conſtructed ſo that it may always deſcribe a *Conical Surface*, and its Ball perform its Gyrations in a *parabolical Superficies*. To this End let K H * be a Verge or Axis perpendicular to the Horizon with a Pinion at K moved by the laſt Wheel in the Train of the Clock; at H it has a hardened ſteel Point in a Pivot of Agate, to render the Motion as free as poſſible.

3424. Let it be propoſed that the Pendulum ſhall perform each Revolution in a *Second* of Time; then it is plain, the Paraboloid Superficies it moves in, muſt be ſuch whoſe *Latus Rectum* is double the Length of a Pendulum vibrating *Half-Seconds* in a Cycloid, (3422.) Let O be the Focus of the Parabola MEC, and M Ç the *Latus Rectum*; and make A E = M O = $\frac{1}{2}$ M C = the Length of a common *Half-ſecond* Pendulum.

3425. At the Point A of the Verge, let a thin Plate A B be fixed at one End, and at the other End B let it be faſtened to a Bar or Arm D B, ſtanding out from the Verge at right Angles and to which it is fixed at D. This *Lamina* or Plate A B is the *Semi-cubical Parabola* or *Evolute* of the given *Parabola* M E C, ſuch as deſcribed (921 to 926.)

3426. The Equation of this *Cubical Parabola* A B was $\frac{27}{16} p x x = y^3$. Let $\frac{27}{16} p = P$, then $P x x = y^3$ and in the Focus, $P = 2y$; in that Caſe, $2 x x = y^2 = \frac{1}{4} P^2$; therefore $x^2 = \frac{1}{8} P^2$, and $x = P \sqrt{\frac{1}{8}} = \frac{27}{16} p \sqrt{\frac{1}{8}}$ = the Diſtance of the Focus from the Vertex A. By aſſuming the Value of x (or putting $x = \frac{1}{10}$ of an Inch) you will find all the correſpondent y's or Ordinates of the Curve A B, by which it may very eaſily be drawn.

3427. If the Pendulum is to make its Gyrations in $\frac{1}{2}$ Seconds, then the Parameter is M C = 4,9 Inches, or 49 Tenths. $\frac{27}{16} p = P = 82,7$ Tenths, and by aſſuming $x =$ to the Number in the firſt Column of the following Table, we ſhall have $y =$ to thoſe in the ſecond reſpectively.

Abſciſſæ

Fig. 1. in the PLATE of the THEORY of *Circulating* PENDULUMS, &c.

Abſciſſæ	Ordinates.	*Abſciſſæ*	Ordinates.
0,05	0,274	1,4	2,531
0,1	0,435	1,7	2,880
0,2	0,692	2,0	3,213
0,3	0,906	2,3	3,526
0,4	1,098	2,6	3,823
0,5	1,274	2,9	4,113
0,6	1,438	3,2	4,392
0,8	1,743	3,6	4,748
1,0	2,023	4,0	5,096
1,2	2,284		

3428. The String of the Pendulum muſt be of ſuch a Length that when one End is fixed at B it may lie over the Plate A B, and then from A hang perpendicular with the Center of its Bob in the Point E (or Vertex of the Parabola M E C) when at reſt. Then the Verge K H being put into Motion, the Ball of the Pendulum will begin to gyrate, and thereby conceive a *Centrifugal Force* which will carry it out from the Axis to ſome Point F, where it will circulate *Seconds* or *Half-Seconds*, according as the Line A E is 9,8 Inches, or $2\frac{1}{4}$ only; and A B anſwerable to it.

3429. HUGENIUS tells us, many Clocks of this Conſtruction were made, and with Succeſs; but that they prevailed not ſo much as the common Sort, on Account of their not being ſo *eaſily and expeditiouſly made* —— That they depend on a Principle of EQUABLE MOTION, which the long Pendulum in a Circle has not perfectly —— That the Index ſhewing Seconds moves with a moſt regular and uniform Motion, and not *(Subſultim)* by *Jerks* and *Stops* as in common Clocks —— That this Pendulum is *entirely ſilent*, or without that conſtant and melancholy *Click Clack*, which neceſſarily attends the long Pendulum. —— Moreover, it may be obſerved that the Pendulum to *circulate Seconds* is but a *fourth Part* of the Length to *vibrate Seconds*, or but juſt the Length of a *common Half-Second Pendulum*. —— And, laſtly, a Contrivance may be added to ſtop the Pendulum in any Part of the Circumference, and thereby render it capable of ſhewing *Thirds*.

3430. This Conſtruction of a *conical Pendulum* is undoubtedly the beſt Form for a *Temporary* CHRONOMETER, for meaſuring

the ſmalleſt Parts of Time occaſionally. But the Part AB ſhould be divided into two from the Point A, and proceed divaricating to the End B, where the Ends of two fine Chains (uſed in *Watches*) or ſilken Strings are to be fixed, which Strings ſhould alſo unite after they paſs the Point A in the Verge or a little before they come to the Bob F. By this Means its Motion will be rendered more ſteady and certain than by a *ſingle Chain or String*. The Arm DB ſhould alſo be nicely balanced by a Counterpoiſe on the other Side. And an *endleſs Screw* at K will do better than a *Pinion* with Teeth; but this is ſubmitted to the Experience and Ingenuity of the Artificer.

CHAP. XVII.

The THEORY *of Mr.* SULLY'S INVENTION *of a Horological* ROTULA, *inſtead of a* PENDULUM, *for regulating* CLOCK-WORK; *with an* IMPROVEMENT *thereof.*

3431. AS this INVENTION of Mr. SULLY is upon an entire new Principle, found out by Trial and Experiments directed by a natural Sagacity, without any Aſſiſtance from a *Phyſico-mathematical* THEORY, I have Reaſon to think it may prove a *Novelty* equally acceptable, as ſingular, to the *Engliſh Automatiſt*; not only ſo, but the Omiſſion of ſo curious an Improvement in *Clock-work*, muſt certainly be thought a Defect in a Treatiſe wrote profeſſedly on the THEORY thereof.

3432. Mr. HENRY SULLY was an *Engliſh Watch maker*, and lived many Years at PARIS; he wrote, in the *French* Language, a Treatiſe on Clock-work, which he entitled *Regle artificielle de Temps*, printed at *Paris*, *A. D.* 1717. I never heard that this *Artificial Rule of Time*, was ever tranſlated into *Engliſh*, nor have I ſeen the Original; it ſhould ſeem by the Title as if it was a Deſcription of his new Clock; but we are told by *Profeſſor* EULER (in the *Petropol. Comment.* for the Year 1727,) that Mr. *Sully* had publiſhed an Account of his new Clock but a little more

more than a Year before, *viz.* in the Year 1725. But leaving the Date of its firſt Publication, this learned Profeſſor thought it an Invention worthy of an Explication, which he has accordingly given us under the Title of *A Diſſertation on a* certain *new Kind of Tautochronal Curves*; and which I ſhall here tranſlate and abridge for the Uſe of the *Engliſh* Artiſts.

3433. SULLY's *Invention* conſiſted in a Kind of *Rotula* or ſmall Wheel * A B made to oſcillate about its Center C (Fig. 2.) by Means of a String with a ſmall Weight C P, playing between two curved Plates C E, C F, fixed to the Surface of the *Rotula*. But what Degree of Curvature was neceſſary for thoſe Plates, that the Oſcillations might be all *iſochronal* among themſelves, and to the Vibrations of a Pendulum of a given Length, was what *Sully* wanted to know, and could only find by frequent Trials, and that very imperfectly.

3434. But to ſhew what this particular Species of Curvature is from *Phyſico-Mathematical Principles*, the Proceſs is as follows. Let C M (Fig. 3.) denote one of the Plates in a Situation not natural, or when the *Rotula* is turned on one Side; and let C B be a vertical Line, parallel to the Direction of the Thread M P touching the Curve in M; then from the Point of Contact M draw M T perpendicular to C B, which will alſo be perpendicular to the Curve in M. Draw the Right Line C O containing the Angle B C O by which the *Rotula* is carried out of its natural Poſition. On the Center C with any Radius C B deſcribe the Arch B O, meaſuring the Angle B C O.

3435. Then it is evident the Weight or Power P at the End of the String will endeavour to reſtore the Rotula to its natural Situation by a Force which will be as P × T M (1049.) But as the Application of this Force is continually altering with reſpect to the *Fulcrum* C, we muſt find one equal to it to be applied to the Radius C O, and acting normally at the Point O. Produce P M to N in a Right Line drawn horizontally through the Center C. Then becauſe C N = T M, the Effect of the Power P to turn the *Rotula* will be the ſame as if it was applied to the Radius C N in N, and acting normally thereto.

 3436. But

* See Plate entituled "The THEORY of the Circulating PENDULUM, Oſcillating ROTULA, &c."

3436. But from the Nature of the *Lever* (1051) we have CO : CN :: P : p = the Weight to be applied at O, whoſe Force ſhall be equal to that of P applied at N. But CN = TM, therefore CO : TM :: P : p, and ſo $p = \frac{P \times TM}{CO}$; therefore becauſe P and CO are conſtant, we have p every where proportional to TM.

3437. We are next to conſider that the Power (p) acting normally at O, will make CO a Pendulum; and farther, that if the ſaid Power (p) (which is the accelerating Force,) be proportional to the Space paſſed over, the Oſcillations of the Pendulum will be *iſochronal* (999.) But the Space to be run over is the Arch OB, or the Angle OCB; therefore in order to produce this *Iſochroniſm* in the Pendulum (and conſequently in the *Rotula*) the Affair is reduced to this *Problem, viz. To find a Curve* CME *of ſuch a Property, that having a Right Line* CO *given in Poſition, and in it, from a given Point* C, *another Right Line* CT *be drawn perpendicular to the Radius of Curvature in any Point* M, *the Part* TM *ſhall be always proportional to the Angle* TCO.

3438. Now when any Quantities are proportional, their Fluxions have the ſame Ratio (789.) Therefore let M and m (Fig. 4.) be two Points in the Curve indefinitely near to each other; and draw CM, Cm, and the two Normals MT and mt, occuring in R the Center of Curvature at the Part M. Alſo let fall the Perpendiculars CT, Ct, on the ſaid Normals. Then is TCp the Fluxion of the Angle TCO; and Tp, the Fluxion of the Normal TM. Now it is plain, that in Regard to the fluxionary Angle TCp, the Side TC is Radius, and Tp, the Tangent. Therefore the Fluxion of the Arch OB, which is Bb will conſtantly be as Tp, the Fluxionary Tangent of the ſame.

3439. Let CB be equal to CT = Radius, and as the Radius of a Circle is conſtant, the Fluxion thereof is nothing, that is $pt = 0$; therefore CT is perpendicular to TM in the Point R, or Center of the Circle of *Oſculation* in the Point M.

3440. In this Caſe, the Angle OCT has its Fluction Bb not only *proportional* but *equal* to the Fluxionary Tangent Tp. For the Fluxion ($\dot{z}$) of any Arch z and the Fluxion of the Tangent (i) thereof, are expreſſed by this

Equa-

Equation $\dot{z} = \frac{a^2 \dot{t}}{a^2 + t^2}$ (823.) and therefore when $t = 0$, or the Tangent is in its naſcent or fluxionary State, that Equation becomes $\dot{z} = \frac{a^2 \dot{t}}{a^2} = \dot{t}$.

3441. But the Fluxions of *contemporaneous Fluents* being equal, ſhew thoſe Fluents are alſo equal, or that T M the Radius of Curvature is ever equal to the Arch of the Circle T O deſcribed in the ſame Time.

3442. Hence the Nature of the Curve C M is manifeſt, it being no other than the EVOLUTE of a *Circular Arch* O T (Fig. 5.), whoſe Beginning is at the Point of the Circle O, and is deſcribed by the Evolution of that Circle whoſe common *Radii* are C T = C O. —— In this Proceſs, we have uſed a different Method of Demonſtration from that of *Profeſſor* EULER, which we think is more natural, clear, and conciſe.

3443. Hence then it appears that the *Laminæ* or Plates O F, O E, [Fig. 5.] are formed by the Evolution of the Circle A D O equal to the Periphery of the propoſed *Rotula*, and to begin from the Point O in the Circumference, and not from the Center C, as in *Sully*'s Conſtruction. In the ſame Point O likewiſe the String O P is to be fixed, which by its Weight P will move between the Plates O E, O F, (like a Pendulum between the *Cycloidal Cheeks*) and thereby continue the Motion of the *Rotula*, once begun; and *render the Oſcillations iſochronal, by a Force ever proportional to the angular Space deſcribed in each Oſcillation.*

3444. The next thing to be determined, is, the *abſolute Time of an Oſcillation* in a given *Rotula*; this is done by finding the Length of a *Pendulum* vibrating in a *Cycloid*, *iſochronal* to the *Rotula*. The Method for doing this our celebrated *Profeſſor* has rather *pointed out*, than *demonſtrated*; we ſhall therefore here ſupply the Principles which he aſſumes as known, and on which this Part of the Theory depends.

3445. To this End we muſt conſider that as in reſpect to the Endeavour or innate Force of Bodies to deſcend, it is all one whether we conſider the Quantity of Matter as diffuſed through any Space or all collected into a Point, this Effort to deſcend being conſtantly proportioned to the *Maſs of Matter into*

the

the Force of Gravity (1000.) So with Regard to an *Oſcillatory* or *Angular Motion* of Bodies about a Center or Axis, it matters not how the Maſs of Matter is diſpoſed, whether we conſider it in the Form of a *Circular Area*, or in that of a *Right Line*; but this latter is moſt natural, as it is that by which the ſaid *Angular Motion* is deſcribed.

3446. If therefore the Radius of the *Rotula* be conſidered as charged with its whole Weight (Q) we ſhall find its Force of Reſiſtance to ſuch Angular Motion will be expreſſed by $Q g n$, (2286,) where $g = \frac{1}{2}$ = Diſtance of the *Center of Gravity*, and $n = \frac{2}{3}$ = Diſtance of the *Center of Oſcillation*, the Length of Radius being = 1. The whole innate Force, therefore, of the Radius or *Rotula* to reſiſt to a *Circular Motion* is as $\frac{1}{3}$ Q.

3447. But if we conſider the *Rotula* in Motion its *active Force* then (called the *Vis viva)* will be as the innate Force multiplied by the Velocity of the *Center of Force*. But this Center of Force, in the Sector of a Circle is diſtant from the Center of the Circle, $\frac{3}{4}$ of the Radius (1098,) it is therefore the ſame in the Circular Area, or *Rotula*; therefore if the Velocity of a Point in the Circumference be ſuch as is acquired by the Deſcent of a heavy Body through the Space S, which will be equal to 2 S (993;) then the Velocity of the Center of Force will be $\frac{3}{2}$ S (for $1 : 2\,S :: \frac{3}{4} : \frac{3}{2}\,S$), conſequently the whole Force of the moving *Rotula* will be $\frac{1}{3} Q \times \frac{3}{2} S = \frac{1}{2} QS$, and therefore the Fluxion of its Motion will be $\frac{1}{2} Q \dot{s}$.

3448. But this Fluxion of Motion in the *Rotula* is the Effect of the Action of the Weight P by Means of the String P O upon the Plate O E. Let v = Velocity, (or Space deſcribed) in an infinitely ſmall Particle of Time by the Weight P; then its momentary Force is $P v$; and as this produces the momentary Motion $\frac{1}{2} Q \dot{s}$ in the *Rotula*, it muſt be equal to it, that is, $P \times v = \frac{1}{2} Q \dot{s}$, and ſo $v = \frac{Q \dot{s}}{2 P}$.

3449. This is the Caſe of the *Rotula* and its Weight P, at the Commencement of an Oſcillation, when the Poſition of the Whole is as ſhewn in Fig. 6. Where B D T is the *Rotula*, T A one of the Plates, when the Radius of Curvature B A is in a Horizontal Poſition, A P is the perpendicular Direction of the Weight P; and let fall C B perpendicular to A B. Then

let

let the *Rotula* by the Action of the Power P move in the firſt Moment with the angular Motion T C *t*, and then will *t a* be the Part of the Plate on which the Power acts, and B *a* the horizontal Radius of Curvature; the Line of Direction is now *ap*, and the Power P has moved from P to *p*, through the Perpendicular Space P *q* = *v*.

3450. On the Center C with the Radius C *a* deſcribe the ſmall Arch *a d*, and draw C *a*, C *d*; then is A *d* = P *q*; and the right-angled Triangles A *a d* and B *a* C are ſimilar; therefore A *d* : A *a* :: B *a* : B C; hence $A d = v = \frac{B a \times A a}{B C}$: But A *a* = T *t*, whence $\frac{B a \times T t}{B C} = v = \frac{Q \dot{s}}{2 P}$ (3448;) therefore 2 P × B *a* × T *t* = B C × Q × $\dot{s}$ = (becauſe of B *a* = BT) 2 P × BT × T *t*. Therefore $\frac{\dot{s}}{T t} = \frac{2 P \times BT}{Q \times BC}$.

3451. But T *t* is the Space deſcribed by the Point T in the firſt Particle of Time, and is therefore as the inceptive Velocity $\dot{v}$; conſequently $\frac{\dot{s}}{T t}$ is as $\frac{\dot{s}}{\dot{v}}$; but $\frac{\dot{s}}{\dot{v}} = \dot{T}$, the Fluxion of the *Time of an Oſcillation*, ſince in all Caſes of deſcending Bodies, we have $\frac{S}{V} = T$ (991.)

3452. Having thus determined the Fluxion of the Time of Oſcillation in the *Rotula*, we ſhall readily diſcover the Fluent, or Time itſelf, by comparing ſuch an Oſcillation with a *Synchronal Vibration* in the *Cycloid*. Thus (Fig. 7.) let A N be the *Semi-cycloid* juſt equal to the Semi-arch of Oſcillation B T, and therein take N *n* = T *t* and through N, *n*, draw P N, *p n*, perpendicular to the Axis of the Cycloid, and let A O be the Length of the Pendulum deſcribing the ſame by Evolution from the *Cycloidal Cheeks*, as taught (1120.)

3453. Then from (*n*) draw *n t* perpendicular to P N, and the Fluxion of the Time of a Vibration will be as $\frac{t n}{N n}$ (3451,) which therefore muſt be equal to the Fluxion of the Time of the Oſcillation, *viz.* $\frac{t n}{N n} = \frac{\dot{s}}{T t} = \frac{2 P \times BT}{Q \times BC}$. But from the Nature

of

of the *Cycloid* (1117 to 1126,) we have $\frac{nt}{Nn} = \frac{AN}{AO}$; therefore becauſe $AN = BT$, it is $\frac{1}{AO} = \frac{2P}{Q \times BC}$ (3450) which gives $Q \times BC = 2P \times AO$, and therefore this Analogy $Q : 2P :: AO : BC$; that is, *The Weight of the* Rotula *is to twice the Weight* P, *as the Length of a Pendulum vibrating in a given Time, is to the Diameter of the* Rotula, *which ſhall oſcillate in the ſame time.*

3454. For Example, ſuppoſe it be required to find the Radius BC of the Rotula that ſhall oſcillate preciſely in a *Second of Time.* Then $AO = 392$ Tenths of an Inch; and ſuppoſe $Q : P :: 100 : 1$, then $BC = \frac{2P \times AO}{Q} = \frac{784}{100} = 7{,}84$ Tenths of an Inch. Or the Diameter $2BC = 1{,}568$ or a little more than $1\frac{1}{2}$ Inch.

3455. It was neceſſary to ſuppoſe the Weight P very ſmall in Compariſon of the Weight Q of the Wheel or *Rotula*, that its own *Vis Inertiæ* might be inconſiderable; for where that to be taken into the Account, it muſt be deducted from the Force which we have all along appropriated to move the *Rotula*; and this would embarraſs the Theory, or leave the practical Execution of it imperfect.

3456. Another Caution neceſſary in a Regulator of this Nature, is, that the Direction of the String below the Cheeks be always *perpendicular*, or that the Weight P has not the leaſt oſcillatory Motion in itſelf, as it would make a Difference in its Force upon the *Rotula*, and induce an Irregularity in its Motion, and Time of its Oſcillations. A principal Thing to prevent this is to have the String of a conſiderable Length below the Plates that the Radius BC of the Wheel may bear but a ſmall Proportion to it.

3457. But as a long String will be inconſiſtent with the Deſign of a Clock in a *portable Form*, there is another Remedy for both the forementioned Evils, and which will at the ſame Time be a great Improvement upon SULLY's Invention. This conſiſts in adding a very large Wheel or *Rotula* ADG [Fig. 8.] to *Sully*'s ſmall *Rotula* BHI; and then as a much greater Force will be required to move the compound *Rotula*, ſo there will be a greater

a greater Disparity between its Weight and that of P, and a much less Motion of the String B P because of the very slow Motion of the *Rotula* thus altered.

3458. That the Reason of this may better appear, let T C T (Fig. 5.) be the fluxionary Sector of the *Rotula*, and T $t = \dot{z}$ the Fluxion of the Periphery, then the Area of the said Sector will be $\frac{1}{2}$ C T $\times \dot{z}$, the Distance of the Center of Gravity $\frac{2}{3}$ C T, and that of Oscillation $\frac{3}{4}$ C T; then (putting C T $= r$) we have its Force to resist circular Motion (3287,) thus expressed, $Q g n = \frac{1}{2} r \dot{z} \times \frac{2}{3} r \times \frac{3}{4} r$ (1098) $= \frac{1}{4} r^3 \dot{z}$. Hence the Force of the whole *Rotula* will be $\frac{1}{4} r^3 z$. But it is 1 : C :: r : z :: Radius : Circumference; therefore $z = Cr$, and so the Force of any Wheel or *Rotula*, will be $= \frac{1}{4} r^4$ C, or as the 4th Power of the Radius.

3459. Therefore if R = Radius of the large additional *Rotula*, the Force of the Whole will be increased in Proportion to R^4, that is, the Length of the *Synchronal* Pendulum will be as

$$A O = \frac{Q \times B C \times R^4}{2 P} = A O \times R^4. \quad (3453.)$$

3460. Since the Time of a Vibration of the Pendulum A O is 1, we shall have the Time T of a Vibration in the Pendulum A O $\times R^4$, thus $1^2 : T^2 :: A O : A O \times R^4 :: 1 : R^4$, therefore $T^2 = R^4$, and $T = R^2$. So that if T = 2, R $= \sqrt{2}$, if T = 3, R $= \sqrt{3}$; if T = 4, R = 2; and so on. Whence it appears how very slow the *Rotula* will move when thus enlarged, and how little the Spring B P will on that Account be liable to be put out of a vertical Position.

3461. Such is the THEORY and CONSTRUCTION of the *Automaton Sullianum* with its improved *Rotula*; and besides this Kind of Regulation, which is entirely new and peculiar to this Piece of Clock-work, there is one Property of it which no other Time-Piece has, and that is, the slow Motion of the *Rotula*, by which the Time of the Oscillations become very long, and consequently by that Means more equable; which induced Mr. EULER * to think a *Time-Piece* constructed in this Manner

* For referring to this Improvement of an additional large *Rotula*, he says,— *Et tandem alium evolvam Casum, qui non contemnendum in* RE NAUTICA *usum mihi præstare videtur.*

might be of Service at Sea in finding the *Difference of Time* or *Longitude*, beyond any other Sort of *Clock-work* yet attempted.

CHAP. XVIII.

The CONSTRUCTION *of an* ANGULAR PENDULUM, *of a general* NATURE, *and adapted to a* Planetary TIME-PIECE, *which, in the ſame* LENGTH, *ſhall vibrate in any given* TIME:

3462. THERE has been, as yet, no other Sort of Pendulum in Uſe, but ſuch whoſe *Time of Vibration depends upon its Length only*; for this is the Caſe of the *Circular*, as well as the *Cycloidal Pendulum*. However the *Phyſical Theory* leaves us not deſtitute of a Method of regulating Clock-work, by a *Pendulum*, which ſhall in *any given Length* vibrate *in any Time required*.

3463. To demonſtrate this we need only conſider the Nature of a common Pendulum of the moſt ſimple Form, *viz.* that of an *uniform Rod or Parallelogram* whoſe Length ſuppoſe $= a$, and Weight $= w$; then the Force of the Rod is $\frac{1}{3} a^2 w = w \times \frac{1}{2} a \times \frac{2}{3} a$. (3286.) Now if the Pendulum be transformed, or divided into two equal Parts, and both theſe Parts move or oſcillate about the ſame Center as before, then if thoſe Parts be ſeparated from each other to an equal Diſtance from the Vertical Line, the Center of Gravity (which we will now called x) will approach nearer to the Point of Suſpenſion, and the Center of Oſcillation (n) will recede farther from it in the ſame Proportion, for it will always be $\frac{\frac{1}{3} a^2 w}{x w} = \frac{\frac{1}{3} a^2}{x} = n$; and ſo $x n = \frac{1}{3} a^2$, a given Quantity. Therefore (n) will be inverſely as (x).

3464. Therefore alſo the Time of a Vibration will be greater, in the ſame Pendulum of two Parts, as thoſe Parts contain a greater Angle with each other; or ſtrictly ſpeaking, *the Time*

of

of a Vibration will be in the inverſe duplicate Ratio of the Co-Sine of half that Angle.

3465. To illuſtrate this by an Example; let ACB (Fig. 9.) be a common *Joint-Rule*, whoſe two Parts ACD and BCD are cloſe together; through the Center C of the *Joint*, let a ſmall Hole be drilled, and the Rule ſuſpended thereby on a poliſhed Pin or Wire, to vibrate freely. Let $AC = a$, the Diſtance of the Center of Gravity $CG = \frac{1}{2}a$; and that of the Center of Oſcillation $CN = \frac{2}{3}a$. Alſo let the Weight of the whole Ruler be $= w$. Then (as was ſaid) the Length of an iſochronal ſimple Pendulum will be $\frac{\frac{1}{3}a^2}{\frac{1}{2}a} = n = \frac{2}{3}a$. (1095.)

3466. But now let the Ruler be opened, or the two Legs AC and BC removed from the Perpendicular CD to an equal Diſtance on each Side, *viz.* to HC and IC; and let $CK = CL\ (= \frac{1}{2}AC) = CG$; and on the Center C deſcribe the Arch KGL; and draw KL cutting the vertical CD in O. Then is the Point O the new Center of Gravity in the Rule thus opened; and $CO = x$; and the new Diſtance of the Center of Oſcillation will be $\frac{\frac{1}{3}a^2 w}{x w} = \frac{\frac{1}{3}a^2}{x} = n$.

3467. Examples will make this Doctrine plain; ſuppoſe it required to open the Legs of the Ruler to ſuch an Angle that the Time of a Vibration ſhall be juſt equal to that of a ſimple Pendulum whoſe Length is equal to that of the Rule CD; in this Caſe we have $n = a$, therefore $x = \frac{1}{3}a = CO$; which will give this Analogy, as $CO : CK\ (:: \frac{1}{3}a : \frac{1}{2}a) :: 10 : 15$ :: Radius : Secant of 48° 12′ = HCD, half the Angle HCI, as required.

3468. For a ſecond Example, ſuppoſe $CN = 9{,}8$ Inches or $CD = 14{,}7$; then will the Ruler cloſed vibrate *Half-Seconds* preciſely; let it now be required to find what Angle it muſt be opened to, that each Vibration may be performed in *One Second?* In this Caſe it is evident $n = 39{,}2$ Inches. Conſequently $\frac{1}{3}a^2 = 39{,}2\,x$; therefore $a^2 = 117{,}6\,x$, conſequently $117{,}6 : 14{,}7 :: 14{,}7 : 1{,}837 = x = CO$. Then $CO : CK :: 1{,}837 : 7{,}35$ :: *Radius* : *Secant* of 75° 32′ = HCD, half the Angle required.

3469. When $x = o$, the Diſtance of the Point N becomes infinite, and the Time of Vibration alſo; that is, the two Legs C H and C I are then in a Right Line, and at *Reſt*. But this Form of a Parallelogram angular Pendulum will meet with too much Reſiſtance from the Air; and beſides, is a *third Part* longer than neceſſary; therefore that ſuch a Pendulum may have all Advantages it muſt be in the common Form of a *Rod and Ball* at the End.

3470. The *Rod* and *Bob* of this new Pendulum muſt therefore be double. The Bob may conſiſt of two Hemiſpheres or two Plano-convex Lenſes, put together by their plain Sides, ſo as to make a compleat Sphere, or elſe a double and equally *Convex Lens*. Let A B, A C, (Fig. 10.) be the two Parts of ſuch an angular Pendulum ſuſpended on the Point A, and opened to the Angle B A C, exactly biſected by the vertical Line A D. But when the two Parts are cloſed, they make the ſimple Pendulum A E. And here let the Weight of the Rod A E (= Weight of A B + A C) = b, and the Weight of the Bob = c.

3471. Then when the Pendulum is opened to the Angle B A C, the Center of Gravity is raiſed from a Point near E to O, in the vertical A D, as appears by drawing the Line B C; then the Center of Oſcillation (n) is removed to a greater Diſtance in Proportion; for putting A E (= A B) = a, and A O = x; we have $\frac{\frac{1}{3}a^2 b \times a^2 c}{x \times \overline{b+c}} = n$ (1094,) whence $\frac{\frac{1}{3}b+c}{b+c} \times a\,a = x\,n$.

3472. Put B O = y, then $a^2 = (AB^2 =)\ x^2 + y^2$, and ſo $\frac{\frac{1}{3}b+c}{b+c} \times \overline{x^2+y^2} = x\,n$; therefore it will be every where $x^2 + y^2$ as $x\,n$; conſequently $y^2 = x\,n - x^2$ which is an Equation to *a Circle* (819,) and ſhews the *Locus* of the Bob B is the Periphery of a Circle whoſe Diameter is n = A D the Length of a Pendulum iſochronal to the angular Pendulum B A C.

3473. Hence alſo it appears that the whole Circle A B D C, (ſuppoſed heavy) or any Part B A C (the Points B, C, being equidiſtant from A), will vibrate in the ſame Time with a Pendulum equal to the Diameter A D. All which *Hugenius* has demon-

demonſtrated from far different Principles, which he himſelf underſtood very well.

3474. Now it is manifeſt the Time of Vibration in this angular Pendulum may be altered at Pleaſure, and be made greater or leſs as occaſion requires; and conſequently, that it is capable of being adapted to meaſure Time univerſally in a Clock from the *Sidereal* (or ſhorteſt) Day, to the longeſt *Lunar One*, and thereby to anſwer the ſame Purpoſes as the univerſal Pendulum before deſcribed in Chap. X. and XI.

3475. In order to this, we are to conſider the Value of x, when the Pendulum is cloſed in A E, is $\frac{2ac+ab}{2\times c+b}=g$, (3027) but it is always equal to $\frac{\frac{1}{3}b+c}{n\times b+c}\times aa$ (3471,) therefore $\frac{\frac{1}{3}b+c}{n\times b+c}\times a^2=\frac{2ac+ab}{2\times c+b}$, which will give this Analogy $a:n::c+\frac{1}{2}b:c+\frac{1}{3}b$. Alſo by the firſt Equation it is $a:g::c+b:c+\frac{1}{2}b$; the ſame Analogies as before in (3029.)

3476. Let t = Time of the Pendulum's Vibration when accommodated to the *Sidereal Day* of 1436′; and T = Time of its Vibration when adapted to the *Solar Day* of 1440′ (3313.) Then it is $t^2:T^2::t:\frac{T^2}{t}=N$, = the Length of the iſochronal Pendulum for the *Solar Day*.

3477. Let $p=\frac{c+\frac{1}{3}b}{c+b}$; then $pa^2=Nx$; therfore $\frac{pa^2}{x}=N=\frac{T^2}{t}$; therefore $tpa^2=T^2x$; but we have $g:x::$ Radius : Co-Sine of the Angle BAO $::r:c$; hence $\frac{gC}{r}=x=\frac{tpa^2}{T^2}$; therefore $gCT^2=rtpa^2$; conſequently $\frac{rpa^2}{gT^2}=C$, the Radius (r) being Unity. Hence ſince t, p, a, g, are given Quantities, *we have* C as $\frac{1}{T^2}$, or the *Co-Sine of the Angle* BAO *as the Square of the Time of Vibration inverſely.*

3478. If

3478. If we make $b : c :: 1 : 10$ then we ſhall have $c + \frac{1}{3}b : c + \frac{1}{2}b :: t : a :: 1436 : 1463{,}4 = a$ (3313) = Length of the Rod A E, when the Pendulum is cloſed; and the Length as well as Time of Vibration of an iſochronal Pendulum are both repreſented by (t) for the Caſe of the *Sidereal Day*; and T is $= 1440$.

3479. Again, ſince in that Caſe, $c + b : c + \frac{1}{2}b :: a : g$ (2475) $::$ 1463,4 : 1397 $= g$. Whence we have $\frac{t\,p\,a^2}{g\,T^2} = C = 83° 58' =$ A B O, whoſe Complement B A O $= 6° 02'$; therefore B A C $= 12° 04'$, the Angle to which the *Sidereal* Pendulum muſt be opened to vibrate *Solar Time*, for Uſe in common Clocks.

3480. Let the Time of Vibration of the Pendulum for the longeſt *Lunar Day* be $T = 1500$; then the Angle to which the Pendulum muſt be opened will be $C = \frac{T^2\,C}{T^2} = 66° 25'$, whoſe Complement is $23° 35'$; the double of which is $47° 10' =$ B A C, the Angle to which the Pendulum muſt be opened for the *longeſt Lunar Day*.

3481. By the ſame Theorem, putting T = to any Number of Minutes in the Day propoſed for a given Planet (3323,) you find the Angle to which the Pendulum muſt be opened to vibrate true Time for that *Planetary Day*. But to ſave the Reader Trouble, I have here ſubjoined a Table by which the *Planetary and Lunar Scales* may be added to a Pendulum of this new Form.

′		° ′	′		° ′
1436	——	0 0	1455	——	13 5
1436¼	——	1 31	1460	——	14 40
1436½	——	2 8	1465	——	16 5
1436¾	——	2 37	1470	——	17 24
1437	——	3 3	1475	——	18 36
1438	——	4 17	1480	——	19 42
1439	——	5 14	1485	——	20 45
1440	——	6 2	1490	——	21 45
1445	——	9 2	1495	——	22 41
1450	——	11 15	1500	——	23 35

3482. Ac-

3482. According to this Table the Arch of a Circle may be divided into *Degrees* on one Part, to 23° 35′ on each Side the Vertical Line; and then into 64′ of Time on another Part, contiguous to the Degrees; by which a proper Scale will be conſtructed for the Pendulum which may thereby be adjuſted to the Time or Length of any given *Planetary Day*.

3483. It muſt be left to the Ingenuity of the Clock-maker to contrive its Application in the beſt Manner, and to determine the Length of the Pendulum, which is done by taking the Length in *Tenths* of an Inch proportional to the Numbers before made uſe of for the Lengths of ſimple and compound Pendulums. Thus the Length of $\frac{1}{2}$ Second Pendulums being 98, you ſay, as 1444 : 1463,4 :: 98 : 99,32 = a = A B, in Tenths of an Inch; the Length A B is therefore $9,\frac{93}{100}$ Inches.

3484. Hence it appears how well adapted this *Angular Pendulum* is for meaſuring *Planetary Days*; and perhaps it may be found in ſome Reſpects more uſeful for that Purpoſe, than that we have formerly deſcribed (Chap. XI.) However, this has a Property which that had not, *viz.* of having the Time of a Vibration protracted or augmented to what Degree you pleaſe. For Example; ſuppoſe the above Pendulum A B were required to vibrate *Seconds* exactly. Then $T : \mathcal{T} :: 1 : 2$; and $T^2 : \mathcal{T}^2 :: 1 : 4$; therefore $\frac{T^2 C}{\mathcal{T}^2} = C$ (3480,) the Co-ſine of the Angle B A O = 76° 54′ in this Caſe; in that of the uniform Rod or Ruler it was 75° 32′. (3468.)

CHAP. XIX.

The THEORY *of ſeveral new* Eliptical *and* Horizontal PENDULUMS *of different* FORMS, *which ſhall vibrate in any given* TIME.

3485. BESIDES the Methods laid down for conſtructing a *Univerſal* or *Planetary Pendulum*, there is yet another, which as it contains ſome other Properties very ſingular and

and curious, I imagine the ſpeculative as well as the practical *Horologiſt* will be pleaſed with the following Account of it.

3486. Suppoſe A B an inflexible Line (without Weight;)* and in any Point taken therein, as L, let it be propoſed to affix thereto at right Angles, a heavy Rod or Bar F L G, biſected in L, which being moved *(in latus)* ſideways, ſhall oſcillate in the ſame Time with the ſimple Pendulum A B, the Point A being the Point of Suſpenſion to both. Let m be the generating Point by which the Rod is deſcribed; and join A m. Put A L $= x$, L $m = y$, A $m = d$, and A B $= n$.

3487. Then $dd = x^2 + y^2$; and $d^2 \dot{y} = x^2 \dot{y} + y^2 \dot{y} =$ the Fluxion $\dot{y}$ of the Particles in the Rod F G multiplied by the Square of its Diſtance; the Fluent whereof is $x^2 y + \frac{1}{3} y^3$ which therefore is as the Sum of all the Particles, each multiplied by the Square of its Diſtance from the Point of Suſpenſion A; the Weight of the Rod is as its Length, or half its Length, *viz.* as $(y;)$ therefore $\frac{x^2 y + \frac{1}{3} y^3}{xy} = n = \frac{x^2 + \frac{1}{3} y^2}{x}$ $=$ A B. (1094.)

3488. Hence we have $x^2 + \frac{1}{3} y^2 = nx$; and therefore $\frac{1}{3} yy = nx - x^2$. Now this Equation is evidently that of an Ellipsis (767,) as that of the Angular Pendulum was an Equation to a *Circle* (3472.) Whence it appears the two Extremes F and G of the Rod are in the Perimeter of an *Ellipſis*.

3489. To determine the Species of this Ellipſis, in the Equation $y^2 = 3nx - 3x^2$, let $x =$ A C $= \frac{1}{2} n$, the Semi-Conjugate Diameter, and then $y =$ C E $= \frac{1}{2} a$, the Semi-Tranſverſe, which Values ſubſtituted for x and y in the Equation give $3n^2 = a^2$, whence $a^2 : n^2 :: 3 : 1 :: a : p =$ *Parameter* of the Ellipſis (764.) Therefore $p = 1$; $a = 3 =$ D E; and $n = \sqrt{3} =$ A B; by which the Ellipſis is truly limited or ſpecified.

3490. It may be proper here to obſerve to the young Reader that there is a Diſtinction made (by Mathematicians) of Analytic Problems, *viz.* into *Plane* and *ſolid Problems*. They call that a Plane Problem when the Quantities in the Equation are but of *two Dimenſions* (as a *Plane* has only *Length* and *Breadth*;) thus the

* See Fig. 1. in the Plate entituled, New Pendulums of different Kinds.

the Equation $y^2 = xn - x^2$, is a *Plane one*, becauſe $y^2 + x^2 = nx$ = Rectangle of two Dimenſions, n and x. But the Equation we have juſt now found $yy = 3nx - 3xx$ (or $y^2 + 3x^2 = 3nx$) (2488,) has three Dimenſions, (*viz.* 3, n, x;) and is therefore a *ſolid Problem*, as a *Solid* has *Length*, *Breadth*, and *Thickneſs*.

3491. The *Plane Problem*, we obſerved (3473) makes the whole Circle, or any two Points or Parts of it equidiſtant from A (Fig. 11.) iſochronal Pendulums. But the preſent *ſolid Problem*, ſhews that any Line terminated by the Ellipſes at Right Angles to the Conjugate A B, and conſequently that the Plane of the Ellipſis itſelf, or any Part of it between one or two right Lines, (normal to the conjugate Axis A B) will all perform their Oſcillations in equal Time.

3492. Now this Form of a Pendulum is very well ſuited for ſhort Vibrations, and may therefore be applied to Clocks of a *portable* or *Table Form*, and in a Size as ſmall as you pleaſe. For there are three different Sorts of Pendulums that you may make out of this *Elliptic Plane*.

3493. The *Firſt* is a Pendulum A M N with *horizontal Length* only, oſcillating on the Point A; and which therefore has this peculiar Excellence, that its Vibrations are not liable to be diſturbed by either the *different Force of Gravity* in different Regions of the Earth, or by different *Degrees of Heat and Cold*, as thoſe of all long Pendulums are. Conſequently this Pendulum does not ſtand in Need of that conſtant Correction neceſſary in all that have vertical Length. It is alſo on this Account better adapted to Sea-uſe; and is much leſs liable to be agitated or put out of Order than long Pendulums by the Motion and Toſſings of the Ship.

3494. The *Second Pendulum* which this elliptic Plane affords, has the Length of the Conjugate A B, and a Bob O P B, which is a Segment of the Plane on its loweſt Part. This Bob has two Peculiarities, *viz.* (1) it has properly no Center of Magnitude or Oſcillation which the Clock-maker has any Need to concern himſelf about, as in common Bobs (3292 to 3296.) Then (2) it has a Form extremely well ſuited to avoid the Reſiſtance of the Air, as is evident by Inſpection. On both theſe Accounts

this Pendulum (which is both *vertical* and *horizontal)* may claim Preference of all ſimple Pendulums of the ſame Length A B.

3495. The *Third Pendulum* derived from the Ellipſis is one of the *univerſal* Kind, or that may be adapted to a Clock for ſhewing *Planetary Time*. For this Purpoſe we are to take the longeſt Line or *Tranſverſe Diameter* D E of the Ellipſis. For it is evident from the Equation (2487,) that while A C = *x* continues the ſame, the Length of an iſochronal Pendulum (*n*) will encreaſe with an Increaſe of C E = *y*; for Example, if A B = 98 Tenths of an Inch, (to ſwing $\frac{1}{2}$ Seconds,) then C E = 84,7; and A C = 49. for *Solar Time*; but if it be required to ſhew the Time for the longeſt *Lunar Day*, the Time of Vibration muſt be increaſed in the Ratio of 1440 to 1500 (3323,) and conſequently the Lengths of the iſochronal Pendulums in the duplicate Ratio thereof, or as 1440^2 to 1500^2, as ſhewn (1116,) this will give A B = 106 = *n*, for the ſaid Lunar Day when we get *y* = C Q = 91; then C Q — C E = E Q = 6,3 = the Length of the Part to be added at each End to accommodate the Pendulum for the Extent of the *Lunar Scale*. And thus may the Increaſe of Length be found for *Mercury*, *Venus*, *Mars*, &c.

3496. I have already obſerved, that a *Fourth Sort* of *Pendulum* reſulting from this Elliptic Equation is the Plane A E B D of the Ellipſis itſelf; nicely ſuſpended on the Point A; for then vibrating Sideways from C towards D and E, it will be perfectly iſochronal to the ſimple Pendulum, whoſe Length is A B. And thus much, at preſent, may ſuffice for the *Theory of Elliptical and Horizontal Pendulums*, of which we ſhall make ſome further Uſe hereafter.

CHAP.

N. B. *The Reader is deſired to correct an Error in Inſtitution* 3384, *by writing* $\frac{CE^2}{CN}$ *for* $\frac{2CN}{CE^2}$, *and* $\frac{NC^2}{CE}$ *for* $\frac{2CE}{NC^2}$. *Alſo to craſe Inſtit.* 3383 *entirely, as being an Over-ſight.*

New PENDULUMS of different KINDS.

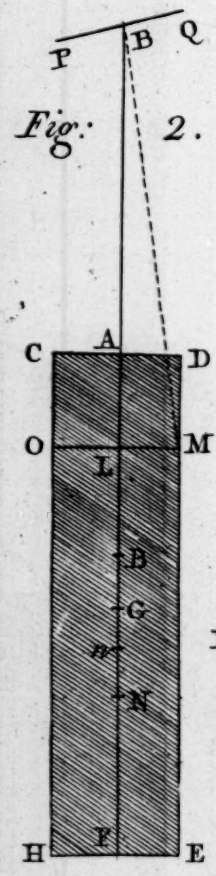


Fig: 2.

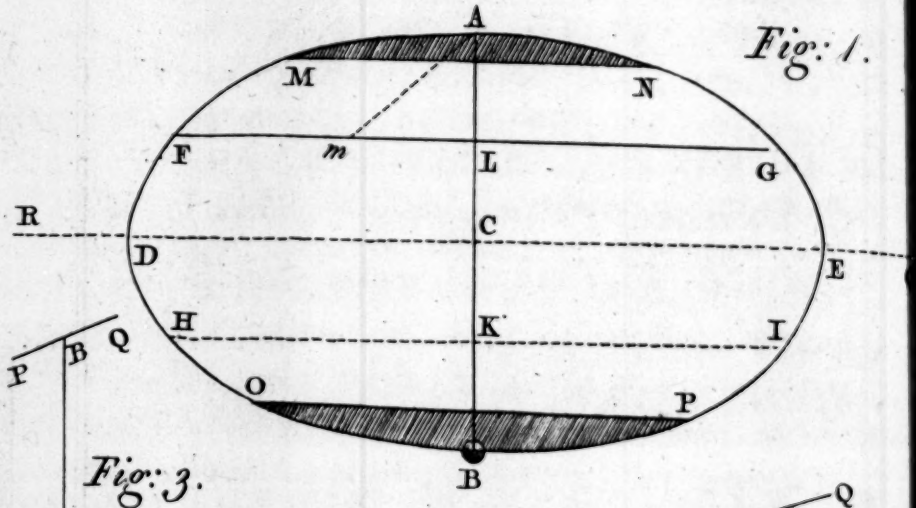


Fig: 1.

Fig: 3.



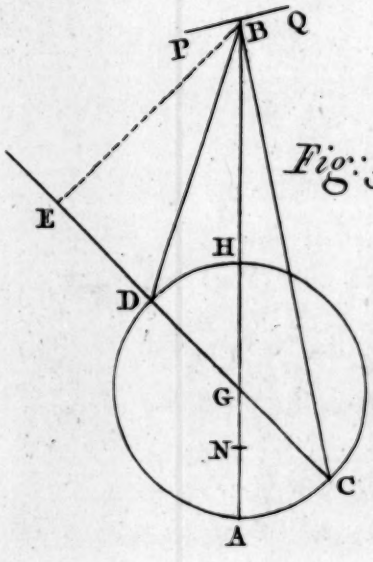


Fig: 5.



Fig: 6.



Fig: 4.

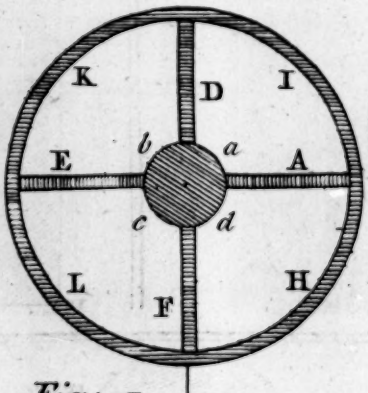


Fig: 7.

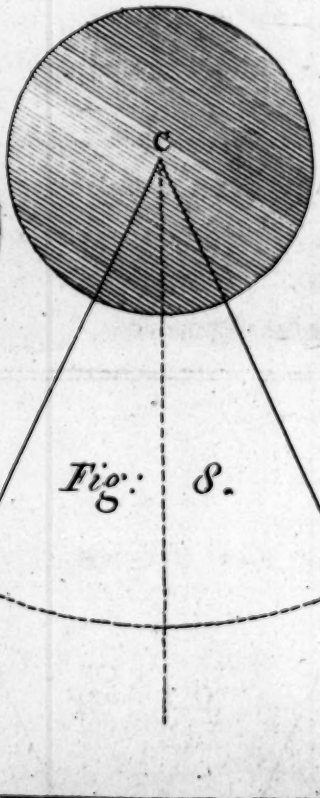


Fig: 8.

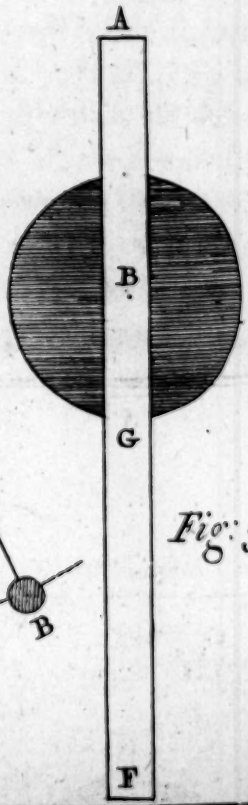


Fig: 9.

CHAP. XX.

The THEORY *and* CONSTRUCTION *of different Sorts of* UNIFORM PENDULUMS, *which, in a* given LENGTH, *ſhall vibrate in any* given TIME; *with ſeveral curious Particulars relative to* CLOCK-WORK, *regulated by ſuch* Pendulums.

3497. MR. GRAHAM (ever memorable for his ſingular Skill and Execution in *Clock and Watch Work*) having tried many Experiments with different Sorts of Metals combined in the Rod of a Pendulum to correct their Irregularities by their different Expanſion or Contraction with Heat and Cold, found them all to little or no Purpoſe; he thought of another Method of effecting the ſame, which was by Means of a Pendulum conſiſting of one Part ſolid and the other Fluid, viz. a *Glaſs Tube filled with Quickſilver*.

3498. This *Mercurial Pendulum*, 'tis eaſy to underſtand, would have its Dimenſions encreaſed or contracted each Way from the Center of *Oſcillation* by Heat and Cold; for the Tube, as a *Solid*, would encreaſe a little downwards, and the *Mercury*, being a Fluid, would have its Column encreaſed upwards from the ſame Cauſe; and therefore the Center of Gravity in one being carried down, and in the other raiſed upwards, 'tis plain, if theſe Increments different Ways of the ſolid and fluid Parts could be nicely proportioned as they ought, the Center of Oſcillation, would not be in the leaſt affected thereby, but would conſtantly be at the ſame Diſtance from the Point of Suſpenſion, and the Pendulum vibrate, of courſe, in equal Time.

3499. Accordingly he conſtructed a Pendulum of this Sort, and after many Trials, ſucceeded ſo well, that when he compared one with a Clock which he kept for a *Standard* or *Regulator* (whoſe Weight of Pendulum was 60lb) and which he obſerved

altered not more than 12 or 14″ in 24 Hours between Winter and Summer, he found by 3 Years and 4 Months constant Observation, that the Irregularity of the Clock with the *Quickſilver Pendulum*, did not, when greateſt, exceed a 6th Part of that of the other Clock, and for the moſt Part of the Time it did not exceed an 8th or 9th Part. In this Clock, the Tube was Braſs and varniſhed on the Inſide.

3500. But as a Pendulum of this Form, to beat Seconds, muſt be near 5 Feet long, it will be worth while here to ſhew how any *uniform Solid Pendulum* may be conſtructed, that, in a *given Length*, ſhall vibrate in a *given Time*; eſpecially as many curious Particulars will reſult from the Demonſtration of the Theory, and ſuch as we think not a little intereſting in a *Treatiſe* of *Clock-work*.

3501. Let CDEH (Fig. 2.) be a *Rectangle* ſuſpended on the Point B by a Thread AB, in its middle Point A; produce BA to F; and drawing OLM parallel to CD; put $AB = a$, $AL = x$, $LM = y$, and $BM = d$. Then, ſuppoſing the Rectangle DH to vibrate (*in latus*, or) Sideways about the Axis PQ, the Force of all the Particles in the Line LM will be $\overline{a + x}^2 \times y + \frac{1}{3}y^3$ (3487) $= a^2y + 2axy + x^2y + \frac{1}{3}y^3$, which multiplied by $\dot{x}$ gives $a^2y\dot{x} + 2ax\dot{x}y + \dot{x}x^2y + \frac{1}{3}y^3\dot{x}$, whoſe Fluent $a^2yx + ax^2y + \frac{1}{3}x^3y + \frac{1}{3}y^3x$ is the Force of the Rectangle CM, or of the Rectangle CE, when x or AL becomes AF.

3502. Then let G be the Center of Gravity in the Rectangle, and becauſe the Fluxion of the Weight $\dot{x}y$ multiplied by the Diſtance $\overline{a + x}$ is $a\dot{x}y + yx\dot{x}$, the fluxionary Moment of the Rectangle, therefore the Moment itſelf will be the Fluent $axy + \frac{1}{2}yx^2$. Conſequently $\frac{a^2yx + ayx^2 + \frac{1}{3}x^3y + \frac{1}{3}y^3x}{ayx + \frac{1}{2}yx^2}$ $= \frac{a^2 + ax + \frac{1}{3}x^2 + \frac{1}{3}y^2}{a + \frac{1}{2}x} = Bn$, the Diſtance of the *Center of Oſcillation* from the Point of Suſpenſion B, when $x =$ AF.

3503. Let $a = 0$, or let the Rectangle be ſuſpended in its Vertex

Vertex A, then will the Diſtance of the Center of Oſcillation $AN = \frac{\frac{1}{3}x^2 + \frac{1}{3}y^2}{\frac{1}{2}x} = \frac{2}{3} \times \frac{x^2 + y^2}{x} = \frac{2}{3}x + \frac{2}{3}\frac{y^2}{x}$, and when the Breadth of the Rectangle is infinitely ſmall, or $y = 0$, it becomes a Right Line AF, and then $\frac{2}{3}\frac{x^2}{x} = \frac{2}{3}x = \frac{2}{3}AF$, as before ſhewn (1129.)

3504. If the Point of Suſpenſion B be within the Rectangle, then $AB = a$, will be negative, but a^2 will be poſitive; and the Diſtance of the Center of Oſcillation will be $\frac{a^2 - ax + \frac{1}{3}x^2 + \frac{1}{3}y^2}{\frac{1}{2}x - a}$.

3505. If $2y = x$, then the *Rectangle* becomes a *Square*; and the Equation for the *Center of Oſcillation* becomes $\frac{a^2 \pm ax + \frac{5}{12}x^2}{\frac{1}{2}x \pm a}$; and if $a = 0$, or the Square be ſuſpended in the vertical Point A, the Theorem is $\frac{\frac{5}{12}x^2}{\frac{1}{2}x} = \frac{5}{6}x =$ Diſtance of the Center of Oſcillation in the Line AF, from the Point A.

3506. If we ſuppoſe x leſs than $2y$, then is the Rectangle to be conſidered as a *Horizontal* one; but the ſame Equation (3504) gives the Diſtance of the Center of Oſcillation in the Line AF, continued out, if required.

3507. If $x = 0$, this Rectangle then becomes an *horizontal Line* $CD = 2y$; and the Equation (3502) becomes $\frac{a^2 + \frac{1}{3}y^2}{a}$, the ſame Expreſſion as was found for the Center of Oſcillation in ſuch a Line before. (3488.)

3508. Let us now ſuppoſe the Plane CDEH (Fig. 3.) to move parallel to itſelf in the Direction of the Line IK, and thereby generate the Solid or Parallelopiped CS in flowing from A to K, or the Solid CT in moving from A to I. Put AI, or AK, $= z$; then the Force of the Rectangle or Plane CE, which is $a^2yx + ax^2y + \frac{1}{3}x^3y + \frac{1}{3}y^3x$ (3501) being multiplied by $\dot{z}$, will be the Fluxion of the Force of the Solid, *viz.* $a^2yx\dot{z} + a^2x^2y\dot{z} + \frac{1}{3}x^3y\dot{z} + \frac{1}{3}y^3x\dot{z}$. And $axy\dot{x} + \frac{1}{2}x^2y\dot{z}$ is the Fluxion of the *Momentum* thereof; therefore the Fluent of the former divided by that the latter, is

$\frac{a^2xyz + ax^2yz + \frac{1}{3}x^3yz + \frac{1}{3}y^3xz}{axyz + \frac{1}{2}x^2yz} = \frac{a^2 + ax + \frac{1}{3}x^2 + \frac{1}{3}y^2}{a + \frac{1}{2}x}$

the Distance of the Center of Oscillation from the Point B, and is the very same as was found for the Rectangle or Plane C E (3502.)

3509. From all which it is evident that the Center of Oscillation is not affected by the *Thickness* of the Parallellopiped, but only by its *Length* A F $= x$, and *Breadth* C D $= 2y$; also, that the Increase of its Distance from the Point of Suspension is always as $\frac{2}{3}\frac{y^2}{x}$, or $\frac{2}{3}$ of a *Third Proportional to its Length and half Breadth* (2503.) Therefore in an *horizontal* Plane, when $y = x$, we have its Distance $= \frac{2}{3}x + \frac{2}{3}\frac{y^2}{x} = \frac{4}{3}x$, or $\frac{1}{3}x$ below the Plane in the Line A F continued out.

3510. If $z = 0$, and $y = 0$, the Solid degenerates into a Line or Rod A F, suspended by a String (without Weight) A B $= a$; and then the Distance of the Center of Oscillation will be $\frac{a^2 + ax + \frac{1}{3}x^2}{a + \frac{1}{2}x}$ (3502.)

3511. When the Rod A F is suspended at one End A, then the Distances of this Point A and the Center of Oscillation N from the Center of Gravity G, we called g and d (3288.) And if the said Rod be suspended from the Point B by the inflexible Line A B, put the Distance B G $=$ G and the Distance G $n =$ D; then will G D $= gd$, in all Cases. For $\frac{a^2 + ax + \frac{1}{3}x^2}{a + \frac{1}{2}x}$ $=$ B $n = N$. Therefore, $a^2 + ax + \frac{1}{3}x^2 =$ G N; and $\frac{a^2 + ax + \frac{1}{3}x^2 - \overline{a + \frac{1}{2}x}^2}{a + \frac{1}{2}x} = \frac{GN - G^2}{G} =$ D; whence $GN - G^2 =$ G D, but $GN - G^2 = \frac{1}{12}x^2 = \frac{1}{6}x \times \frac{1}{2}x = gd$; therefore G D $= gd$; and consequently G : g :: d : D.

3512. After the same Manner it is proved for the Parallellogram C E; and all other vibrating Bodies, *that the Distances of the Points of Suspension and of the Centers of Oscillation from the Center of Gravity are in an inverse Ratio*, and are therefore, *mutually interchangeable, or convertible into each other.* Thus for Example, if

if the Rectangle C E was to be ſuſpended upon the Point N (which is the Center of Oſcillation when ſuſpended at A) then the Plane will be inverted, and the Center of Oſcillation will be in the End or loweſt Point A of the middle vertical Line.

3513. Or thus; Let O be the Center of Oſcillation in the Plane or uniform Pendulum C E; then $AO = \frac{2}{3} AF$, when the Point of Suſpenſion is in the Vertex A. Then let the Pendulum be ſuſpended on the Point B, ſo that $AB = \frac{1}{3} AF$ (Fig. 4.) then will the Center of Oſcillation be removed from O to the End or Point F in N, and the Diſtance B N being equal to A O, the Time of Vibration on the Centers A and B will be the ſame.

3514. If the Point of Suſpenſion be taken nearer to the Center of Gravity G, the Center of Oſcillation will go out of the Plane to ſome diſtant Point in the Right Line A F produced. Thus if *(h)* be the Point of Suſpenſion, *B* will be the Center of Oſcillation. And if $Gh = Gn$ (in Fig. 2.) then will GB here be equal to G B there (according to 3511, 3512.) As it will ever be $AG \times GN = BG \times Gn = Gh \times GB$, and ſo on.

3515. Hence it is evident, ſince in the Fraction $\frac{AG \times GN}{Gh} = GB$, the Numerator is conſtant, we ſhall have GB as $\frac{1}{Gh}$; and therefore GB will be infinite, when $Gh = 0$, or when the Plane is ſuſpended on the Center of Gravity G. Conſequently, *in any given uniform Pendulum* C E *a Point of Suſpenſion (h) may be found or aſſigned between* B *and* G, *ſuch that the Time of Vibration ſhall be equal to that of a common Pendulum of any given Length not leſs than the Minimum of this Sort.*

3516. As the Time of a Vibration in a Pendulum of this Form, is, when a *Maximum*, infinite; it may be proper to determine its Quantity when a *Minimum* or the leaſt it can be. To this End put $BG = x$, and $Gn = y$ (Fig. 2.) and $x + y = p$, an iſochronal Pendulum; this, when a *Minimum* gives $\dot{x} + \dot{y} = 0$; and therefore $\dot{x}y = -y\dot{y}$; but ſince $xy =$ a conſtant Quantity, (3514,) we have $\dot{x}y + x\dot{y} = 0$; and hence $x\dot{y} - y\dot{y} = 0$, and ſo we find $x = y$ or $BG = Gn$, when the Vibration is of the leaſt Time poſſible.

3517. But

3517. But it is always $xy = dg$ (3511,) and therefore in this Case $x = y = \sqrt{gd} = \sqrt{\frac{1}{12}S^2}$ (3287,) therefore $x + y = p = 2x = 2\sqrt{\frac{1}{12}}$, putting $= S = AF = 1$; that is, the whole Length AF is to the Length of the ſhorteſt Pendulum (p) as 1 to 0,577.

3518. That this Form of a Pendulum may be adapted to geeral Uſe, or to meaſure any Time from the *Sidereal* to the *Lunar Day*, in a given Length thereof S, Let $\frac{a^2 - aS + \frac{1}{3}S^2 + \frac{1}{3}W^2}{\frac{1}{2}S - a} = p$, the Length of an iſochronous Pendulum proper for the Time of the given Day (3504.) (Here $S = x = AF$ the Length, and $W = y = \frac{1}{2}CD$, half the Breadth of the given Pendulum.) Then $a^2 - aS + \frac{1}{3}S^2 + \frac{1}{3}W^2 = \frac{1}{2}pS - ap$; and $a^2 - aS + ap = \frac{1}{2}pS - \frac{1}{3}S^2 - \frac{1}{3}W^2 = s$; and putting $p - S = t$, we have $a^2 - ta = s$; and $a^2 - ta + \frac{1}{4}t^2 = s + \frac{1}{4}tt$; and therefore $a = \sqrt{s + \frac{1}{4}t^2} - \frac{1}{2}t^2$; and thus the Diſtance of the Point of Suſpenſion from A is found for any Value of (p) from that of a Pendulum of 1436′ for a *Sidereal*, to 1500′ for the longeſt *Lunar Day* (3323.)

3519. Hence then it appears, (1) That an uniform Pendulum entirely ſolid, may be made to vibrate in a given Time. (2) That it may be of any given Length and Width. (3) That it may be adapted to regulate Clocks for meaſuring different Lengths of Days. (4) And that the Errors they are ſubject to from *Heat and Cold*, or *Difference* of *Gravity* in different Regions of the Earth, are very ſmall, if not altogether inconſiderable; as the Point of Suſpenſion is here at ſo ſmall a Diſtance from the Center of Gravity; and the Length of the Pendulum itſelf ſo much leſs than an iſochronal One of the common Sort of 58,8 Inches (3298.)

CHAP.

CHAP. XXI.

The THEORY *of* Rotulary PENDULUMS, *conſtructed in a new* METHOD, *which, in very* ſhort LENGTHS, *ſhall vibrate in any* given TIME.

3520. WE cannot diſmiſs the Subject of Pendulums, till it is rendered more compleat by a few Theorems and Obſervations which yet remain to be demonſtrated, as upon them, ſome conſiderable Improvements in the Doctrine of Pendulums will appear to depend. The great Lengths of a *Second Pendulum* requiring ſo large and cumberſome a Clock-caſe has long been an Objection, which the *Rotulary Pendulum* of Mr. SULLY was deſigned to remove; but this may be done by Methods more facile and natural, as we ſhall now proceed to ſhew.

3521. Though our Methods of ſhortening the *Second Pendulum* will require the Uſe of the *Rotula*, yet its Application will be to the *natural*, and not to an *artificial Pendulum*, as *Sully*'s was. And theſe Methods will admit of as much eaſier Conſtructions as they are in themſelves more natural and better adapted to Uſe, eſpecially at SEA, where *Sully*'s Pendulum will be much more affected by the Motion of the Ship, than any of theſe, which we have now to propoſe.

3522. The THEORY on which theſe Methods of reducing the Size of Second-pendulum Clocks are founded, are as follow. Let it be required to find the *Center of Oſcillation* of the *Periphery* A D H C of a Circle ſuſpended at B by an inflexible Line B H and vibrating (*in latus*) about an Axis P Q. Suppoſe C and D two Particles or Weights in the Periphery, and draw the Lines C B, D B, and the Diameter C D, which continue out to E, and let fall the Perpendicular B E upon it; laſtly, through the Center G draw the Vertical B H A.

3523. Then (putting $CG = GD = r$, $GB = a$, and $DE = e$,) we have $CB^2 = CG^2 + BG^2 + 2CG \times GE$ (639) $= r^2 + a^2 + 2r \times \overline{r + e}$. Alſo we have $BD^2 =$

$BG^2 - GD^2 - 2GD \times DE = BG^2 + GD^2 - 2GD \times GE = a^2 + r^2 - 2r \times \overline{r + e}$; therefore $CB^2 + BD^2 = 2a^2 + 2r^2$; or the Sum of the Forces of the *two* Particles C and D is $\overline{a^2 + r^2} \times 2$. Therefore putting P = Periphery of the Circle, the Force of the whole Periphery will be $\overline{a^2 + r^2} \times P$.

3524. In a Circle whoſe Diameter is 1, the Periphery is $3{,}14159 = p$; then $1 : p :: 2r : 2rp = P$, which ſubſtituted for P in the foregoing Expreſſion, gives $2rpa^2 + 2pr^3$, for the Force of the Periphery to reſiſt to angular Motion from the Diſtance BG. When $a = r$, or the Suſpenſion is at the Vertex H, the Force is, as $r^3 \times 2p$, or as $2rr \times P$. But when $a = 0$, or the Circle is ſuſpended on its Center G, the Force thereof is $r^3 \times 2p$, or $r^2 \times P$.

3525. The *Momentum* of the Periphery is aP; therefore $\frac{a^2 + r^2 \times P}{aP} = \frac{a^2 + r^2}{a} = BN = C$, the Diſtance of the *Center of Oſcillation*. When $a = r$, then $2r = HA = C$, as we before ſhewed from other Principles. When $a = 0$, then $\frac{a^2 + r^2}{0} = C$, infinite; or the Circle will in that Caſe be at reſt.

3526. If we conſider the Radius AG as a variable Quantity then ſince the Force of the Periphery is $2prra^2 + 2pr^3$, or $a^2P + Pr^2$; therefore $Pa^2\dot{r} + Pr^2\dot{r}$, or $2pr\dot{r}a^2 + 2pr^3\dot{r}$ will be the Fluxion of the Force of the circular Area ACHD (Fig. 6.) The Force then of that Area will be $Pra^2 + \frac{1}{3}Pr^3$, or $pr^2a^2 + \frac{1}{2}pr^4$. Wherefore when $a = 0$, or the Circular Area or *Rotula* ADHC moves on the Center of Gravity G, the Force is $\frac{1}{2}pr^4$ as we formerly found by another Method. (3458.)

3527. Then becauſe the Area $ACED = \frac{1}{2}rP = rrp$, therefore $aprr$ will be the Momentum thereof; conſequently $\frac{a^2pr^2 + \frac{1}{2}pr^4}{aprr} = a + \frac{r^2}{2a} = BN = C$, the Diſtance of the Center of Oſcillation from the Point of Suſpenſion B.

3528. What we have hitherto ſaid is upon Suppoſition that the Rod BG, by which the Circle is ſuſpended is without

Weight

Weight; but if its Weight be conſiderable, call it *(b)*; then its Force will be $\frac{1}{3}a^2 b$ (3306.) And the Diſtance of the Center of Oſcillation will, in this Caſe, be $\frac{\frac{1}{3}a^2 b + 2a^2 + r^2}{\frac{1}{2}ab + 2a} = C$. And, if $r = 0$, then $\frac{\frac{1}{3}a^2 b + 2a^2}{\frac{1}{2}ab + 2a} = C$. Which Expreſſion is analogous to that for the Pendulum formerly deſcribed (3306.)

3529. We may now proceed to the Application of the Wheel or *Rotula* to the natural Pendulum for reducing Clocks to a *conciſe and portable Form*, and at the ſame Time to have the Advantage of a ſlow Vibration, and, of Courſe, a much greater Equability of Motion, than common ſhort Pendulums can be expected to have. Firſt, Let a WHEEL be made uſe of for this End (as in Fig. 7;) then if the Thickneſs of its Perimeter be inconſiderable, let the Weight thereof be called P, and the Radius $= R$, and we ſhall have its Force as R^2 P (3524.)

3530. Again, the Force of any heavy uniform Rod S ſuſpended on its Middle Point or Center of Gravity will be as $\frac{1}{12}$ S^3 (3286.) Alſo the Weight of the Rod is as S, and $\frac{1}{2}$ S $= R$, therefore the Force of ſuch a Rod or double Radius to reſiſt angular Motion will be $\frac{1}{3}RRS$; therefore, if the Weight of all the Radii A, D, E, F, *&c.* be called W, then their Force will be $\frac{1}{3}RRW$.

3531. The central circular Part of the Wheel *(abcd)* may be conſidered as a ſmall *Rotula* whoſe Radius is r, and its Force is $\frac{1}{2}wr^2$ (3529) putting $w =$ to its Weight: For the *Weight* is as the *Area*, which is $\frac{1}{2}$ rP (3458.) $= pr^2 = w$. Therefore the Sum of the Forces of all the Parts of the Wheel will be R^2 P $+ \frac{1}{3}R^2$ W $+ \frac{1}{2}wr^2$.

3532. And becauſe the Weight of the Pendulum muſt, in this Caſe be conſidered in every Part, and particularly expreſſed, we will put the Weight of the circular Area MNO $=$ p $= pr^2$ (3527.) Whence the Center of Oſcillation of the whole Pendulum will be $\frac{\frac{1}{3}a^2 b + a^2 p + \frac{1}{2}pr^2}{\frac{1}{2}ab + ap}$.

3533. Therefore by connecting the Wheel and Pendulum

together, the Center of Oſcillation reſulting from both will be $\frac{R^2 P + \frac{1}{3} R^2 W + \frac{1}{2} w r^2 + \frac{1}{3} a^2 b + a^2 p + \frac{1}{2} p r^2}{\frac{1}{2} a b + a p} = C.$

3534. We may hereafter ſee more of the Uſe of the foregoing Theorem in other Parts of Mechanics; but in the Buſineſs of Pendulums the *Rotula* is much preferable to the Wheel and the Theorem is more ſimple; for putting the Weight of the Rotula = W, and its Radius = R, then its whole Force of *Inertia* will be $\frac{1}{2} W R^2$ (3530,) and when the Pendulum is connected with it in the beſt Manner, its Weight will be inconſiderable, therefore $b = 0$; and the Theorem for the Center of Oſcillation is $\frac{\frac{1}{2} W R^2 + a^2 p + \frac{1}{2} p r^2}{a p} = C.$

3535. If we put $a^2 + \frac{1}{2} r^2 = n$, we have $\frac{\frac{1}{2} W R^2 + n p}{a p} = C$; and therefore $\frac{1}{2} W R^2 = a p C - n p$; whence this Analogy, $p : W :: \frac{1}{2} R^2 : a C - n$.

3536. If we put $\frac{1}{2} W R^2 + \frac{1}{4} p r^2 = S$, we have $\frac{S + a^2 p}{a p} = C$, therefore $S + a^2 p = a p C$, therefore $a p C - a^2 p = S$, and $a C - a^2 = \frac{S}{P}$; and by compleating the Square, and extracting the Root, we have $a = C - \sqrt{\frac{S}{P} + \frac{1}{4} C^2}$.

3537. In the ſame Manner may the *Angular Pendulum* (deſcribed Chap. XVIII.) be connected with the Rotula, and its Time of Vibration thereby prolonged, or its Length very much ſhortned for a given Time of Vibration. For $\frac{\frac{1}{2} W R^2 + \frac{1}{3} a a b + a^2 c}{x b + x c} = n$ (3471,) or putting $\frac{1}{3} b + c = s$, and $c + b = v$; we have $\frac{\frac{1}{2} W R^2 + s a^2}{v x} = \frac{T^2}{t}$ (3477.) Whence $\sqrt{\frac{1}{2} W R^2 + s a^2} \times t = T^2 v x = T^2 v C g$; therefore $\frac{\sqrt{\frac{1}{2} W R^2 + s a^2} \times t}{T^2 v g} = C =$ Coſine of the Angle to which the Pendulum muſt be opened, that its Vibration may be as the given Time T (3478, 3479.) This Conſtruction of the Pendulum and Rotula is ſhewn Fig. 8.

3538. Laſtly,

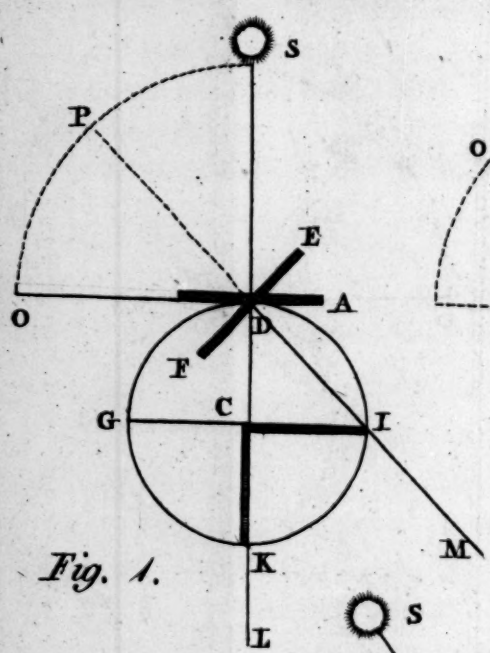


Fig. 1.

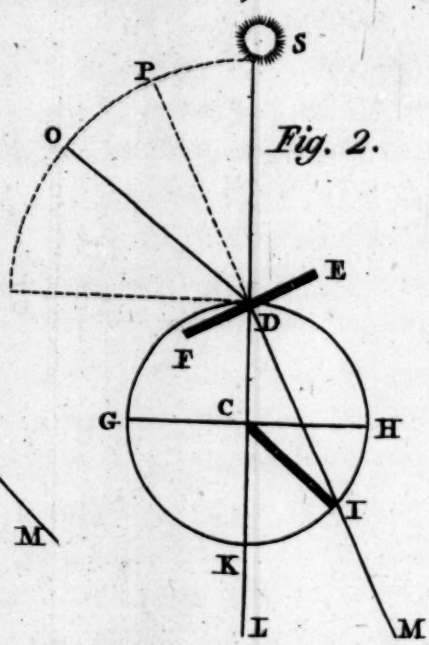


Fig. 2.

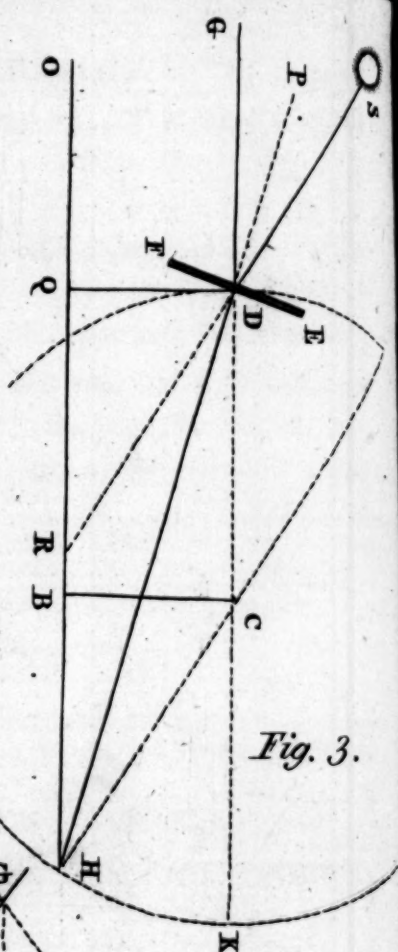


Fig. 3.

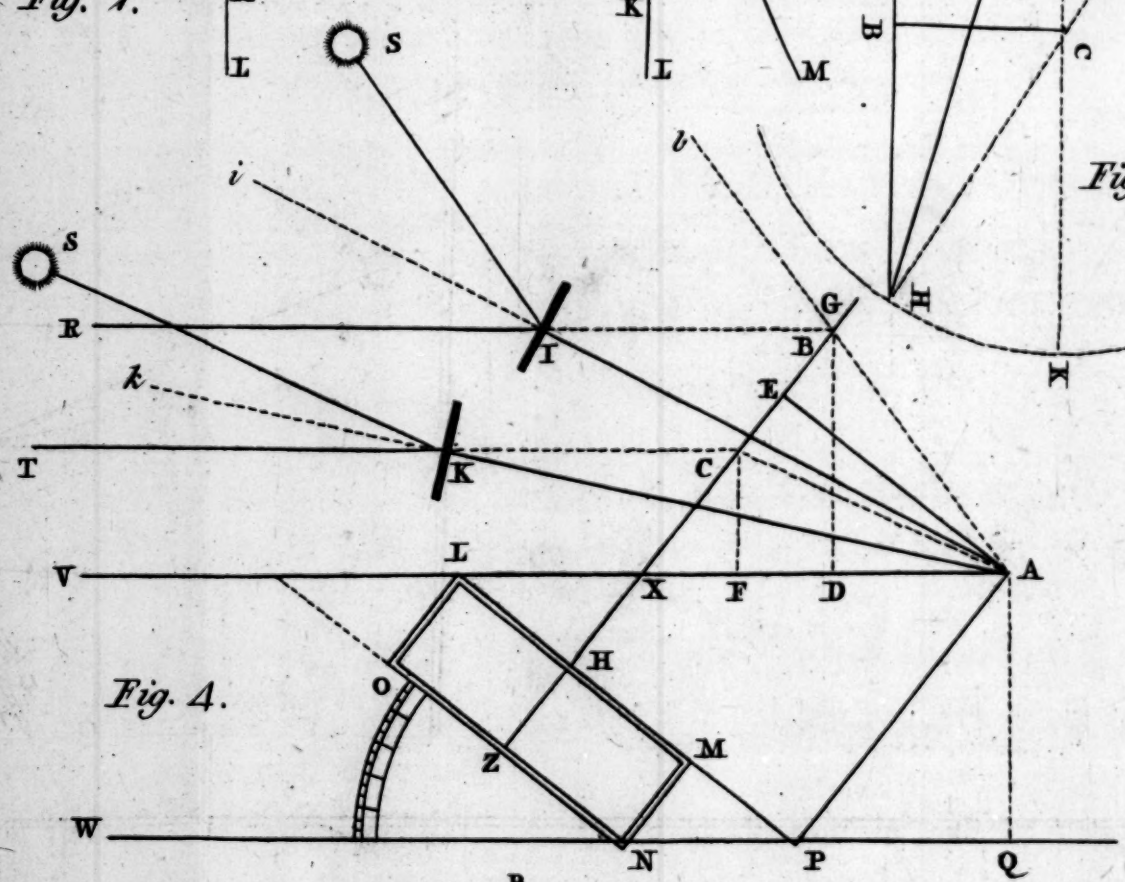


Fig. 4.

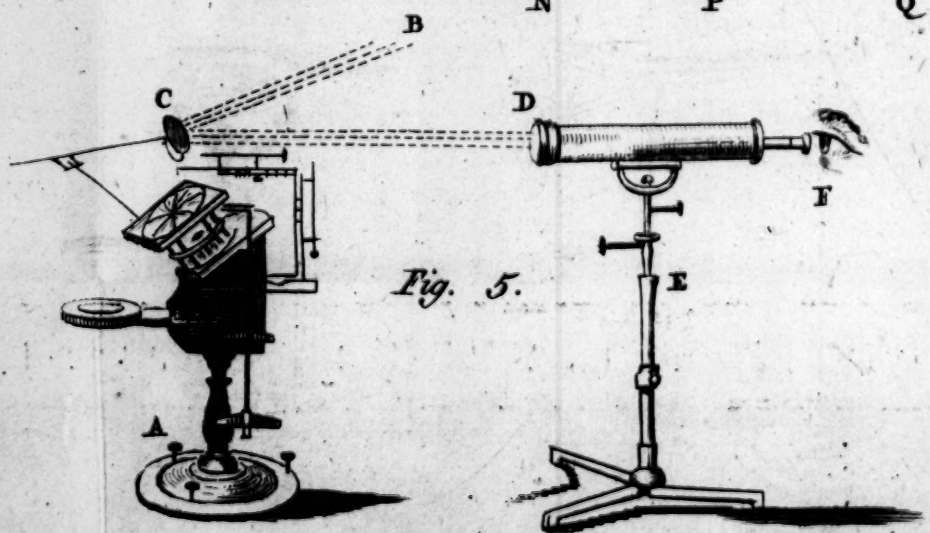


Fig. 5.

3538. Laſtly, the uniform Pendulum, deſcribed in the laſt Chapter, may with Eaſe be adapted to the *Rotula* (as in Fig. 9.) and the Theorem (3504) may be accomodated thereto; for $2yx$ is as the Area or *Weight* of the Pendulum itſelf, whence, putting $2yx = w$, we have $\frac{1}{3}y^2 = \frac{1}{3}x^2 \times \frac{1}{4}w^2$, and putting $1 + \frac{1}{4}w^2 = s$, we ſhall have the Theorem $\frac{\frac{1}{2}WR^2 + a^2 - ax + \frac{1}{3}sx^2}{\frac{1}{2}x - a}$ = C, the Length of an iſochronal Pendulum.

3539. And thus it appears how eaſy it is for the expert *Clock-Maker* to chuſe what Lengths and Forms of Pendulums he pleaſes, to vibrate in a given Time; and by this Means to render his Clocks portable and conciſe in any Degree. And further, that by ſuch compendious Conſtructions they are in the beſt Manner fitted for *Uſe at* SEA.

CHAP. XXI.

The PRINCIPLES *of* Celeſtial MECHANICS *explained, and applied to the Conſtruction of a* New HELIOSTATA, *or* Planetary CLOCK *for* fixing *the* RAYS *of* LIGHT *proceeding from the* SUN, MOON, *and* PLANETS, *by which thoſe Objects are rendered apparently at* REST *for* Aſtronomical OBSERVATIONS.

3540. HAVING ſhewn the Conſtruction of a Variety of *Planetary Pendulums*, we now proceed to conſider the Nature, Form, and Diſpoſition of a PLANETARY CLOCK, that is of a much more general Nature than the *Helioſtata* of *S'Graveſand*, which I have at large explained in my *Philoſophica Britannica*; for that is wholly confined to the *Sun*, as its Title implies; and is of a much coarſer, and more cumberſome Conſtruction, than that we have here to propoſe.

3541. It muſt be confeſſed the *Helioſtata* has been greatly improved by the learned C. G. KRATZENSTEIN in what he calls *Mechanicæ Cœleſtis Specimen Primum*, or Firſt Specimen of *Celeſtial*

Celeſtial MECHANICS, in which he has in ſome Meaſure provided for viewing the *Moon*, and *Planets* as well as the *Sun*; but we can by no Means think the Mechaniſm of his Clock, or the Manner of applying it, ſo conciſe and delicate as the Nature of the Subject both require and will admit of; this we preſume will be readily granted by any one who will pleaſe to compare the Account we here give of the Clock of our own Conſtruction with that you find of his in the *Petropolitan* COMENTARIES, for the Years 1747 and 1748.

3542. The *Rationale* he has given of this new Kind of *Celeſtial Mechaniſm* (as he very properly calls it) is exceeding good; and as it only wants to be a little ſoftened and dilated, to ſuit it to an *Engliſh* Genius, we ſhall here undertake that Office, and, we hope, to the Satisfaction of the curious Reader.

3543. The Deſign of ſuch a *Celeſtial* MACHINE being to render the SUN, MOON, and PLANETS, apparently *quieſcent*, or *at Reſt to the View*, it cannot but be eſteemed the moſt uſeful Invention in Mechanics, and beſt adapted for aſtonomical Obſervations; where the conſtant Motion of the celeſtial Bodies and the difficult and irkſome Application of long Teleſcopes create perpetual Moleſtation to the ingenious Student; and gerally, not only diſcourage, but very often entirely defeat the Enterprizes of induſtrious Obſervers.

3544. The firſt Principle of this new Theory of *Celeſtial* MECHANICS is that of *Catoptrics*, where it is ſhewn, *that the Angles contained by the incident and reflected Ray with the Perpendicular Plane are always equal to each other* (1282.) Hence it will follow, that if the incident Ray be conſidered as fixed, and the reflecting Plane be moveable, then the reflected Ray will alſo be moveable and its angular Motion will be juſt double to that of the Plane.

3545. For ſuppoſe the Radiant Object at S, and S D a Ray of Light, incident perpendicularly upon a reflecting Plane A B; with any Radius C D deſcribe the Circle D I K G to touch the Plane in the Point of Incidence D; and let D K be the vertical Diameter, and I G the horizontal one. Then it is evident that S D will repreſent not only the incident Ray, but alſo the reflected Ray as well as the Perpendicular to the Plane, for in this Caſe they all coincide.

3546. Then

3546. Then ſuppoſe the reflecting Plane A B moveable about the fixed Point D by an inflexible ſtrait Wire D L, and that it was thus moved about, till it came into the Poſition E F making an Angle A D E with its firſt Poſition, of 45°; laſtly, ſuppoſe this Motion was produced by an Arm C K moveable on the Center C, and having a Fork at the End K to receive the Wire D L; then it is plain, (1.) That while the Plane is moved from the Situation A B into E F, the Wire D L is moved into the Poſition D M interſecting the Circle in the Point I, or End of the horizontal Diameter. (2.) The Arm C K has moved into the Poſition C I. (3.) That becauſe in the Triangle D C I, the Sides D C and C I are always equal, therefore the Angles at D and I will be ever equal; and conſequently (4.) The Angle K C I will ever be equal to 2 C D M, or the angular Motion of the Arm C K is twice that of the Wire D L or of the Plane A B. (5.) That the incident Ray S D, the Perpendicular P D, and the reflected Ray D O, will ever be reſpectively parallel to the three Sides C D, D I, and I C of the Triangle C D I. And that therefore (6.) The angular Motion of the reflected Ray O D will be double that of the Plane or Glaſs A B.

3547. And *(vice verſa)* the ſame Things are equally evident with regard to the *incident Ray* being conſidered as *moveable*; and then, if its Motion be juſt twice as much as that of the Plane A B, the Poſition of the reflected Ray will be conſtant, or always the ſame. Thus for Example, if E F be the Poſition of the reflecting Plane, and O the Radiant be ſuppoſed to move through a Quadrant from O to S, then in that Time the Wire D M will move into the Poſition D L and the Plane into the Poſition A B, which angular Motion M D or E D A being but half ſo great as the angular Motion of the Radiant O D S (or I C K) it is evident, *the reflected Ray* D S *will be always in the ſame Poſition, and ſo may be conſidered as fixed or immoveable.**

3548. This is the fundamental Principle on which the Theory of the *Machina Cæleſtis*, as alſo that of the *Newtonian* SEA-OCTANT (uſually called *Hadley*'s *Quadrant)* do entirely depend. This,

* See Fig. 1 and 2 of the Plate entituled The PRINCIPLES of *Celeſtial* MECHANICS.

This, with Regard to the ſaid OCTANT, has been already ſhewn in a ſmall Treatiſe on that Subject; † and with reſpect to this CELESTIAL CLOCK, we ſhall now a little expatiate in its Illuſtration.

3549. If the SUN or PLANET were to move or circulate about us in the Horizon, it is evident, ſince the Wire or Ray D M is, in this Caſe, always in the Plane of the Horizon; and perpendicular to the reflecting Plane, therefore the reflecting Plane will always be Vertical to the Plane of the Horizon; and its Perpendicular D P will deſcribe a Quadrant while the Sun or Planet deſcribes a Semi-circle in the Horizon, at the ſame Time that it appears quieſcent or fixed, when viewed in the Reflecter E F, in the Direction S D (3547.)

3550. But if we ſuppoſe the *Sun* or *Planet* to move in a Circle parallel to the Horizon, and at a given Altitude above it, while the Eye continues in the horizontal Plane; then the Motion of the Radiant and Reflecting Plane will be in the ſame Ratio as before, but the Poſition of the ſaid reflecting Plane will now be oblique to the Plane of the Horizon, in order that the reflected Ray may be parallel to it, and continue fixed and immoveable as before.

3551. Thus if H O be the Horizon, (Fig. 3.) and the Angle S R O be that of the Sun's Altitude above it, then if we take any Diſtance R Q for Radius, and on the Point Q erect the Perpendicular Q D, interſecting the Ray S R in D, it will be the Tangent of the Sun's Altitude to the Radius Q R. Through the Point D draw G C parallel to O H; and laſtly, draw P D H to biſect the Angle of Altitude S D G. Then it is evident, if a Plane *Speculum* E F be placed with its Center on the Point D, and in a Poſition perpendicular to the Line P H, it will be that which is required for reflecting every incident Ray S D from every Part of the diurnal Circle of the Sun into one conſtant Direction D G parallel to the Horizon O H.

3552. For continue G D to K, and take D C = D R = Secant of the Altitude, and then C will be the Center of a Circle parallel to that of the Sun's Motion; and drawing C H (= C D) it will repreſent the Arm, which by its Fork at the End will always keep the Wire D H conſtantly biſecting the Angle

† The THEORY of HADLEY'S QUADRANT demonſtrated.

Angle SDG, and thereby render the reflected Ray DG permanently fixed. The above *Phænomena* belong to the *Parallel Sphere*, or to a Perſon under either Pole.

3553. But to the Inhabitants of an *oblique Sphere* the *Equator* itſelf and all its *Parallels*, make an Angle with the Horizon; and thereby cauſe that every Day, and every Part of every Day, the Altitude of the Sun above the Horizon will be variable; and conſequently there muſt, in this Caſe, be another Figure for the Conſtruction or Diſpoſition of the Speculum for fixing the reflected Ray in the Plane of the Meridian, and parallel to the Horizon.

3554. Thus let AV (Fig. 4.) be an horizontal Line, and the Angle EAV be the Elevation of the Plane of the EQUATOR EA above the ſame. Alſo let the Angle EAG be the *North Declination* of the SUN or PLANET, and the Angle EAC the *Declination, South*. Then, of Courſe, EXA is the Latitude of the Place. Now AE being Radius, EB is the *Tangent* and AB the *Secant* of the North Declination or Elevation of the Planet above the Equator EA; therefore by drawing BR parallel to AV, and therein taking BI = AB, it will give the Point I for the Place of the Center of the *Speculum* (3551,) by which the Sun-beam SI will be reflected into IR parallel to the Horizon by a Wire IA perpendicular thereto, and which continued to *(i)* biſects the Angle of the Sun's Altitude SIR = BAV = 2BAI.

3555. In like Manner, if through the Point C, a Line CT be drawn parallel to the Horizon AV, and in that we take CK = AC, then K will be the Place of the Speculum to reflect the Sun-beam SK parallel to CA, (in South Declination) into the Direction KT, by the Wire KA, biſecting the Angle CAV = SKT.

3556. From whence it is evident, the Center of the Circle parallel to the Sun or Planet's diurnal Motion, and in which the Speculum is placed, will always be in the Line BX parallel to the Axis of the World; and therefore if the ſaid Line be continued each Way to G and Z indefinitely, it may repreſent the Axis ZH of a Clock LMNO, continued out; which Clock, being placed in a Poſition parallel to the Equator AE, if it be furniſhed with an Arm HP moving on the Center H, and carrying on

its Extremity P an upright Wire or Stem P A with a Fork at A to receive the Wire I A of the Speculum, then by the Mechaniſm of the Clock the Index or Arm H P will be carried with a Motion ſimilar to that of the Sun or Planet whoſe Beam, S I, will thereby be always rendered permanent in the ſame horizontal Direction, from the riſing to the ſetting of the ſame; all which is evident from the preceding Articles.

3557. It is manifeſt alſo from the Conſtruction of the Figure that the Poſition of the Speculum being conſtantly variable, the Diſtance thereof from the Point A in the horizontal Line A V, and its Altitude above the ſaid Line, will be variable likewiſe, with the different and variable Altitude of the Luminary above the Horizon of the Place; and therefore it is neceſſary to calculate the horizontal Diſtances and Altitudes of the Speculum from the greateſt to the leaſt Quantities thereof for any particular Latitude, that it may be readily adapted to any given Declination of the Sun or Planet for the Day it is uſed.

3558. With Reſpect to the Firſt of theſe, *viz.* the Diſtance of the Foot of the Speculum I from the Point A in the horizontal Line A V, it is eaſily computed thus; from the Center or Point B let fall the Perpendicular B D; then in the Triangle A B D, the Angle B A D is known, being the Sum of the Co-Latitude (E A D) and the Declination (E A B); and the Side A B (= B I) being the Secant of the given Declination to the Radius A E, from whence A D (the Baſe) is found; which added to B I, gives (B I + D A) the horizontal Diſtance required, ſuppoſe for the greateſt Declination propoſed E A B. Thus in the Triangle C A F we find A F; and then A F + C K, is the horizontal Diſtance for any other Declination E A C on the other Part, or below the Equator.

3559. The *Altitude* of the Speculum above the horizontal Line A V is the Perpendicular B D, for the Declination E A B; and C F for the Declination E A C; which are Parts of the given Triangles; and therefore are readily found for all Declinations. And thus the Speculum is adjuſted for ready and conſtant Uſe at all Times.

3560. If it be required to conſtruct this *Celeſtial* CLOCK univerſally, or ſo as to adapt it for all Latitudes, it is only making it to move on a ſtrong and well wrought Hing on the Fore-part

at

at N, and then it may be elevated to any Degree of Latitude on a graduated Arch on the back Part at O, to which it may be fastened by a Screw.

3561. But then it must be considered, that as the Perpendicular Height A P, of the Fork A, above the Arm H P, is the Tangent of the Co-Latitude, or Elevation of the Equator, *viz.* A L P = L P W to the constant Radius L P; therefore that Height (A P) will be constantly variable with the Latitude in an inverse Ratio, or is greater as the Latitude is less; and *vice versa*. And this must be provided for by Calculation also, and constantly adjusted to the Latitude by a *graduated Scale*. All which is easily done by the Triangle A L P, wherein L P is of a given Length, and P A is found for any given Latitude L A P = A P Q. So that for every particular Latitude, the three Quantities A D, B D, and A P must be calculated, and the *Clock* and *Speculum* thereby adjusted.

3562. According to the Luminary you propose to view or observe, the Pendulum of the Clock must be peculiarly adapted; the Methods of doing which, for the various Sorts of *Planetary Pendulums*, we have already described in their several Constructions, and must leave the Artist to chuse out of them all that which he thinks best; for by this Time we presume he must be very sensible the SUN, the MOON, a STAR, and each particular PLANET must have a Pendulum properly qualified to render it *quiescent to the View*.

3563. In this *Celestial* CLOCK it will moreover be very expedient to have the Hour-circle upon the Face of the Clock *moveable*, and divided into 24 Hours, or twice XII; for by this Means the Hour and Minute of the Luminary's Culminating, or being on the Meridian, may be brought into the Vertical Line or *Meridian of the Clock*; and then the Hands being placed to the present Hour and Minute in that moveable Circle, will constantly shew the Time and Motion of that Luminary; for the Hour-hand will always keep Pace with it in the Heavens, and shew its *Right Ascension*, or Difference from Solar Time, which will be still *more* evident if the moveable Hour-Circle be added to that which is fixed, and both divided into twice XII Hours. (See 3325, *&c.*)

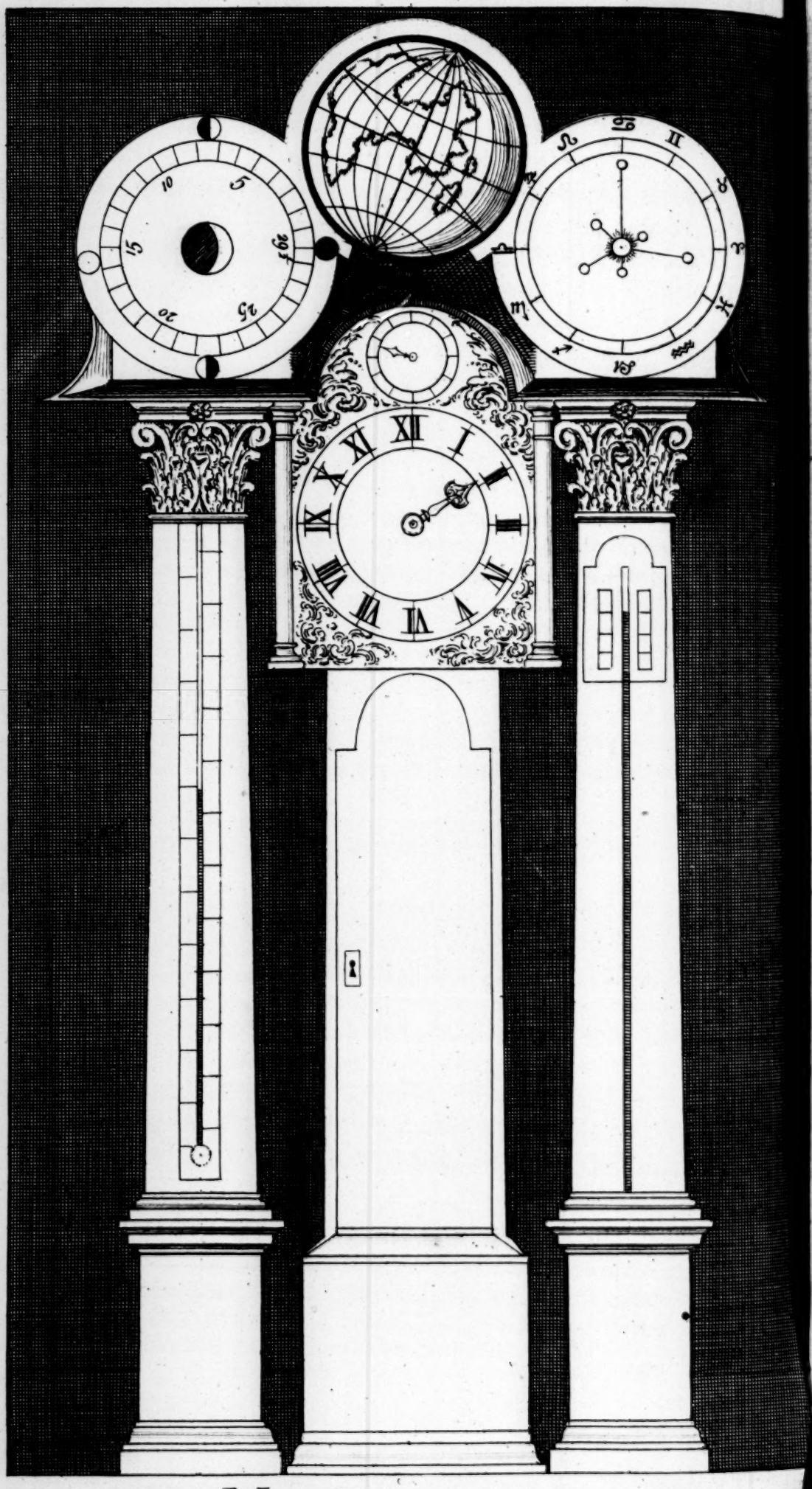
3564. Before any Obſervation is begun, the Clock muſt be very nicely placed in the Meridian Line, or ſo as that the Meridian Line of the Clock may coincide with the Meridian of the Place; to which End this Clock muſt be furniſhed with a very good *Magnetical Compaſs and Needle*, and the VARIATION thereof muſt be accurately found by Experiment for the Latitude where it is uſed. And by that the Clock may be readily adjuſted to the Meridian. Alſo two *Spirit Levels* muſt be placed in a proper Part of the Machine at right Angles to each other, by which it may be always reduced to a truly horizontal Poſition.

3565. By Means of this Machine duly adjuſted to the SUN, if the Room be darkened and a Hole made in the Window-Shutter to let in a Beam of the Sun's Rays upon the Speculum, it will continue to be reflected in one conſtant Direction all the Time it can fall on the Speculum, which may be ſo contrived as to be long enough for moſt *Optical Experiments*, either with PRISMS, or the SOLAR MICROSCOPE, *&c.* Beſides it will be no difficult Matter to follow the incident Beam with the Speculum by having *Caſtors* at the Bottom, or Foot of the Frame, ſupporting the Clock, and thereby moving and adjuſting it, as required, with the greateſt Eaſe.

3566. But the nobleſt Purpoſe to be anſwered by this *Celeſtial* MACHINE is fixing the heavenly Bodies, or rendering them ſtationary or quieſcent in the Field of the Teleſcope for aſtronomical Obſervation; an Advantage hitherto wanted (and unattainable by any other Means,) for advancing the Science of Aſtronomy to its true Summit of Perfection. The Teleſcope for this Purpoſe ſhould be of the reflecting Sort, and furniſhed with a Micrometer of different Forms, *viz.* the *Lattice*, the *parallel Wires*, the *divided Object-glaſs*, the *fine Screw*, *&c.* for meaſuring and delineating the Surfaces of the SUN, MOON, and PLANETS.

3567. I need not add, that the Foot and Stem of the Teleſcope ſhould be ſo contrived, that it may be always placed truly horizontal, exactly in the Meridian of the Clock, and juſt of the ſame Height of the Center of the Speculum, by this Means the Obſerver will ſit at Eaſe, and, without the leaſt Diſturbance, ſurvey the various and wonderful Phænomena of each celeſtial

Body,



The MICROCOSM.

Body. And thus we have finiſhed the Theory, or general Principles, of the ſublimeſt Piece of Machinery, that the Invention of Man has yet been able to produce.*

3568. It may alſo be obſerved, that if the Speculum at I or K be placed in the Direction perpendicular to the former, or parallel to that of the Wire A I or A K then will the Ray S I, S K be reflected the contrary Way, or in the Direction I B, or K C, in the ſame Right Line R B, or T C as before; and in many Caſes ſuch a Reflection or Poſition of the Ray will be more convenient than the other on the Part towards the Sun.

3569. Laſtly; The Speculum C to be uſed in all Obſervations of the heavenly Bodies muſt be a *plain One*, and as perfectly true as Art can make it; and indeed unleſs it be extremely true and well poliſhed, it cannot anſwer any ſuch Purpoſe. No *Concave* or *Convex Speculum* can in ſuch a Caſe be uſed, for Reaſons mentioned in the Theory of Catoptrics; though in many Experiments which require the Convergency of Rays for illumination only, a Concave Reflector may be applied to anſwer many uſeful Deſigns, which the Ingenuity of Artiſts will naturally ſuggeſt.

CHAP. XXII.

The PRINCIPLES *of* Celeſtial MECHANICS *applied to the Conſtruction of a* MICROCOSM, *conſiſting of a* PLANETARIUM, TELLURIAN, *and* LUNARIUM, *for exhibiting the* Celeſtial MOTIONS.

3570. THERE is another Branch of *Clock Work*, which may be properly eſteemed the *Second Part of* CELESTIAL MECHANICS; for by this we exhibit the MOTIONS of the *Heavenly Bodies*, as well as the TIME thereof, by the Clock

* See a further Account of the practical Uſe of this new *Helioſtata*, or Planetary Clock in another Part of this Work, *viz. The Young* GENTLEMAN *and* LADY'S PHILOSOPHY, Page 302, *&c.*

Clock, and proper Mechaniſm for that Purpoſe, to be added thereto.

3571. The *Celeſtial Bodies* whoſe Motions are ſhewn by Clock-Work are the PLANETS, the EARTH and the MOON; the Motions of the *Secondary Planets*, or *Satellites* of *Jupiter* and *Saturn*, are ſometimes ſhewn in ſuch a Syſtem of Mechaniſm as is uſually called an ORRERY; but the common Methods of conſtructing theſe large Inſtruments render them very expenſive, and conſequently they are but rarely made. But we ſhall endeavour to remove this Difficulty by propoſing other Conſtructions and Forms of Orreries that will be leſs coſtly, and perhaps more elegant and natural, as well as much more conciſe.

3572. The Calculations of the *Celeſtial Motions* depend on conſidering the Times of the Revolutions of the heavenly Bodies about their proper Centers, and taking the Ratios of thoſe Periods in ſuch Numbers as will beſt anſwer for toothing the Wheels and Pinions of the Work by which thoſe artificial Motions or Revolutions are to be produced; and then how or in what Manner they are to be connected with the Movement of the Clock for their Continuation.

3573. We ſhall firſt begin with the Motions of the *Primary* PLANETS about the SUN; and by the beſt aſtronomical Obſervations, their mean Revolutions or periodical Times are as follow, *viz.*

SATURN	revolves in	10759,3 Days.
JUPITER	—— ——	4332,5
MARS	—— ——	686,9
EARTH	—— ——	365,25
VENUS	—— ——	224,7
MERCURY	—— ——	88.

3574. Now the Ratio of the Earth's Period to thoſe of the Reſt of the Planets is hereby given, and therefore may be expreſſed in leſſer Numbers which may be at the ſame Time Integers, and conveniently adapted for the Teeth of the Wheel-Work for the intended Movement. And upon Trial thoſe which

which are found to anſwer beſt are as in the following Table, *viz.*

As 365,25 :	88	:: 83 :	20	for MERCURY.
365,25 :	224,7	:: 52 :	32	— VENUS.
365,25 :	686,9	:: 40 :	75	— MARS.
365,25 :	4332,5	:: 7 :	83	— JUPITER.
365,25 :	10759,3	:: 5 :	148	— SATURN.

And according to theſe Numbers, the Wheel appropriated to the EARTH muſt have 50 Teeth.

3575. Then if on one Arbor you fix ſix Wheels with the following Number of Teeth, 83, 52, 50, 40, 7, 5, to drive another ſet of Wheels each moveable upon another Arbor, and with the Number of Teeth in the other Column reſpectively, 20, 32, 50, 75, 83, 148; and laſtly, if on the ſeveral moveable Wheels proper Sockets are fixed (moving within, and independent of each other,) then will round ivory Balls, at the End of ſlender Wires fixed to thoſe Sockets, be carried about the Arbor in the ſame Periods of Time, reſpectively to each other, with thoſe of the real Planets in the Heavens.

3576. If this Syſtem of Movements be properly diſpoſed in a Box; and on the Top or Cover be drawn or engraved the Calendar and Ecliptic, and a braſs or lacquered Ball be placed on the Top of the Arbor of moveable Wheels, then, by a Handle or Winch, the Whole may be put into Motion; and it will then become a *Planetarium* or *manual* ORRERY of a moſt uſeful Form, and exhibit all the *Phænomena* of the SOLAR SYSTEM, or that according to *Copernicus*.

3577. If inſtead of the lacquered Ball, you put a three-inch *terreſtrial* GLOBE on the ſaid Arbor, and a ſmall braſs Ball upon the Arm carrying the ivory Ball, repreſenting the Earth among the Planets, then will the Machine be changed into the *Ptolomaic Syſtem* of the *World*; by which all the Abſurdities of that ſenſeleſs Hypotheſis may be eaſily exhibited and refuted.

3578. Such a Syſtem of Planetary Movements may eaſily be applied to a Piece of *Clock Work* by which it may be made to go any required Time, as an Hour or two; in which the Earth may revolve two or three Times about the Sun (or the Sun about

about the Earth,) which is a ſufficient Interval for exhibiting all the conſiderable Appearances in either Syſtem, relative to the *Primary Planets*. But if it be required that the Motions of this artificial Syſtem ſhould be the ſame with thoſe in the real Syſtem of the World, then we muſt proceed as follows.

3579. Let it be required to conſtruct a Movement for a Motion to be performed in the Space of *one Year*, or $365\frac{1}{2}$ Days; for we muſt not pretend to the Accuracy of an Hour or two in a Year. In $365\frac{1}{2}$ Days there are 730 Half-days of 12 Hours. This Number is produced by the Factors $18\frac{1}{4}$, 8, and 5; for $5 \times 8 \times 18\frac{1}{4} = 730$. Therefore any Number of Wheels and Pinions which will produce theſe Quotients, will produce the Motion deſired.

For Example,

4) 73 ($18\frac{1}{4}$	4) 73 ($18\frac{1}{4}$	8) 146 ($18\frac{1}{4}$
4) 40 (10	4) 32 (8	8) 64 (8
5) 20 (4	4) 20 (5	12) 60 (5

3580. Thus ſuppoſe a Wheel (A) with a Pinion of 5 Teeth drives a Wheel (B) of 20; and this Wheel carrying a Pinion of 4 Teeth drives a Wheel (C) with 40; and this with a Pinion of 4 drives the laſt and largeſt Wheel (D) with 73 Teeth. Then it is evident (3180.) that in one Turn of B there will be 4 Turns of A; and 40 Turns of A in one Turn of C; and 730 Turns of A in one of the laſt Wheel D. And therefore this Wheel D moves round once in a Year, and conſequently if the firſt Wheel A be connected with another equal Wheel fixed on the Arbor of the Great Wheel (or Fuſee) of an 8 Day Clock, then by Circles on its extreme Parts the *Day of the Month*, or *Sun*'s *Place* in the Ecliptic will be very exactly ſhewn, through the whole Year.

3581. Now this little Syſtem of Wheels may be very eaſily connected with the Movement of a Clock on one Part, and with that of the *Planetarium* on the other; ſo that the Clock (with a proper *additional Weight*,) ſhall conſtantly keep the Planets moving about the artificial Sun juſt with the ſame angular Velocities as the real Planets themſelves move with in the Heavens; and if in the Beginning of the Year the Places of the

the Planets in the Orrery be duely adjuſted or rectified by an *Ephemeris*, they will nicely correſpond to thoſe they reſpectively repreſent in the Heavens, and continue ſo to do ever after.

3582. It will be readily allowed, that this Method of conſtructing a *Planetarium* is the moſt natural and ſimple that can be, as not a Wheel, or even a Tooth, need here be uſed more than what is abſolutely neceſſary. By this *Planetary* CLOCK, it will every Day be ſeen, what Part of the Ecliptic the *Sun*, the *Earth*, or any *Planet* is in —— the various mutual *Aſpects* of Planets —— their *Conjunctions*, *Quadratures*, and *Oppoſitions* —— the *apparent Direction* of their Motion, *direct*, *ſtationary*, or *retrograde* —— whether they are *above or below the Horizon* —— whether they *riſe or ſet before or after the Sun:* —— In ſhort you have by this Means the whole *Syſtem of Celeſtial Phænomena* conſtantly in View.

3583. By the ſame ſmall Compages of Wheels, deſcribed (3575,) placed horizontally, it is evident the Axis of the Terreſtrial Globe may be made to move about the Axis of the Ecliptic once in a Year; and thereby all the *Phænomena* of the Earth's ANNUAL MOTION, with Regard to the *Seaſons* of the Year in different Latitudes and Climates, may be ready ſhewn, and obſerved. For ſince the Poſition of the Wheel D is ſuppoſed horizontal, if an Axis be fixed in its Center perpendicularly, it will repreſent the Axis of the Ecliptic. But if at a proper Height above the Wheel D the ſaid Axis be bent out of the Perpendicular ſo as to make an Angle therewith of 66° 30′ in the Form of an *Arm*, and on this Arm another Axle be placed at right Angles, this will make, with the forementioned Axle, an Angle of 23° 30′, and conſequently will repreſent the *Axis of the Earth*; and will be carried on round the Perpendicular Axis of the Ecliptic once in a Year, if properly connected with the Clock.

3584. Then for the *Diurnal* MOTION of the EARTH at the ſame Time, it may be eaſily effected thus; let the Axis of the Earth be moveable in a Socket fixed Perpendicular to the forementioned ſtrong Arm; on the under Part of which (below the Arm,) let a Wheel (G) be fixed, ſuppoſe of 72 Teeth; then let the Teeth of this Wheel play in the Teeth of another Wheel (F) which muſt be but half the other Number, *viz.* 36; and this Wheel muſt be placed horizontally about the Stem or per-

pendicular Axis juſt below the Arm, and ſupported on a Bridge to render it independent of the Work for the *Annual Motion* below. Laſtly, this Wheel F muſt have another equal Wheel (E) of 36 Teeth, to connect it with the Wheel A on the Arbor of the great Wheel of the Clock, and this Wheel A muſt alſo have 36 Teeth.

3585. Therefore, if upon the Terreſtrial Axle, the Globe of the Earth be placed, ſo as to be moveable thereon, or fixed with a Nutt and Screw at the End, as Occaſion requires, then it is plain, it will be carried about once in 24 Hours, becauſe the Wheels A, E, F, each move round in 12 Hours, and the Wheel F of 36 Teeth driving the Wheel G of 72 muſt move it round, (and the Earth on the Axis to which it is fixed,) once in 24 Hours.

3586. With Regard to the Wheel E, as it drives the Wheels above the Bridge for the diurnal Motion, ſo by an Arbor going through, and carrying a Pinion of 5 Leaves, it drives the Wheels B, C, and D, below for the annual Motion at the ſame Time. Alſo by the Interpoſition of this Wheel E, the Globe is made to revolve the right Way, *viz.* from *Weſt to Eaſt*, as the Earth itſelf really does.

3587. Then by a *Circle of Illumination*, and an *artificial Sun* at a proper Diſtance in the Axis thereof, with ſome other Apparatus, all the *Phænomena* of the *annual and diurnal Motions* of the *Earth* are moſt naturally exhibited for every Day throughout the Year. Thus you ſee the North and South Pole alternately in the illumined and dark Hemiſphere; —— the Sun riſing and ſetting each Day upon different Parts of the Horizon; —— the variable Lengths of the *diurnal* and *nocturnal Arches* deſcribed by the City of LONDON or any other Place: —— If a *Crepuſcular Circle* be added, the Time of Beginning and End of *Twilight* will be ſhewn for each Day; with many other important Particulars in *Geography* and *Aſtronomy*.

3588. 'Tis moreover eaſy to contrive, that this Mechaniſm for the Globe may be diſengaged from the Clock, or be put on and taken off at Pleaſure; in which Caſe it may be turned with a Winch ſo as to repreſent all the Phænomena of a Year or a Day, in a very ſhort Time; and then ſuch an *Apparatus* may be added, as will exhibit the *Solar and Lunar Eclipſes*, *&c.* the

Manner

Manner of doing this, together with a Print of the Globe thus conſtructed, may be ſeen in the *Young* GENTLEMAN's *and* LADY's PHILOSOPHY.

SCHOLIUM.

3589. We ſhall hereafter give a Conſtruction of a *Lunarium*, to ſhew the *Monthly* MOTION and PHASES of the MOON, as alſo the Motion of her APOGEE *and* NODES; if then this lunar Movement be connected with that of the Clock (as it eaſily may) and the whole be diſpoſed in a proper Form, it will make what may be more truly called a MICROCOSM, than any thing which has hitherto borne that Name; and to aſſiſt the young Artiſt's Imagination I have preſented to his View, in a Copper-Plate, the Form of a Specimen of ſuch a Piece of *Celeſtial* ARCHITECTURE in which are the following diſtinct Parts, *viz.* (1.) A large CLOCK for a *Primum Mobile*. (2.) Two PILLARS, (or *Pilaſters*) on one is placed a *Barometer*, and on the other a *Thermometer*. Theſe Pillars ſupport an *Architrave* on which is placed (3.) An ORRERY or *Planetarium*, ſuch as before deſcribed. (4.) A TELLURIAN, or *Terreſtrial* GLOBE with the annual and diurnal Motions. (5.) A LUNARIUM, ſhewing the Motion, Phaſes, and Age of the MOON. Such a ſuperb Structure, we preſume, would well become the Study or *Muſæum* of the Opulent and the Great; but ſo little Genius, Taſte, or Encouragement is now found for ſuch Works of Art, that it is not worth while to inſiſt further on their Uſe or Excellency. By ARTS and ARTISTS we, at this Time, underſtand only *Engraving*, *Painting*, and *Sculpture*, and thoſe who practice them; and LEARNING and LEARNED MEN, mean no more than *Novels*, *Romances*, *Plays*, and their *Authors*, in the preſent *Triſtram-Shandy* Age.

CHAP. XXIII.

The MECHANISM *and* CONSTRUCTION *of a* TELLURIAN, *with Three* Terreſtrial GLOBES; *illuſtracting the* THEORY *of the* EARTH'S MOTION; *the* Rationale *of the* SEASONS; *and the* Inequality *of* DAYS *and* NIGHTS.

3590. IN the Conſtruction of the ORRERY deſcribed in the laſt Chapter, the Earth's *annual Motion* was ſhewn, but not the *diurnal Motion* at the ſame Time; we ſhall now explain a new Piece of *Celeſtial Mechaniſm*, which being applied to the foregoing ORRERY (now ſuppoſed in a *Horizontal Poſition*) will not only produce both theſe Motions of the Earth together, but at the ſame Time, by Means of *two other Globes*, the *Rationale* of all the various Phænomena and Affections of the Earth will at once moſt evidently appear.

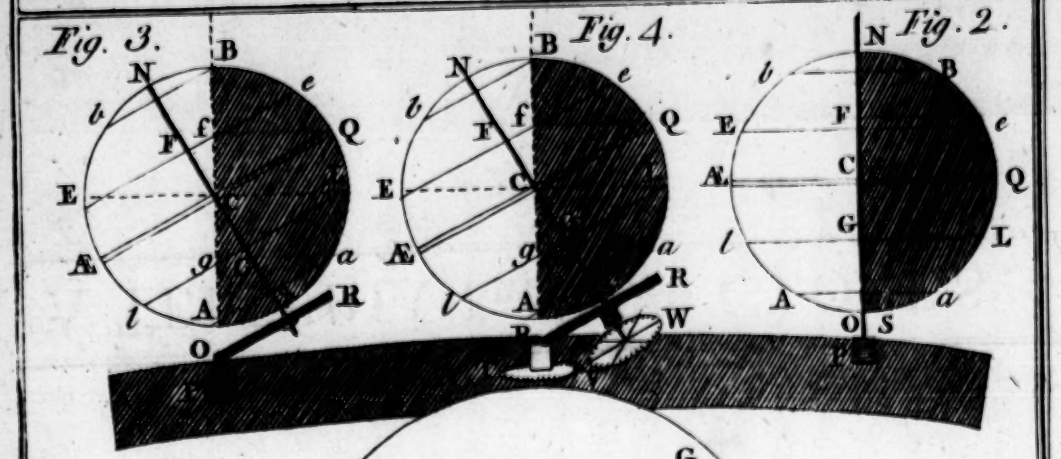
3591. I preſume it is needleſs to mention to the intelligent Reader, *that if any Body deſcribes the Circumference of a Circle, with the ſame Part always directed to the Center, it will neceſſarily be moved or turned once about its own Axis or Center in the ſame Time*; for by this Means the Body has that very Part directed to every Point in the Circumference of the Circle, which would be impoſſible without *one entire Revolution about its Axis*; this is therefore an *Axiom* or *ſelf-evident Truth*.

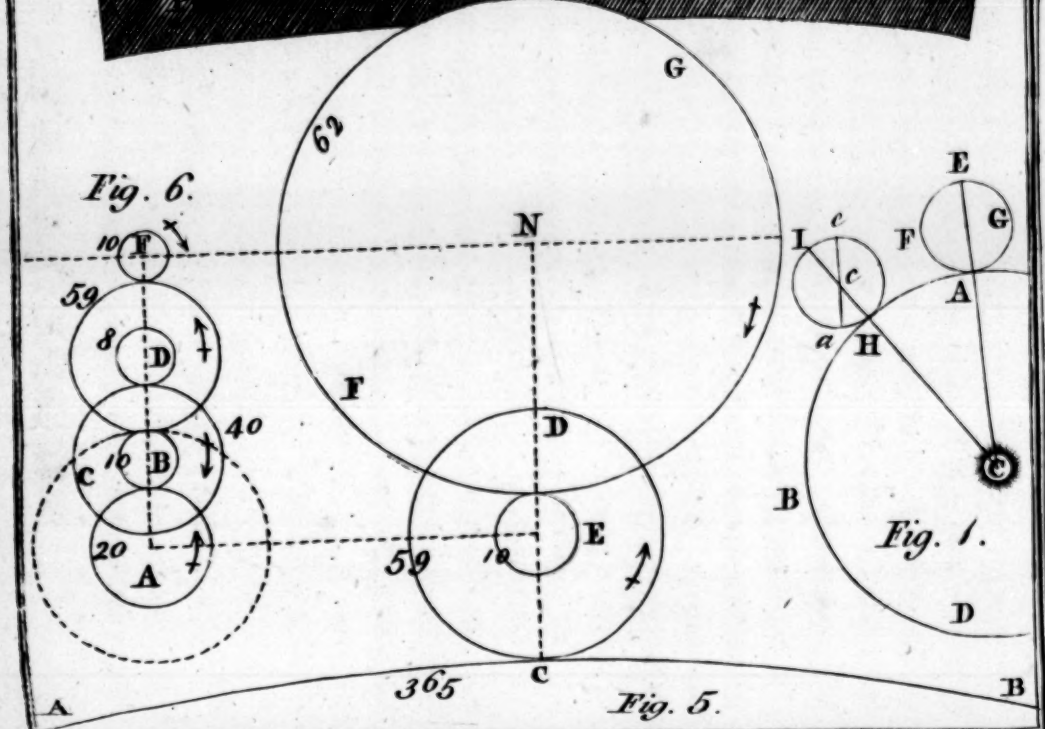
3592. Alſo, it is farther evident, that the Direction of the Motion of ſuch a Body, both about the Center of the Circumference it deſcribes, and about its own Center or Axis, is the ſame, or towards the ſame Part. But to illuſtrate theſe Poſitions, let A B D be the Circle deſcribed by any Plane F E G fixed upon an inflexible Rod or Arm E C, moveable about the Center C.* Let this Arm (or *Radius Vector*) revolve from the Situation E C to any other I C, then will the angular Motion

* See Fig. 1. in the Plate of the TELLURIAN with three GLOBES.

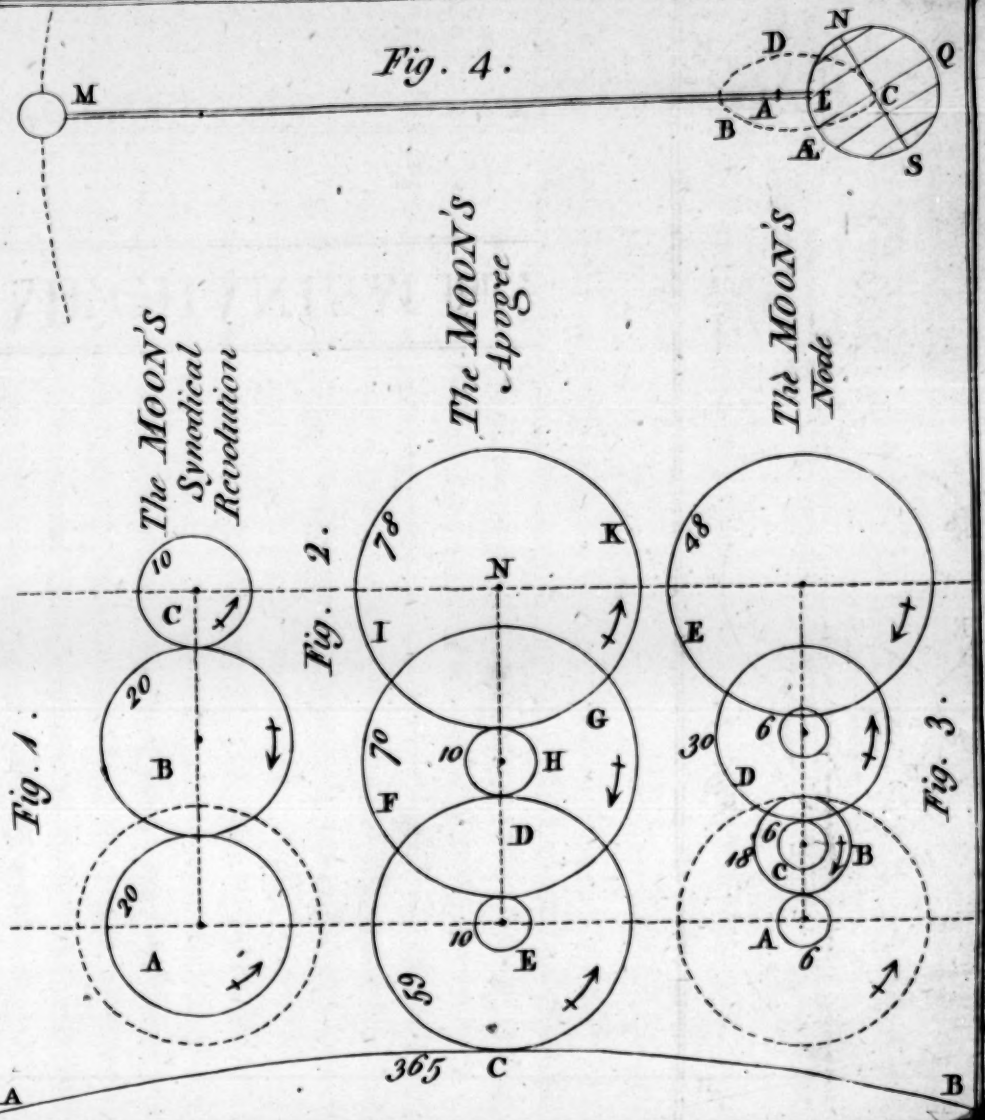


A TELLURIAN with three GLOBES.





MECHANISM for a LUNARIUM.



tion of the Body F G be = A C H about the Center C in that Time.

3593. Upon the Plane in the Situation I H, draw *a e* parallel to A E, and it will make the Angle I *c e* with the Radius I C; and thereby it appears that the Line A E (or *a e*) has deviated from its firſt to the preſent Poſition I H by the Quantity of the Angle I C E; and therefore by ſo much has the Plane moved about its own Center *(c)* in the Time of deſcribing the Part of the Circumference A H. But ſince C E and *a e* are two parallel Lines cut by a right Line I C, the Angles A C H and I *c e* will be equal; that is, the angular Motion of the Plane about the Centers C, and *c*, will be conſtantly equal; and the Direction of the Motion from *e* to I is the ſame as that from A to H.

3594. Hence it will follow, that if any luminous Body be placed in the Center C of a Circle A B D, and a Globe F G be moved in that Circle ſo as to have the ſame Part always turned to the Center, then will the ſame Hemiſphere or Part of the Globe be conſtantly enlightened, and the other Part will be always dark. ——— Thus ſuppoſe Æ N Q S (Fig. 2.) were a Planet (our *Earth*, for Inſtance) moving in the Manner abovementioned about the Sun in the Center at an immenſe Diſtance, then would the ſame Hemiſphere S Æ N be perpetually illuminated; and the other S Q N conſtantly in the Dark. Whence it is evident, a Planet in ſuch Circumſtances could never be a proper Seat, or Place for Habitation, for ſuch a Species of rational Creatures as the *Human Race*.

3595. Let ſuch a Globe, then, be ſuppoſed to move about its Axis N S placed perpendicular to the Plane of its Motion about the Sun. Such a Motion we call its *Diurnal Motion*, as it produces DAY and NIGHT alternately to all Parts of the Surface, in the Space of one Revolution which in our Globe is 24 Hours. Then it is plain, that every Part on the Surface of ſuch a Globe would be one half of the Revolution in the illumined Hemiſphere, and the other half, in the dark One; ſo that in ſuch a Condition of the Globe, there would be a *perpetual Equality of Day and Night*.

3596. Again, ſince the Axis N S of the *Diurnal* Motion is perpendicular to the Plane of the *Annual Motion*, which is ſuppoſed

poſed to paſs through the Centers both of the Earth and Sun; then it is evident, the Rays of the Sun will be perpendicular to the middle Part of the Globe at the Equator Æ Q, and will fall obliquely on every other Part; and the more ſo as the Part is nearer the Poles N, S, on either Side the Equator. Whence it will follow, that there will be *one and the ſame Degree of Light and Heat* to the ſame Part of the Surface of the Globe *conſtantly*; and therefore *no Variation of Seaſons* throughout the Year. Hence ſuch a Poſition or Conſtitution of the Globe of the Earth would by no Means ſuit with the Condition of the *Human Species*.

3597. Let us next ſee what will be the Caſe of a diurnal Motion of the Globe about an Axis in a Poſition inclined to the Plane of its annual Motion, ſuch as S N (Fig. 3.) and further let us ſuppoſe that the Axis N S is conſtantly in the ſame Plane with the Line E C L which connects the Centers of the Sun and Globe. Then it will follow, that the Parts of the Globe about the upper or North Pole N, will be conſtantly inclined to the Sun, while the oppoſite Parts about the South or lower Pole S are as conſtantly turned from it.

3598. The Conſequence of this *inclined Poſition* of the Globe will be, an *Inequality of Days and Nights*; and *different*, or *contrary Seaſons* to Parts at an equal Diſtance from the Equator on either Side. This is evident by Inſpection; for Æ C is the *Semidiurnal Arch* in the Equator; E f is the ſame in the Parallel E *e*, which is longer (in Time) than the other by the Time of deſcribing F f. But the ſemidiurnal Arch *(l g)* in the ſame Parallel *l* L of South Latitude is ſhorter than Æ C by the ſame Quantity *g* G = F f. And the Parts about the North Pole N to the Diſtance of N B (which meaſures the *Inclination of the Axis*, or Angle N C B) are perpetually in the illumined Hemiſphere and have conſtant Day; and on the Contrary, thoſe Parts at the ſame Diſtance about the South Pole S have an eternal Night.

3599. Then with Reſpect to the Seaſons, they are by this Means very different in oppoſite Parallels of Latitude; for now the Parts at E receive the perpendicular Rays, and have the *Summer Seaſon*; but ſince the Parts within the *Arctic Circle* B *b* have an inoccidual Sun, or enjoy perpetual Day, they will have the *hotteſt Seaſon*, and the North Pole N will be the hotteſt

Part

Part of ſuch a Globe. On the other Hand, the Parts within the *Antarctic Circle* A *a* will never feel the enlivening Beams of the Sun, and muſt be doomed to *perpetual Winter* and the intenſeſt Cold. Such a State of the Globe would therefore be very uneligible to every Sort of Inhabitants of the preſent *Terreſtrial Globe*.

3600. Now it will be readily allowed, that an *Alternation and Variety* in the *Seaſons* of the Year and *Lengths of Days and Nights*, will be the moſt deſirable and delightful State to rational Inhabitants of the Globe, and ſupply them with every Bleſſing equally and interchangeably through every revolving Year. And this wonderful and univerſal Effect is produced in our Terraqueous Globe by one ſimple Contrivance (a moſt ſtriking Inſtance of *divine Mechaniſm*,) *viz.* by giving the inclined Axis SN of the Earth's Motion an equable Movement about the Axis of the Ecliptic A B once in the Time of its annual Revolution about the Sun, and in a Direction contrary to the Earth's Motion; and this will produce the abovementioned neceſſary Effects.

3601. For we have ſhewn (3593) that while the Globe is carried about the central Sun, with the Axis S N, or the ſame Side A E B always turned towards it, the ſaid Globe and its Axis, will in that Time be alſo turned once about their proper Center C and in the ſame Direction with the Globe's annual Motion; therefore, if the ſame Motion be impreſſed on the Globe in a contrary Direction, it will have all the Parts of its Surface turned to the Sun, of Courſe, in the Space of one Revolution or Year, and its Axis S N will always keep a Direction parallel to that which it had at firſt; ſo that all Effects and Phænomena will be the very ſame as if the Earth were really at Reſt on its inclined Axis, and the Sun itſelf were to move about it (as it appears to do) in the Ecliptic Circle E C L.

3602. For then, when the Sun was in the Equinoctial Point or Beginning of the Ecliptic at C, it will, at the ſame Time, be in the Equator Æ Q, and conſequently illumine the Globe from Pole to Pole and produce an Equality of Days and Nights in every Part thereof; and this is called the SPRING-SEASON of the Year.

3603. After

poſed to paſs through the Centers both of the Earth and Sun; then it is evident, the Rays of the Sun will be perpendicular to the middle Part of the Globe at the Equator Æ Q, and will fall obliquely on every other Part; and the more ſo as the Part is nearer the Poles N, S, on either Side the Equator. Whence it will follow, that there will be *one and the ſame Degree of Light and Heat* to the ſame Part of the Surface of the Globe *conſtantly*; and therefore *no Variation of Seaſons* throughout the Year. Hence ſuch a Poſition or Conſtitution of the Globe of the Earth would by no Means ſuit with the Condition of the *Human Species*.

3597. Let us next ſee what will be the Caſe of a diurnal Motion of the Globe about an Axis in a Poſition inclined to the Plane of its annual Motion, ſuch as S N (Fig. 3.) and further let us ſuppoſe that the Axis N S is conſtantly in the ſame Plane with the Line E C L which connects the Centers of the Sun and Globe. Then it will follow, that the Parts of the Globe about the upper or North Pole N, will be conſtantly inclined to the Sun, while the oppoſite Parts about the South or lower Pole S are as conſtantly turned from it.

3598. The Conſequence of this *inclined Poſition* of the Globe will be, an *Inequality of Days and Nights*; and *different*, or *contrary Seaſons* to Parts at an equal Diſtance from the Equator on either Side. This is evident by Inſpection; for Æ C is the *Semidiurnal Arch* in the Equator; E f is the ſame in the Parallel E *e*, which is longer (in Time) than the other by the Time of deſcribing F f. But the ſemidiurnal Arch *(l g)* in the ſame Parallel *l* L of South Latitude is ſhorter than Æ C by the ſame Quantity *g* G = F f. And the Parts about the North Pole N to the Diſtance of N B (which meaſures the *Inclination of the Axis*, or Angle N C B) are perpetually in the illumined Hemiſphere and have conſtant Day; and on the Contrary, thoſe Parts at the ſame Diſtance about the South Pole S have an eternal Night.

3599. Then with Reſpect to the Seaſons, they are by this Means very different in oppoſite Parallels of Latitude; for now the Parts at E receive the perpendicular Rays, and have the *Summer Seaſon*; but ſince the Parts within the *Arctic Circle* B *b* have an inoccidual Sun, or enjoy perpetual Day, they will have the *hotteſt Seaſon*, and the North Pole N will be the hotteſt

Part

Part of ſuch a Globe. On the other Hand, the Parts within the *Antarctic Circle* A *a* will never feel the enlivening Beams of the Sun, and muſt be doomed to *perpetual Winter* and the intenſeſt Cold. Such a State of the Globe would therefore be very uneligible to every Sort of Inhabitants of the preſent *Terreſtrial Globe.*

3600. Now it will be readily allowed, that an *Alternation and Variety* in the *Seaſons* of the Year and *Lengths of Days and Nights*, will be the moſt deſirable and delightful State to rational Inhabitants of the Globe, and ſupply them with every Bleſſing equally and interchangeably through every revolving Year. And this wonderful and univerſal Effect is produced in our Terraqueous Globe by one ſimple Contrivance (a moſt ſtriking Inſtance of *divine Mechaniſm,*) *viz.* by giving the inclined Axis SN of the Earth's Motion an equable Movement about the Axis of the Ecliptic A B once in the Time of its annual Revolution about the Sun, and in a Direction contrary to the Earth's Motion; and this will produce the abovementioned neceſſary Effects.

3601. For we have ſhewn (3593) that while the Globe is carried about the central Sun, with the Axis S N, or the ſame Side A E B always turned towards it, the ſaid Globe and its Axis, will in that Time be alſo turned once about their proper Center C and in the ſame Direction with the Globe's annual Motion; therefore, if the ſame Motion be impreſſed on the Globe in a contrary Direction, it will have all the Parts of its Surface turned to the Sun, of Courſe, in the Space of one Revolution or Year, and its Axis S N will always keep a Direction parallel to that which it had at firſt; ſo that all Effects and Phænomena will be the very ſame as if the Earth were really at Reſt on its inclined Axis, and the Sun itſelf were to move about it (as it appears to do) in the Ecliptic Circle E C L.

3602. For then, when the Sun was in the Equinoctial Point or Beginning of the Ecliptic at C, it will, at the ſame Time, be in the Equator Æ Q, and conſequently illumine the Globe from Pole to Pole and produce an Equality of Days and Nights in every Part thereof; and this is called the SPRING-SEASON of the Year.

3603. After

3603. After the first Quarter of the Revolution, or the Sun has passed from C to E, and reached the *Tropic of Cancer* E *e*, then the Globe will be illuminated from A to B, and the North Pole N, with all northern Latitudes from E to B, will be now turned towards the Sun and receive the *greatest Quantity of Light and Heat* they can ever have, and consequently this will be the Midst of their *Summer Season*. The Day E f in every northern Parallel E *e*, is longer than the Night f *e*; and all within the Arctic Circle *b* B have no Night at all on that *Midsummer Day*.

3604. Just the Contrary happens in southern Latitudes; for there the Sun-beams reach no farther than to the Antarctic Circle A *a*; and all within it are in total Darkness for one whole diurnal Revolution of the Globe. And the Length of the Day (*l g*) in any Parallel *l* L, is shorter than the Night *g* L by the Quantity *g* G; consequently in these Latitudes, the Light and Heat of the Sun is now the least, or the Cold and Darkness the greatest it can be, and therefore make their WINTER-SEASON.

3605. When the Sun apparently descends from the northern Climates to the Equator again on the other Side the Globe to the first Point of *Libra* opposite to C, it will then again make equal *Day and Night* throughout the Globe, by enlightening it from Pole to Pole, as before (3602.) The Light and Heat is now at a Mean, and make the other temperate SEASON called AUTUMN.

3606. As the Sun passes from the Autumnal Equinox thro' the next three Signs of the Ecliptic, the *southern* Parts of the Earth will be turned gradually to the Sun; till, at Length, the Sun being arrived at L, the first Point of *Capricorn*, it will then be neareſt of all to the South Pole S and illuminate all that Hemisphere A L B, which before was dark when the Sun appeared at E; and consequently every Thing with respect to Day and Night, and the Seasons, is just the reverse of what it was then. All *Southern Latitudes* now enjoy the *Midsummer Season*, and in the northern Latitudes it is *Winter*. What was before the Length of their Night, *viz.* *g* L, is now the Length of their Day, and *vice versa*.

3607. The Sun returning again to C, renews the Spring and begins the future Year. —— Thus by giving the Earth an annual

annual retrograde Motion about the Axis A B of the Ecliptic, all Parts of its Surface become equally expoſed to the Sun on each Side the Equator, and equally participate the Advantages which redound from ſo wiſe and beneficent a Contrivance, in the Courſe of one whole Year.

3608. Now this whole Theory (as above explained) of the Earth's *annual and diurnal Motions* may be very eaſily repreſented and illuſtrated by a Piece of Mechaniſm, which I call a TELLURIAN; conſiſting of three ſmall Terreſtrial Globes, placed in the Manner ſhewn in Figs. 2, 3, and 4. The firſt of theſe (Fig. 2.) is moveable with the Hand about an Axis N S, which is perpendicular to the Plane of its Motion about the Sun, and is fixed to the *Radius Vector*, and therefore will exhibit the Phænomena of *equal Day and Night*, and *Identity of* SEASONS through the Year, and nothing more, as before obſerved (3596.)

3609. The *ſecond Globe* (in Fig. 3.) has the Axis S N of its diurnal Motion inclined to the Axis A B of the Ecliptic in an Angle B C N = 23° 29′. But this Axis S N being fixed to a Part P O R, which is itſelf fixed to the *Radius Vector*, will indeed produce a Variety of Seaſons, and Difference in the Lengths of Days and Nights; but then as the Axis S N is conſtantly turned to the Sun, there can be no Viciſſitudes of *Seaſons* nor any Variation of the Lengths of Days and Nights, which remain conſtantly the ſame, in the ſame Place throughout the Year, agreeable to (3599.)

3610. But the *third Globe* (in Fig. 4.) is not fixed to the *Radius Vector*, but is connected with the Mechaniſm contained therein, in ſuch Sort, as to qualify it for exhibiting the real Phænomena of our own Globe, and juſt as they happen in Nature. To this End the firſt Thing to be done is to counteract or nullify the *direct annual Motion* of the Globe (in Fig. 3.) about the Axis A B of the Ecliptic, by fixing its Stem O P upon a Plate that has an equal Motion in a contrary Direction at the ſame Time.

3611. Thus; Let A C B (Fig. 5.) be a Part of the outward Rim of the Plate of the *Orrery* (3575), divided into 365 Teeth; and upon the lower Plate of the *Radius Vector*, let a ſmall Wheel C D be placed with 59 Teeth to move in thoſe of the Orrery-plate. Upon this Wheel a Pinion E muſt be fixed

with 10 Teeth to turn another Wheel F G of 62 Teeth; and then if the Shaft or Stem of the Globe (Fig. 4.) be fixed in the Center N of this Wheel or Plate, it will communicate to the Globe the *Retrograde Motion* required. For 59)365(= 6,2, or the Wheel C D is moved round its Center 6,2 Times while it revolves once round the Orrery-plate; and becauſe 10)62(= 6,2 alſo; it is evident, that the Wheel F G moves once round in one Revolution of the *Radius Vector*, and becauſe the Wheel C D moves *directly*, from Weſt to Eaſt, 'tis plain the Wheel F G moves retrograde, or the contrary Way as the Arrows (↕) ſhew in each.

3612. Hence by theſe two Wheels only, the annual Motion of the Globe (Fig. 4) about the Axis A B is annihilated, and a *conſtant Parallelism* of the Axis N S of its diurnal Motion is produced; and therefore not only the Difference of the Seaſons and Days and Nights is effected, but all the Change or Variety thereof that happens to the natural Globe for every Latitude, and every Day of the Year.

3613. The next Thing to be conſidered, is, the Syſtem of Wheel-work for giving the ſame Globe its proper diurnal Motion about its Axis N S, and in the natural Direction from Weſt to Eaſt. Here we find 365 of theſe diurnal Revolutions to *one* annual One; and therefore ſuch a Syſtem of Wheels muſt give 59 Turns of the Globe in one Turn of the Wheel C D, which muſt be the *Primum Mobile*, or firſt Mover to that Syſtem.

3614. Therefore the Number 59 muſt be broke into three Quotients, becauſe the laſt Wheel muſt have a retrograde Direction, as will appear by and by; ſuppoſe theſe Quotients are 10)20(= 2; 8)40(= 5; and 10)59(= 5,9; then it is evident, if upon a Bridge or Plate fixed over the Wheel-work of Fig. 5. the following Wheels are properly diſpoſed, *viz.* a Wheel A of 20 Teeth (Fig. 6.) fixed on the Arbor of the Wheel C D (Fig. 5.) to drive a Pinion B of 10 Teeth on a Wheel C of 40, which drives a Pinion D of 8 Teeth fixed upon a Wheel of 59, and this at laſt drives a ſmall Wheel F of 10 Teeth; then, I ſay, it is evident this laſt Wheel F will be turned round 59 Times while the firſt Mover C D turns once; and its Direction will be *retrograde*, as ſhewn by the Arrows.

3615. The

3615. The Wheels being ſo proportioned, that the Center of the laſt Wheel F (Fig. 6.) lies exactly over that of the Wheel FG (Fig. 5.) a Socket is to be fixed into the ſaid Wheel F, moveable about the Stem or Fulcrum of the Globe which is fixed firmly in the Center N of the Plate F G. This Socket paſſing through a Plate (faſtened over all the Machinery) will appear above it, as at P T. And if on this upper Part of the Socket there be fixed a Wheel T V with 10 Teeth playing in another V W of an equal Size and Number of Teeth fixed to a Part of the Axle N S below the Piece P R (in which the ſaid Axle moves) then will the Globe (Fig. 4.) be turned round 59 Times in each Revolution of the Wheel C D of Fig. 5. and conſequently it will turn upon its Axis once every Day or 24 Hours. And this Motion will be *direct*, becauſe that of the Wheel T V (or F below) is *retrograde*.

3616. And thus the TELLURAN applied to the ORRERY will by its three Globes together give a full Demonſtration and Illuſtration of the *Genuine* THEORY of the annual and diurnal Motions of the Earth in every Reſpect; *one Globe* ſhews (Fig. 2.) that by a *parallex Axis without Inclination*, there can be no Difference in the Seaſons or Quantity of Day and Night ——— A *ſecond Globe* ſhews (Fig. 3.) that an *inclined Axis without Parallelism*, can only ſhift the Scene of the Seaſons, and produce a different Length of Day, and Night in contrary Latitudes; but can never make any Alteration in either ——— But the *third Globe* with an Axis, at the ſame Time, *always inclined and parallel to itſelf*, produces every Variety we find in Nature, and exhibits an exact Conformity to the Motions and various Phænomena of the natural Terraqueous Globe.

CHAP. XXIV.

The CONSTRUCTION *and* MECHANISM *of a* LUNARIUM, *for ſhewing the* MOTION *and* PHASES *of the* MOON, *as alſo of her* APOGEE, *and* LINE *of* NODES,

3617. THE Conſtruction of a LUNARIUM, or Inſtrument to ſhew the Phænomena of the MOON in regard to her own *Motion* and *Phaſes*, and the Motion and Poſition of the *Apſides* and *Nodes* of her *Orbit*, will be more complicated than that of the *Tellurian*; but as the periodical Revolutions are known, it will be eaſy to contrive ſuch Syſtems of Wheel-Work as ſhall repreſent them adequately, and produce Appearances every Way ſimilar, and correſponding to Nature itſelf.

3618. The LUNARIUM, to exhibit the Phænomena of Lunar Motions, may be contrived and conſtructed in the following Manner. ACB here again repreſents the dentated Rim of the Plate of the Orrery of 365 Teeth, in which the Wheel CD (Fig. 2.*) moves as before; and is now alſo to be conſidered as the *Primum Mobile*, or that which gives Motion to all the Machinery of the *Lunarium*. As this Wheel has 59 Teeth, it is evident, that if to its Arbor below, there be fixed a Wheel A of 20 Teeth, to drive another equal Wheel B of 20, and this to move a 3d Wheel C of 10 Teeth; then this laſt Wheel C will turn round once in the Time of a *Lunation*, or 29 $\frac{1}{2}$ Days; for it will turn twice in each Revolution of A, which is in 59 Days. And the Direction of its Motion will be the ſame, *viz.* from *Weſt* to *Eaſt*.

3619. In the Center of the Wheel C (Fig. 1.) let there be fixed a ſtrong perpendicular Wire to paſs through, and to a convenient Height above, the Upper Plate or Top of the *Lunarium*. Then let there be provided another Wire EM (Fig. 4.) and to one End E thereof, let therebe fixed an Ivory Ball Æ N Q S, with proper Circles to repreſent the EARTH; and at the other End, ano-

* See the Plate of MECHANISM for a LUNARIUM.



The THEORY of the EQUATION of TIME.

Fig. 3.

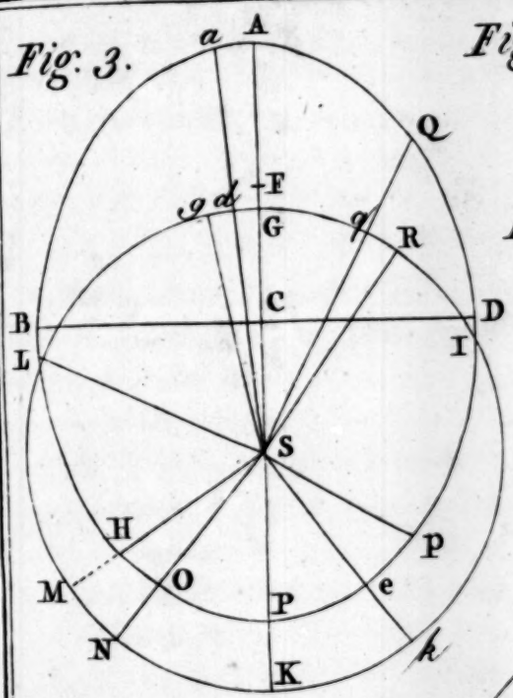


Fig. 7.



Fig. 1.

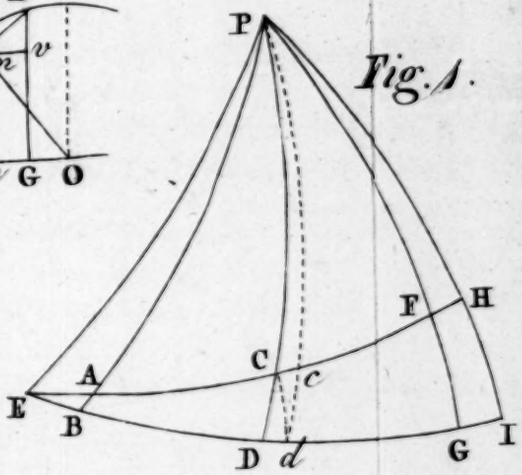


Fig. 5.

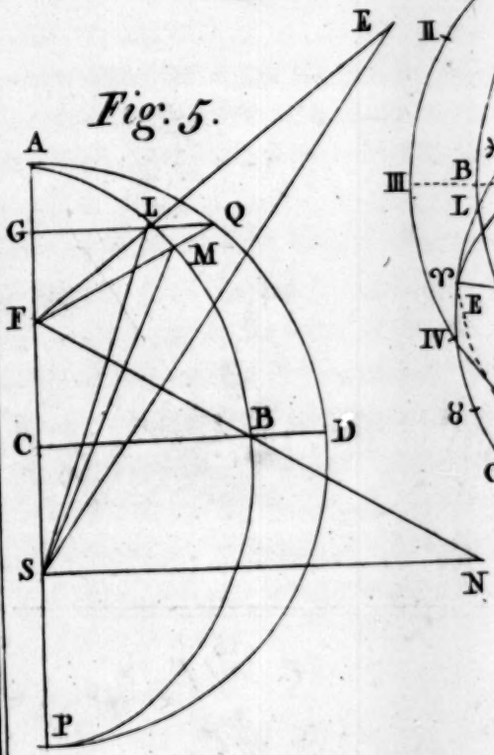


Fig. 8.

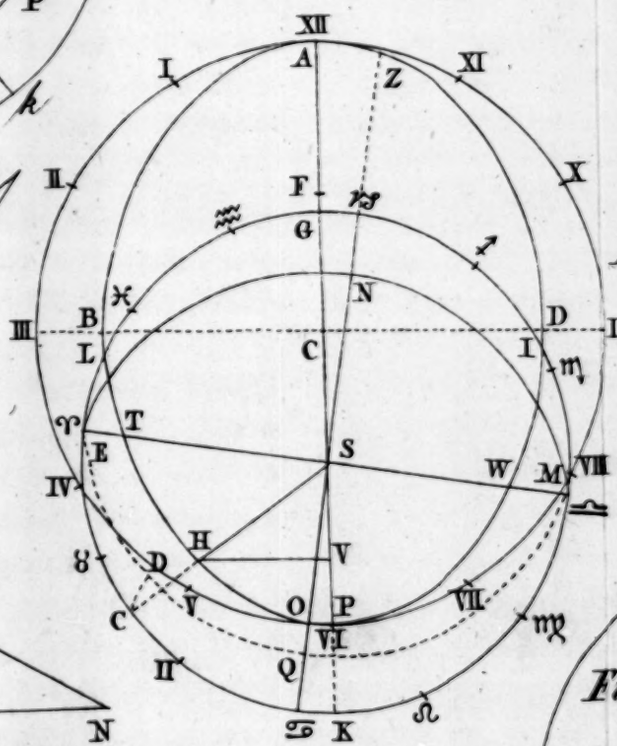


Fig. 4.

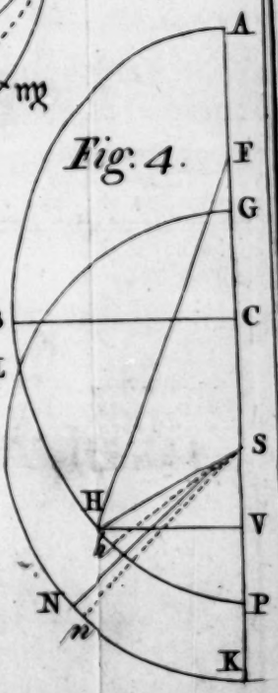


Fig. 6.

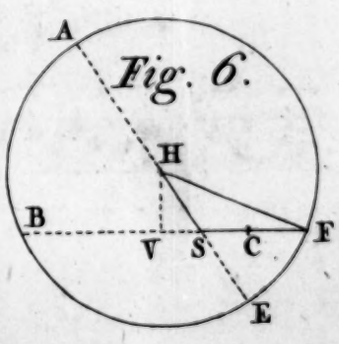
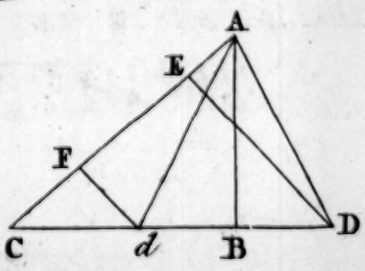


Fig. 2.



another Ball M to repreſent the MOON, one fourth Part of the Earth's Diameter. For the *Diameters of the Earth and Moon* are known to be nearly as 7964, to 2192 Miles.

3620. The Magnitude and Denſity of the Earth and Moon being duely conſidered, the common Center of Gravity A between them will be found to fall very near the Surface of the Earth; this then will be the Point on which thoſe two Bodies will be in Equilibrio; and therefore if this Point A be fixed upon the Top of the Wire or Axle of the Plate C, both the Moon and Earth will be carried round their Center of Gravity in $29\frac{1}{2}$ Days; and in 12 Lunations or 354 Days they will be carried once about the Sun, which Revolution is called the *Lunar* YEAR, which is 11 Days ſhorter than the Solar Year of 365 Days; and this Deficiency of 11 Days is what the *Chronologiſts* call the EPACT.

3621. This Conſtruction of a Lunarium preſents us at the ſame Time with *another Motion of the Earth, viz.* its *Menſtrual Motion* about the common Center of Gravity A; and hence we learn (1.) That it is not the Center of the Earth C which deſcribes what is commonly called the *Orbis Magnus* or annual Orbit about the Sun, but the Center of Gravity A between the Earth and Moon. (2.) That the Moon alſo in its menſtrual Motion does not regard the Center C of the Earth, but the Center of Gravity A, as the Center of her proper Motion. (3.) That the Center C of the Earth is fartheſt from the Sun at the *New Moon*, and neareſt at the *Full Moon*. (4.) That in the Quadratures the *Menſtrual Parallax* of the Earth is ſo ſenſible as to require a particular *Equation* in *Aſtronomical Tables*. Theſe Points, though of ſo great Importance, we have not yet ſeen repreſented in *Orreries* or *Lunariums* hitherto made.

3622. Again, if a *Candle* or *Lamp* be placed in the Center of the Orrery, then will the artificial Moon, revolving about the Earth, be enlightened in the ſame Manner as the real Moon in the Heavens is by the Sun, and all her Phaſes in each Lunation with reſpect to the Earth will be the very ſame as we obſerve them in the Heavens.

3623. But further, if the Balls for the Earth and Moon were properly furniſhed with Magnets within Side, and then inſtead of being fixed to the Ends of the Wire E M, they were nicely

ſuſpended

ſuſpended on fine Points of Needles placed there, with magnetical Cards ſhewing the ſeveral Points of the Compaſs, as might eaſily be done, then we might agreeably obſerve that the Earth and Moon would both conſtantly ſhew the ſame Part or Face to each other as they moved about the Center A; and that in turning once round, the Ends of the Wire E M would apply ſucceſſively to every Point of the Card or Compaſs, and thereby ſhew that each Ball in one Revolution about the Center A, did alſo move once about its own Axis.

3624. But though this be the real Caſe of the Moon M, it is not of the Earth, which (to anſwer other Purpoſes) has another Motion communicated to it, *viz.* the *diurnal Motion* in 24 Hours, every Way independent upon the menſtrual Motion about A. And therefore that the *Lunarium* may be quite ſimilar to Nature, the Ball Æ N Q S ſhould be ſtill poſſeſſed of an internal Magnet, and ſo placed upon an Axis fixed in the End of the *Radius Vector*, as to be voluble about it at Pleaſure.

3625. In this Conſtruction, the general Phænomena of the Earth and Moon will be mutually evident; for hence it will appear (1.) That the Moon muſt always neceſſarily ſhew the *ſame Face* or Side to the Earth. (2.) That the Earth turns every Part of its Surface to the Moon once in 24 Hours: (3.) In Conſequence of this, the light and dark Parts of the Moon appear permanent and fixed to us. (4.) But a great Variety of luminous and opake Parts preſent themſelves in a conſtant Rotation over the Surface of the Terreſtrial Ball to the Inhabitants of the Moon, if any there be. (5.) All the *Phaſes* of the Moon appear as in the Heavens. (6.) Alſo the ſame *Lunar* Phaſes will be obſerved in the Earth by the *Lunarians*, for we are mutually a Moon to each other. (7.) Thoſe Phaſes are contrary in the Earth and Moon, for when the Moon is *new* or *dark* to us, our Earth is wholly *enlightened* or *full* to them. (8.) Our terreſtrial Ball being four Times larger in Diameter, will preſent the *Lunarians* with a *Full moon* 16 Times as large as ours. (9.) Only one Hemiſphere of the Moon being turned to the Earth, the Inhabitants of that alone can ſee our Globe. (10.) Our Earth appears to them always in *one Place*, or fixed in the ſame Part of the Heavens. (11.) The Moon has an apparent

parent diurnal Motion through the Heavens, by the Earth's real Motion on its Axis. (12.) The *Lunarians* in the oppoſite Hemiſphere never ſee our Terreſtrial Globe, as we can never ſee them.

3626. With reſpect to the Sun, it is evident by the *Lunarium*, (1.) That it muſt always enlighten one half her Globe. (2.) That every Part of the Lunar Ball is turned to the Sun in the Space of her Monthly or periodical Revolution. (3.) Therefore the Length of the Day and Night in the Moon is ever the ſame, and equal to $14\frac{3}{4}$ of our Days. (4.) When the *Sun ſets*, the *terreſtrial Moon riſes* to the *Lunarians*, in the Hemiſphere next the Earth. (5.) To them therefore there can never be any dark Night at all. (6) While thoſe in the other Hemiſphere can have no Light by Night but what the *Stars* afford them.

3627. Not only the Moon herſelf but even the very Orbit ſhe deſcribes about the Earth, has a ſenſible Motion, and is found to make one Revolution in a little leſs than *Nine Years*; for by the Aſtronomical Tables the Place of her Apogee was

		S	D	′	″
in the Year 1755	in	0	15	38	20
1764	—	0	21	50	17
Motion in Years 9	—	12	6	11	57 = 366,2

Therefore ſay, as 366°,2 : 9 :: 360° : $\frac{9 \times 360}{366,2}$ = Time of one Revolution of the *Apogee*; then $\frac{9 \times 360}{366,2} \times 365 = 3229,5$ Days.

3628. Now in order to compute the Syſtem of Wheel-work for producing this Motion of the Apogee, we muſt divide $3229\frac{1}{2}$ Days by 59, the Teeth in the firſt Wheel CD (Fig. 2.) and the Quotient is 54,7; this muſt be broke into two other Quotients, *viz.* 7 and $7\frac{8}{10}$, both on Account of its being too large in itſelf, and eſpecially becauſe it is neceſſary to have two Wheels move, *viz.* F G and I K, that the latter may have a *direct Motion*, as the Apogee really has. Therefore a Pinion E of 10 Teeth on the Axis of the Wheel C D muſt drive a Wheel F G of 70 Teeth; and another Pinion of 10 on that Wheel muſt

muſt drive a Wheel IK of 78 Teeth; and then a braſs Socket fixed about the Center of this laſt Wheel, will carry about an *Ellipſis*, repreſenting the Moon's Orbit, in the Time required.

3629. Again, the Orbit of the Moon is not parallel to the Ecliptic or Earth's Orbit, but makes an Angle therewith of 5° 18′ at a Mean; now from different Forces of the Sun and Earth in different Situations, the Moon will in each Revolution be various attracted, and her Orbit agitated; being ſometimes drawn down, at other Times elevated; ſometimes it is moved forwards, and at others, backward; and thus there will be a variable Motion generated in the *Line of the Nodes*, but upon the whole the *Retrograde* Motion is greateſt, and conſequently the Nodes will move backwards in the Ecliptic, and make one Revolution in, nearly, 19 Years.

3630. For the Place of the aſcending Node ☊ in

			S	°	′	″
the Year	1746	was	11	27	1	4
	1765	is	11	19	30	32
Years, 19			11	22	38	28 = 352½ Degrees.

Therefore ſay, as 352°,5 : 19 :: 360° : $\frac{19 \times 360}{352,5}$ = Time of one Revolution in Years, or $\frac{19 \times 360}{352,5} \times 365 = 7082,5$ Days.

3631. Then 59)7082,5(= 120, very nearly; this 120 muſt be broke into *three Quotients*, that the laſt Wheel may have a *retrograde Motion*. Theſe Quotients may be 3, 5, 8; and then a Pinion A (Fig. 3.) fixed on the Axle of the Wheel CD (Fig. 2.) having 6 Teeth may drive a Wheel B of 18; this with another Pinion C of 6, drives a Wheel D of 30; and this alſo with a Pinion of 6 moves a Wheel E of 48 Teeth; and in a retrograde Direction; and therefore a Socket fixed in this Wheel will carry a Line pointing to the Place of the Nodes through all the Signs of the Ecliptic *in Antecedentia*, in the Space of 7082½ Days as required.

3632. In the laſt Place, upon the upper Part of the Axis of the firſt Mover CD (Fig 2.) let a Pinion of 10 Leaves be fixed to drive a Wheel of 62; then a Socket (including all the

the reſt) being fixed in this Wheel will carry an *Ecliptic Circle* upon the Plate of the *Lunarium*, which may at any Time be placed in a Poſition ſimilar to the *Ecliptic* on the Plate of the Orrery, and then it will always continue ſo by Virtue of its conſtant Paralleliſm produced by this Mechaniſm. (3611.)

3633. If a Circle be placed about the *Line of Nodes* (3629,) and moveable upon it as an Axle, it may be occaſionally placed horizontally, or inclined to the Horizon in an Angle of 5° 18′; and then if the Moon M (Fig. 4.) be fixed to the End of a fine ſteel Wire, paſſing freely though a ſlender Socket on the End of *Radius Vector* E M, ſo that it may be over the Middle Part of the laſt mentioned Circle and conſtantly move upon it; it will be eaſy to underſtand how the Moon may be made to move either horizontally or in her proper Orbit; in order more particularly to ſhew whatever relates to the Nature, Cauſe, and various Phænomena of ECLIPSES.

3634. Alſo, being now furniſhed with a *Solar* and *Terreſtrial Ecliptic* it will be very eaſy to underſtand the Rationale of the general *Deceptio Viſus*, or *Illuſions of Sight* in regard to the *Motions*, *Places*, *Magnitudes* and other Affections of the heavenly Bodies, and to reſolve them all conſiſtently with Truth. Alſo from hence the *Heliocentric* and *Geocentric* Places of the Moon and Planets will readily, at any Time, appear. And thus all the Great Points and Poſitions in the *Solar and Lunar Aſtronomy* become eaſily explicable by ſuch a Conſtruction of an ORRERY, TELLURIAN and LUNARIUM, as has been deſcribed.

3635. After the ſame Manner a *Jovian* LUNARIUM may be conſtructed with Eaſe, by knowing the Periods and Diſtances of each of *Jupiter*'s Moons, which are as in the following Table.

Satellite.	Diſtance.	Days. H. ′	Ratio to 1 Day.
1	$5\frac{6}{10}$	1 : 18 : 27	24 : 42
2	9	3 : 13 : 13	20 : 71
3	14,3	7 : 3 : 42	12 : 86
4	25,3	16 : 16 : 32	6 : 100

3636. The Numbers in the Ratio of theſe Periods to one Day will ſhew the Diameters and Number of Teeth in a fixed and a moveable Set of Wheels for theſe Satellites in the ſame Manner

as before in the primary Planets (3574.) That is, upon one Arbor there must be fixed four small Wheels, or Pinions, with the Numbers of Teeth 24, 20, 12, and 6, to drive four moveable Wheels of 42, 71, 86, 100 Teeth respectively. Upon these, Sockets are placed to carry the Secondaries round their Primary *Jove*, in Distances measured in Semi-diameters of his Globe expressed by the Numbers in the second Column.

3637. But now to adapt these to the dentated Edge of the Orrery, and from thence derive a proper Motion we need only consider, that if a Pinion of 8 be fitted to the Teeth of the Orrery below, and fixed to a Wheel of 64 Teeth above, playing in another Pinion of 8 fixed on the Arbor of the fixed Set of Wheels, then will that Arbor, and all its Wheels, be turned once round in every natural Day of 24 Hours; and consequently, all the Satellites will move about their Primary in such a Number of Days as they really do in the Heaven, and at proportionable Distances.

3638. If now a wax Candle be placed in the Center of the Orrery, and a Piece of white Paper on the *Radius Vector* beyond the Satellites perpendicularly; then the Motions of the Shadows of the Moons will naturally represent the apparent Motions of these Moons as seen in the Field of a Telescope, *viz.* a *Rectilineal Motion* through the Diameter of the View; as also their *direct*, *retrograde*, and *stationary* Phænomena, and lastly, those of their *Immersions* and *Emersions* in *Eclipses*; their Occultations, and Disapparition on the illumined Disk of their Primary; all which are of such Consequence in finding the Difference of Meridians, or *Longitude* of Places; and many other Points of *Astronomy*, *Navigation*, *&c.*

3639. A *Saturnian* LUNARIUM may be constructed in the very same Manner for 5 Moons; whose *periodical Times* are as follow, from Dr. *Halley*'s Tables.

	Days.	H.	′	Semid. of Ring.
The First revolves in	1 :	21 :	18	2,097
The Second ——	2 :	17 :	41	2,686
The Third ——	4 :	12 :	25	3,752
The Fourth ——	15 :	22 :	41	8,698
The Fifth ——	79 :	7 :	48	25,348

3640. The

A New Equal ALTITUDE INSTRUMENT for drawing a MERIDIAN LINE.

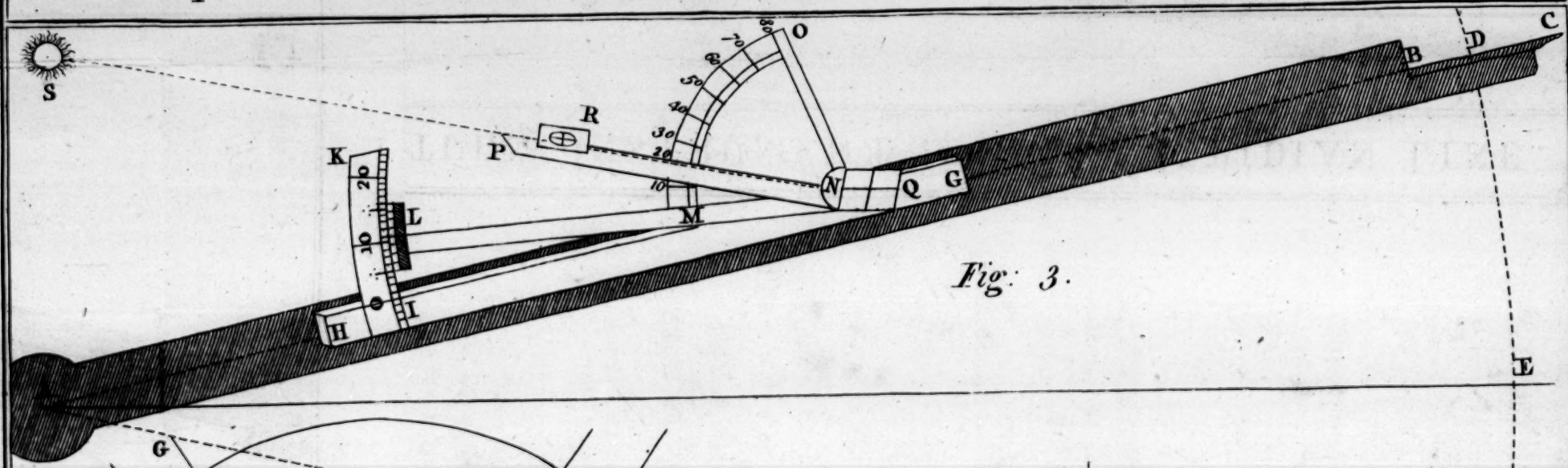


Fig: 3.

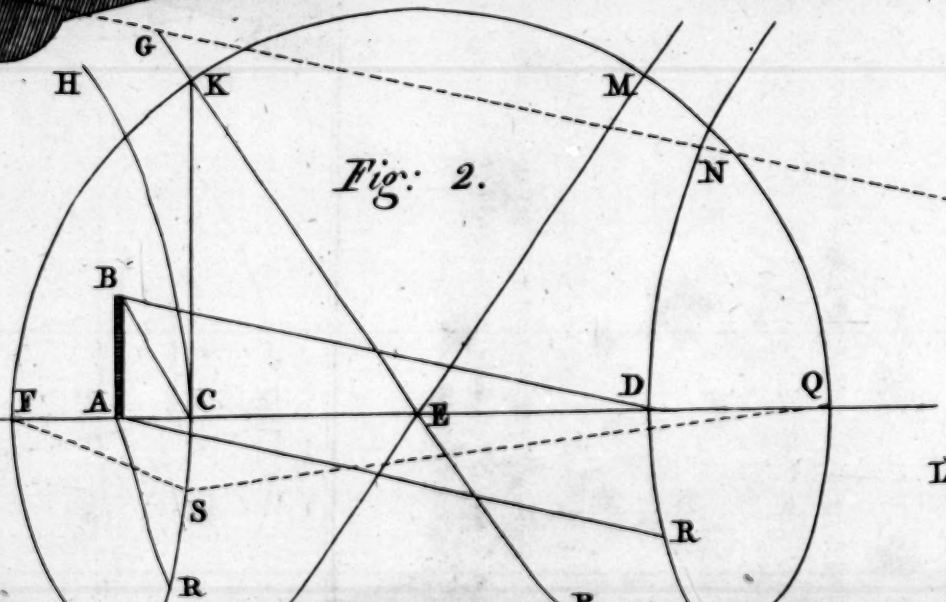
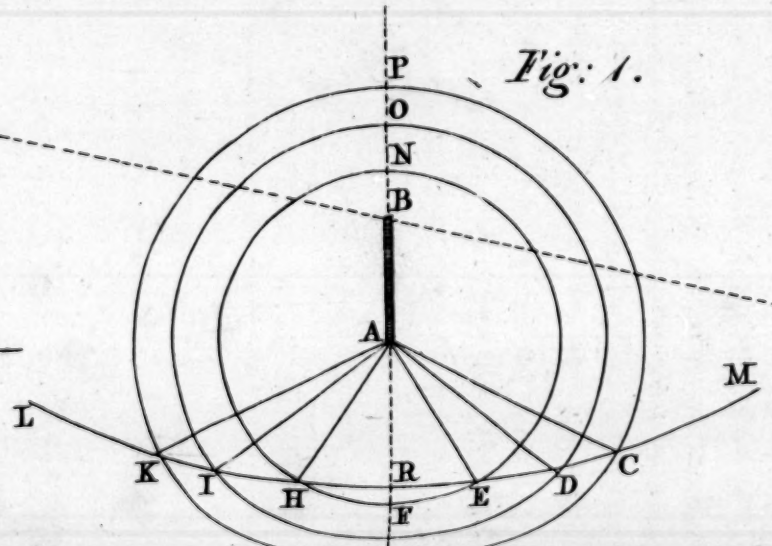


Fig: 2.

Fig: 1.



3640. The four inmoſt Satellites deſcribe their Orbits very nearly in the Plane of the Ring produced; which Plane makes an Angle of about 31 Degrees with the Plane of the Ecliptic; and the Nodes are in 19° 45′ of *Virgo* and *Piſces*. The Orbit of the fifth Satellite is a little wide of the Reſt. Which Particulars being obſerved in the Conſtruction of this Lunarium, it will very naturally exhibit the various Phænomena of this *Saturnian* Syſtem of Moons and Ring, for every Year of *Saturn*'s Period. *Note*, in this Lunarium, the Line of the Nodes muſt be kept always parallel to itſelf, as was ſhewn for the Line of the Earth's Nodes (3611.) And then, as *Saturn* paſſes from the Node in *Virgo* to the oppoſite one, the Sun will enlighten the northern Plane of the Ring; as it will the ſouthern Plane in paſſing through the other ſix Signs.

CHAP. XXV.

The ASTRONOMICAL DOCTRINE *of the* EQUATION *of* TIME *explained.* A CALCULATION *of that Part ariſing from the* OBLIQUITY *of the* ECLIPTIC *to the* PLANE *of the* EQUATOR; *with a* TABLE *of this* EQUATION *for every* DEGREE *of the* ECLIPTIC.

3641. IF a Clock or Watch could be conſtructed with Mechaniſm abſolutely perfect, it would always ſhew or keep *equal Time*, or *go truly*; but as this is not the Caſe it will often want rectifying, or being ſet right. Now as we have no direct or immediate *Standard* or *Index* of equal Time, by which the Error of a Time-Piece can be pointed out and inſtantly corrected, we muſt be content, or rather, we ought to felicitate ourſelves that we have it in our Power to do this by any Means at all. For this Purpoſe we have recourſe to two Methods

thods generally; one is the *Sun-Dial* and an *Equation Table*; the other is by aſtronomical Obſervation and Calculation.

3642. The firſt Method is very eaſy, and therefore adapted to common Uſe, which is now to be explained; as we have before given the Theory of the 2d Method largely from Inſt. 1915 to 1920 incluſive. The *Equation of Time* is a Doctrine reſulting from two Principles, *viz.* (1.) The Sun's *apparent diurnal Motion* not being in the *Equator*, the only great Circle of *equal Motion* with reſpect to the Meridian; and (2.) The *annual apparent* Motion of the Sun not being in *a Circle*, but an *Ellipſis* which cannot admit of *Equal Motion*; as we have abundantly ſhewn (1236.)

3643. We ſhall conſider each Part of the *Equation of Time* ſeparately, and then both together. Therefore, firſt with reſpect to the Motion of the Sun being in the Ecliptic and not in the Equator; it is evident from the View of any Globe revolving upon its Axis under the Meridian, that the Motion, or rather the Velocity of the Motion, by which the ſeveral Points of the Ecliptic and Equator paſs the Meridian will be very different; and that, that of the Equator will be conſtant and equable, and that of the Ecliptic always variable and unequal. But this Matter will be eaſily elucidated by a Figure.

3644. Therefore let E I (Fig. 1.) be the firſt Quadrant of the Equator, and E H the firſt of the Ecliptic; the Angle I E H being 23° 29′. Alſo, let P be the Pole of the Equator P H I, a Quadrant of the Solſtitial Colure and P F G a Poſition of the Meridian very near to it; let P A B be the Meridian very nigh to the Equinoctial Point E, and P E the Meridian paſſing thro' that Point; then it is evident, that upon the firſt momentary Motion of the Globe, the very ſmall Arches E B and E A will paſs under the Meridian in the ſame Inſtant, and therefore thoſe Arches will adequately repreſent the Fluxions or Ratio of the inceptive Velocities of Motion in the Equator and Ecliptic at the Equinox E.

3645. Again, as P H I is the *Solſtitial Colure* and P F G a Meridian exceeding near it, the Arches G I and F H, will be as the Fluxions or ultimate Velocities with which the two Arches E G and E F become equal in the *Solſtice* H. But the infinitely ſmall Arches E B and E A may be conſidered

as

as right Lines, or the Angle AEB as rectilineal; and therefore EB : EA :: Co-ſine of AEB : Radius :: $\dot{x} : \dot{y}$. And GI : FH :: Radius : Co-ſine of IH (= AEB) :: $\dot{x} : \dot{v}$. Therefore, we have $\dot{x} : \dot{y} :: \dot{v} : \dot{x}$, or $\dot{y}\,\dot{v} = \dot{x}^2$, a conſtant Quantity; *therefore the Velocity of Motion in the Beginning of the Ecliptic* E *exceeds the conſtant Velocity of the Equator juſt as much as it falls ſhort of it at the End of the Quadrant in* H.

3646. There is, therefore, ſome intermediate Point C where the Velocity in the Ecliptic is *equal* to that in Right Aſcenſion in the Equator, which is now to be inveſtigated; make the Arch E*d* = EC, and D*d* will be the Difference of the two Arches EC and ED, which, in that Caſe, muſt be a *Maximum*; for let the Arch EC = z, and ED = x; then will D*d* = $z - x$, whoſe Fluxion, when a *Maximum*, is $\dot{z} - \dot{x} = 0$, (818,) or $\dot{z} = \dot{x}$; that is the *Fluxions or Velocities of the Arches* EC *and* ED *will be equal, when their Difference* D*d becomes the greateſt of all.*

3647. Now to determine the Quantity of the Arch EC, when D*d* is a *Maximum*, we have (by *Fluxionary* Spherics 1818,) the Fluxion of EC ($\dot{z}$) to the Fluxion of ED ($\dot{x}$) as the Co-ſine of CD to the Sine of C. Alſo by common Spherics (1799,) we have, the Sine of C to the Co-ſine of E as Radius to the Co-ſine of CD. Or thus in ſhort, $\dot{z} : \dot{x} ::$ *cs* CD : *s*C; and *s*C : *cs*D :: R : *cs*CD; therefore $\frac{\dot{x} \times cs\,CD}{\dot{z}} = sC = \frac{cs\,E \times R}{cs\,CD}$; which will give this Analogy, $cs\,CD^2 : cs\,E \times R :: \dot{z} : \dot{x}$; and therefore when C*c* = D*d*, or $\dot{z} = \dot{x}$, or when D*d* is a Maximum (3646,) we have $cs\,CD^2 = cs\,E \times R$; and ſo $cs\,CD = \sqrt{cs\,E \times R}$. Whence EC, ED, and D*d*, are all known.

3648. The ſame Arch EC may be found without Fluxions by premiſing the following *Lemma*, *viz. The Sum of the Tangents of any two Angles* BAC, BAD, *is to their Difference, as the Sine of the Sum of thoſe Angles is to the Sine of their Difference.* Let BC and BD (Fig. 2.) be the two propoſed Tangents to the Radius AB; and take B*d* = BD, join A*d*, and draw DE and *d*F perpendicular to AC. Then it is manifeſt, becauſe B*d* = BD, that AD = A*d*, and *d*AB = DAB, and conſequently, that CA*d* is the Difference of the two Angles

BAC

BAC and BAD. Then are the Triangles CDE and C*d*F ſimilar, and ſo we have CD (= CB + BD) : C*d* (= CB — BD) :: DE : *d*F; but DE and *d*F are Sines of DAE and *d*AF to the equal Radii AD and A*d*, whence the Truth of the *Lemma* is evident.

3649. Then (*per* Spherics, 1789) Radius : Co-ſine E :: Tangent EC : Tangent ED (Fig. 1.) and by Compoſition and Diviſion (648) we have Radius + *c s* E : Radius — *c s* E :: Tangent EC + Tangent ED : Tangent EC — Tangent ED :: $s\overline{EC + ED}$: $s\overline{EC - ED}$ (*per Lemma.*) But the Ratio of the firſt two Terms of thoſe Analogys is conſtant becauſe the Angle E is ſo, therefore that of the two laſt Terms will be ſo likewiſe; and conſequently, $s.\overline{EC - ED}$ will be a Maximum, when the $s.\overline{EC + ED}$ is ſo, that is, when EC + ED = 90° for then the Sine thereof will be = Radius. Whence we have Radius + *c s* E : Radius — *c s* E :: Radius : $s.\overline{EC - ED}$; whence becauſe the three firſt Terms are known the fourth $s.EC - ED = s.D\,d$, is known alſo.

3650. But there is yet a more direct and ſimple Method of coming at this Equation, becauſe when a *Maximum* it is known to be a *Third Proportional to Radius and Tangent of half the Angle* E; as will thus appear. The Radius is the Sine of 90°, and therefore is the Co-ſine of 0°. Therefore Radius + Co-ſine E : Radius — Co-ſine E :: Co-ſine 0 + Co-ſine E : Co-ſine 0 — Co-ſine E :: Co-tangent $\frac{E + 0}{2}$: Tangent $\frac{E - 0}{0}$ (1829) :: Co-tangent $\frac{1}{2}$ E : Tangent $\frac{1}{2}$ E. But it is $t : r :: r : ct$ (1831), therefore $r^2 = ct \times t$, therefore $r^2 \times t = ct \times t \times t$; whence $ct : t :: r^2 : t^2$, that is Co-tangent $\frac{1}{2}$ E : Tangent $\frac{1}{2}$ E :: Rad.^{2} : $\overline{\text{Tangent } \frac{1}{2} E}^2$. Therefore Rad^2 : $t.\overline{\frac{1}{2} E}^2$ :: Radius : Sine of $\overline{EC - ED} = s\,D\,d$, and dividing by Radius, we have Rad. Tang. $\frac{1}{2}$ E :: Tang. $\frac{1}{2}$ E : Sine of D*d*, as was to be demonſtrated.

3651. The Quantity of this Equation computed is D*d* = 2° 28′ 34″, which converted into *Time*, is 9 Minutes 54 Seconds; and this is the greateſt Difference of Time that would ever be found between the Sun-Dial and a Clock if the Obliquity

liquity of the Ecliptic or Sun's Path were the only Cauſe thereof. Now ſince E C + E D = 90, and D *d* = (E C — E D =) 2° 28′ 34″, therefore E C = 46° 14′, therefore the Equation begins from nothing in *Aries* ♈ or in the Point E, and increaſes to its Maximum in ♉ 16° 14′ and from thence decreaſes till the Sun arrives at ♋ in the Point H, where it again becomes nothing.

3652. In this firſt Quarter of the Ecliptic, it is evident, that had the Sun moved in the Equator it would have been in *d* at the Moment it is in the Ecliptic at C, and conſequently the Sun at C is in the Meridian P D, which is before the Time the *Equinoctial Sun* arrives to it, at *(d)*; as being then *Eaſtward* of it by the Difference in Time every correſponding to the Motion in Right Aſcenſion of the ſmall Arch D *d*. Whence it appears that the *apparent Time* of the *Ecliptic Sun*, or that ſhewn by a *Dial*, is in this firſt Quadrant always *before* the *Mean or Equal Time* in the Equator by the Difference belonging to the Equation D *d*, which, therefore, when found, muſt be ſubducted from the *Solar Time* to have the mean Time for the Watch or Clock.

3653. By making the ſame Conſtruction in the ſecond Quadrant of the Ecliptic, it will appear, that the Point *(d)* is Weſtward of the Point C, and therefore comes firſt to the Meridian Arch. Whence the *mean Time* is now before the *apparent or Solar Time*, and conſequently the Equation D *d* now becomes *Addititious*, or muſt be added to the Solar Time to have the Mean, which now precedes it. Thus when the Sun is in ♌ 43° 46′ (= C H,) if we add 9′ 54″ to the Time ſhewn by the *Dial* it will give the *Mean Time* for the *Watch*; for ſo much does the *true Noon* of that Day exceed the *apparent Noon* on Account of the Sun's *oblique Motion*.

3654. It is evident the ſame Equation will be produced in the 3d and 4th Quadrants of the Ecliptic, and will accordingly be *Ablatitious* and *Addititious*, as is ſhewn in the following Table, where the Quantity of that Equation is computed for the Sun's Place in every Degree of the Ecliptic.

Subtract

Subtract from the Apparent Time.

The true Place of the Sun.

Signs	♈	♎	♉	♍	♊	♐	
Deg.	′	″	′	″	′	″	
0	0	0	8	23	8	45	30
1	0	20	8	34	8	35	29
2	0	40	8	44	8	24	28
3	1	0	8	53	8	13	27
4	1	19	9	2	8	1	26
5	1	39	9	10	7	48	25
6	1	58	9	17	7	34	24
7	2	18	9	24	7	20	23
8	2	37	9	30	7	6	22
9	2	56	9	35	6	50	21
10	3	15	9	40	6	35	20
11	3	34	9	44	6	18	19
12	3	52	9	48	6	2	18
13	4	11	9	50	5	44	17
14	4	29	9	52	5	27	16
15	4	46	9	54	5	8	15
16	5	4	9	54	4	50	14
17	5	21	9	54	4	31	13
18	5	37	9	53	4	11	12
19	5	54	9	51	3	52	11
20	6	10	9	49	3	32	10
21	6	25	9	46	3	11	9
22	6	40	9	42	2	51	8
23	6	55	9	38	2	30	7
24	7	9	9	33	2	9	6
25	7	23	9	26	1	48	5
26	7	36	9	19	1	26	4
27	7	48	9	12	1	5	3
28	8	1	9	4	0	43	2
29	8	12	8	55	0	22	1
20	8	23	8	45	0	0	0
	′	″	′	″	′	″	*Deg.*
	♓	♍	♒	♌	♑	♋	*Signs*

Add to the apparent Time.

CHAP. XXVI.

A CALCULATION *of that Part of the* EQUATION *of* TIME *which ariſes from the* Elliptic FORM *of the* EARTH's ORBIT; *with a proper* TABLE *thereof for every* DEGREE *of the* EARTH's ANOMALY.

3655. THE ſecond Cauſe of the Inequality of Time ſhewn by a DIAL and a CLOCK, was ſaid to be owing to the *Elliptic Form* of the Earth's annual Orbit (3642), and this is known to be Fact from common Obſervation; for through every Year the Diameter of the Sun is found to ſubtend a variable Angle, being ſenſibly greater at one Time than at another; for a few Days after the *Winter Solſtice* it meaſures, by the Micrometer, about $32' 43''$, but the ſame Diſtance after the *Summer Solſtice* it is no more than $31' 38''$.

3656. Now were the Orbit of the Earth truly *circular* it would be always at an equal Diſtance from the Sun, which would therefore always appear of an equal Bigneſs; conſequently, as it does not ſo appear, it muſt be at unequal Diſtances from us to cauſe thoſe unequal apparent Magnitudes (1451.) And becauſe the greateſt and leaſt ſolar Diameters are as $32' 42'' = 1963''$, and $31' 38'' = 1898$; the greateſt and leaſt Diſtances of the Earth from the Sun will be inverſely as the ſame Numbers 1963 and 1898.

3657. Let this Ellipſe Orbit of the Earth be denoted by A B P D, (Fig. 3.) in whoſe Focus S is the Sun; and let the other Focus be F. Upon the Center S deſcribe a Circle G I K L, whoſe Diameter G K is a *Mean Proportional* between the two Axis A P and B D of the Ellipſis; then will the Area of that Circle be equal to the Area of the Ellipſis (894.) And we may now compare the Motions in the Ellipſis and Circle as they would each be deſcribed in the ſame Time; and thus inveſtigate an Equation by which the unequal Motion of the Earth in the former, and the equal Motion of a Point in the latter, may be always equated, or adjuſted to each other.

3658. Suppoſe now the Earth begins its Motion from the *Aphelion* A of the Ellipſe, at the ſame Time that the imaginary Point ſets out with an equable Motion in the Circle from G; then becauſe the Diſtance A S exceeds the Diſtance S G, the Velocity of the Earth will be leſs than that of the Point (1213), and therefore in the Time that the Earth deſcribes the Arch A *a*, the Point will have deſcribed G *g*, making the Area G S *g* equal to the Area A S *a*, purſuant to the general Law of Nature. (1212.)

3659. Becauſe the Area A S *a* = G S *g*, ſuduct from each Side the common Part or Area G S *d*, and there will remain the Area A G *d a* equal to that of the Triangle *d* S *g*; therefore the ſaid Area A G *d a* will be ever proportional to the *Equation of the Orbit*, or the Arch (*g d*). Conſequently when this Area becomes a *Maximum*, there the Equation will be greateſt alſo, which is in the Point L where the Circle interſects the Ellipſe, for then the ſaid Area becomes A L G.

3660. As in the Point L the Earth in its Orbit, and the imaginary Point in the Circle have both the ſame Diſtance from the Sun, L S; they have there the ſame Velocity of Motion; and after they have paſſed that Point L, the Earth approaching conſtantly nearer the Sun, will have a Velocity greater and conſtantly gaining upon that of the Point in the Circle, but ſtill the Point in N will be before the Earth at H, till at laſt the Angle H S N or Equation of the Orbit will vaniſh in the Line S P K, where the Earth arrives to its *Peribilion* P, and the Point in the Circle to K.

3661. The Area A S H being ſtill equal to the Area G S N, we have at length, the Semi-Ellipſis A L P = the Semi-circle G L K; and ſubducting the common Area G L P, there will remain the Area A L G = the Area P L K. And as the Equation aroſe from nothing in the Point A to its greateſt Magnitude at L, by *continual Increments*; ſo after paſſing the Point L, it muſt leſſen by *decreaſing Quantities*, which conſtitute the Area P L K; ſo that what was gained in the firſt Part, is loſt in the latter, and the Equation becomes nothing in the Line S K.

3662. Or thus; continue the Ray S H to M; then the Elliptic Sector A S H = G S N, as being deſcribed in the ſame Time;

Time; from each, take the common Part G L H S, and there will remain A L G = N L O + O S H = M H L + M S N; therefore A L G — M H L = K M H P = M S N, the Measure of which Angle is the Equation M N; and is therefore nothing when M H L becomes equal to A L G or P L K.

3663. In the firſt half of the Ellipſis, or while the Earth deſcends from the higher *Apſis* or Aphelion A to the lower P, the *Mean Anomaly* will ever be greater than the true, or the Place of the imaginary Point *(g)* will be to the Eaſt of the Point *(d)*. And therefore if we ſuppoſe the Earth at reſt in the Center at S, and the Circle G L K I to be the *Primum Mobile*, then *(d)* will be the Place of the Sun in the Ecliptic, and *(g)* that of the Point of *Mean Motion*; alſo A S P will be the *Meridian*, to which when *d* arrives, it will be NOON by the *Sun-Dial*, but when *g* comes to it it will be XII by the Watch. And ſince theſe apparent Motions of the Heavens are from Eaſt to Weſt, it is evident, the *Time of Noon by the Dial, will preced that by the Clock*; and conſequently the Equation or Arch *g d* (turned into Time) muſt be *ſubtracted from the Time by the Dial*, to have the Mean Time by the Clock or Watch, as that is now *ſlower than the Dial.*

3664. But all the Time the Earth aſcends from the lower *Apſis* or *Perihelion* P, to the Point of Interſection I, it will be nearer to the Sun at S than the Ecliptic or Circle, and conſequently its Motion will be ſwifter than that of the imaginary Point; therefore its Place *p* at any Time will be before that of the ſaid Point at *k*, and conſequently will come after it to the Meridian at P, in the *apparent Motions*; therefore the Mean Time will be before the ſolar Time, or the *Clock will now be faſter than the Dial*; and the Equation *e p* muſt be *added to the Time by the Dial* to have the mean Time by the *Watch.*

3665. This will alſo continue to be the Caſe till the Earth arrives at the *Aphelian* A; for though in paſſing from the mean Diſtance at I to the greateſt at A, the Velocity of the Earth at Q is leſs than that of the Point R in the Ecliptic, and is conſtantly decreaſing, yet it will always be before the ſaid Point; and therefore, with Reſpect to the *apparent Motion*, it muſt come later to the Meridian than the Point of equal Motion R, and ſo

the Equation R q is ſtill to *be added to the ſolar or apparent Time by the Dial*, to have the *mean Time by the Clock*.

3666. Here R q is the *Equation*; and the Area or Sector KSR = PSQ, from which take the common Area PSRI and there will remain PKI = QI q + RS q, and therefore PKI — QI q = RS q, is proportional to the Equation R q, (ſee 3662;) therefore when QI q becomes AIG (= PKI), the Equation of the Orbit vaniſhes again.

3667. Let GN be any given *mean Motion* (Fig. 4.) and AH the correſponding *true Anomaly*; draw S h, S n, indefinitely near to SH and SN; and put N n = $\dot{y}$, and H h = $\dot{x}$; then becauſe the fluxionary Triangle HS h = NS n, we have the Angle HS h : NS n :: NS^2 : HS^2 (1270) :: $\dot{x}$: $\dot{y}$. Therefore $\frac{NS^2 \times \dot{y}}{HS^2} = \dot{x}$, Fluxion of the true Anomaly AH.

3668. When the Fluxions of Quantities are equal, their Difference will be a *Maximum* (3646;) therefore in L, where SH = SN = SL we have $\dot{x} = \dot{y}$; and conſequently the *Equation of the Orbit* GN — AH, will there be the greateſt poſſible, as we before ſhewed, (3659.)

3669. We have found Areas, Angles, and Arches, proportional to the Equation of the Orbit; but to compute the real Quantity of this Equation, we muſt firſt ſolve the ſo much famed Problem of KEPLER, *viz. To cut off a Sector* AS a *by the Right Line* S a (Fig. 3.) *that ſhall be to the whole Area of the Ellipſe, as the Time of deſcribing the Arch* A a, *to the Time of a whole Revolution in the Elliptic Orbit.* But as the moſt geometrical Way of doing this, is by *infinite Series*; and there is a much eaſier and very exact Method of computing the Equation invented by the late Biſhop *Ward* we ſhall here explain that, with *Bullialdus*'s Correction thereof.

3670. This Method depends on an Hypotheſis, *that a Ray* FL (Fig. 5.) *drawn from the upper Focus* F *to the Planet at* L, *deſcribes an Angle* AFL *proportional to the Time of the Planet's paſſing through the Arch* AL. Therefore let ABP be the Ellipſe the Planet deſcribes; AP the Line of the *Apſides*; S the Focus in which is the Sun, F the other Focus or Center of equal Motion. The Angle AFL being as the Time, the Place of the Planet will

be

be at L, and the Angle A S L is the *true* or *coequate Anomaly*.

3671. Produce F L towards E, and make F E = A P, and join E S. Then is L E = L S (769,) and the Triangle E L S being iſoſcles, the Angle L E S = L S E, and both together equal to the Angle F L S (632.) Therefore in the Triangle E F S, having the Sides F E and F S, and the Angle of mean Anomaly A F E, or E F S, we find the Angle E, and 2 E = F L S, which is the Difference between the mean Anomaly A F L, and the True A S L, and is therefore the Value of the *Equation ſought*; for this taken from the mean Anomaly A F L leaves the True one A S L.

3672. Alſo in the Triangle F L S, having all the Angles and the Side F S, the Side S L is known, which is the Diſtance of the Earth from the Sun, in its Orbit at L. And thus the Equation of the Orbit, and Diſtance from the Sun, is found in this Hypotheſis very exactly for the Earth; but for the other Planets, eſpecially *Mars* and *Mercury*, it requires ſome Correction, which it received from the celebrated Aſtronomer *Iſmael Bullialdus*, as follows.

3673. Upon the tranſverſe Axis A P deſcribe the Circle A D P, and let A F L be the mean Anomaly, as before. Thro' L draw the Line Q L G perpendicular to the Axis, meeting the Circle in Q; join F Q, cutting the Ellipſe in M, and M will be the Place of the Planet in its Orbit for the mean Anomaly A F L. Let B C be the Semi-conjugate, which continued to D, we have C B : C D :: G L : G Q, which is therefore known; join S M, and then the Angle A S M will be found in the ſame Manner as before the Angle A S L was found, (3671.)

3674. This Correction of *Bullialdus* accelerates the Motion of the Planets in the *firſt and third Quarter*; and in the *Second and Fourth retards* them a very little Matter, in Reſpect of *Ward*'s Hypotheſis, which makes their Places by this Theory agree much better with Obſervations. But the late Mr. SIMPSON has given us a Conſtruction which finds the Planet's Place ſtill much nearer the Truth than either of the foregoing.*

3675. The general Reaſon of this Hypotheſis is this, that the Velocity in the Orbit being every where inverſely as a Perpendicular

* See his *Mathematical Eſſays*, p. 41.

pendicular on the Center of Force S upon a Tangent to the Orbit in a given Point or Place of the Planet (1213), it will follow that A*a*, B*b*, P*p* be Spaces paſſed through in equal Times; then drawing the Lines F*a*, F*b*, F*p*, the Angles A F*a*, B F*b*, P F*p*, will in the Earth's Orbit, be extremely near equal. For at A and P we have A*a* : P*p* :: P S : S A :: A F : F P; therefore A F*a* = P F*p*. Again, A*a* : B*b* :: B S : S A :: F B : F P; but in the Earth's Ellipſe F is very near to C, and we ſhall have F B : F P :: A F : F B, very nearly; conſequently the Angle B F*b* = A F*b*, extremely near. Therefore *the angular Motion about the upper Focus* F *is very near equable.**

3676. When the Planet is in B, the middle Point of its Semi-Ellipſe, then F E becomes F N, and F B = B N = A C = S B; and F S N is a Right Angle, and ſince the Point B very nearly coincides with the Point I (Fig. 3.) where the Equation of the Orbit is a *Maximum*, therefore the Angle F B C will inſenſibly differ from the Half of that Equation. And becauſe (3656) A S is as 1963, and S P as 1898, therefore A C = F B, will be as 1930,5, and A S — A C = C S = C F, will be as 32,5; therefore ſay As F B = 1930,5 : F C :: Radius : Sine of F B C = 58′ 9″½, which doubled, 1° 56′ 19″, = F B S, the greateſt Equation of the Earth's Orbit; and ſuch you find it in the Aſtronomical Tables of Dr. HALLEY. This Equation turned into Time for every Degree of the Earth's Anomaly, is as in the following Table.

* Thro' Inadvertence, the Lines, F*a*, F*b*, F*p*, were forgot to be drawn in Fig. 5. but they can be eaſily ſupplied by the Reader, and it is therefore hoped will occaſion no Obſtruction to his underſtanding this material Point.

Subtract

Subtract from the Apparent Time.

The Mean Anomaly of the Sun.

Signs	0		1		2		3		4		5		
Deg.	′	″	′	″	′	″	′	″	′	″	′	″	
0	0	0	3	48	6	39	7	45	6	47	3	57	30
1	0	8	3	55	6	43	7	45	6	43	3	50	29
2	0	16	4	2	6	47	7	45	6	39	3	43	28
3	0	24	4	9	6	51	7	45	6	35	3	35	27
4	0	32	4	16	6	54	7	45	6	30	3	28	26
5	0	40	4	22	6	58	7	44	6	26	3	20	25
6	0	48	4	29	7	1	7	44	6	21	3	13	24
7	0	56	4	35	7	5	7	43	6	16	3	5	23
8	1	3	4	42	7	8	7	42	6	11	2	58	22
9	1	11	4	48	7	11	7	41	6	6	2	50	21
10	1	19	4	54	7	14	7	40	6	1	2	42	20
11	1	27	5	0	7	17	7	39	5	56	2	34	19
12	1	35	5	7	7	19	7	37	5	51	2	27	18
13	1	12	5	12	7	22	7	36	5	45	2	19	17
14	1	50	5	18	7	25	7	34	5	40	2	11	16
15	1	58	5	24	7	27	7	32	5	34	2	3	15
16	2	6	5	30	7	29	7	30	5	28	1	55	14
17	2	13	5	35	7	31	7	28	5	22	1	47	13
18	2	21	5	41	7	33	7	25	5	16	1	39	12
19	2	28	5	46	7	35	7	23	5	10	1	31	11
20	2	36	5	51	7	36	7	20	5	4	1	22	10
21	2	43	5	57	7	38	7	18	4	58	1	14	9
22	2	51	6	2	7	39	7	15	4	51	1	6	8
23	2	58	6	7	7	41	7	12	4	45	0	58	7
24	3	6	6	12	7	42	7	9	4	38	0	50	6
25	3	13	6	17	7	43	7	5	4	32	0	41	5
26	3	20	6	21	7	43	7	2	4	25	0	33	4
27	3	27	6	26	7	44	6	59	4	18	0	25	3
28	3	34	6	30	7	44	6	55	4	11	0	17	2
29	3	41	6	35	7	45	6	51	4	4	0	8	1
20	3	48	6	39	7	45	6	47	3	57	0	0	0
	′	″	′	″	′	″	′	″	′	″	′	″	*Deg.*
	11		10		9		8		7		6		*Signs*

Add to the apparent Time.

CHAP. XXVII.

The THEORY *of the* Compound EQUATION *of* TIME, *and the* CONSTRUCTION *of one* General TABLE *thereof depending on the* SUN's PLACE *in the* ECLIPTIC.

3678. THE two Equations of Time we have now conſidered muſt be both applied to obtain a general or abſolute Equation, conſiſting of the Sum or Difference of theſe particular Ones, according to their affirmative or negative Quality; but becauſe One requires the Knowledge of the Sun's Place in the Ecliptic, and the other, the Sign and Degree of the Earth's or Sun's mean Anomaly; therefore to render this Matter leſs troubleſome or difficult, Aſtronomers have compoſed one general Table out of the two, making it depend entirely on the Sun's Place in the Ecliptic.

3679. But this general Table can be only a *temporary One*, as ſerving with any Exactneſs only a certain Time, becauſe the Line of the Apſides A P, from which the Numbers in the ſecond Table begin, is not fixed with reſpect to Signs of the Ecliptic, but has a direct ſlow Motion through the ſame at the Rate of one Degree in 59 Years, or 1° 41′ 7″ *per Century*. Conſequently in about 30 or 40 Years this general Table will require to be renewed where any Computations of Accuracy are concerned, but for barely ſetting common Clocks by a Dial, it may ſerve much longer.

3680. By Dr. *Halley*'s Tables, the Poſition of this Line A P is in 8° 42′ 52″ of *Cancer* and *Capricorn* at the Beginning of the Year 1764. But as the Theory of this compound Equation is not quite ſo ſimple and evident as every young Horologiſt might wiſh, I ſhall endeavour to render it plain by an algebraic *Calculus*, and to illuſtrate the ſame by a proper Diagram; both which have been hitherto wanting in Books treating of this Subject. But to facilitate theſe Demonſtations the two following *Lemmata* are to be premiſed.

3681. In

3681. *In any Triangle* FHS (Fig. 6.) *we have* $\overline{FH+SH} \times \overline{FH-SH} = FS \times 2CV$; *ſuppoſing* HV *perpendicular to* FS, *and* FC = CS. For with HF as a Radius, on the Point H as a Center, deſcribe the Circle ABF and continue the Side SH each Way to the Circle, and the Baſe FS to the Circle in B. Then is AH = HF, and AS = FH + SH, and SE = FH − SH; and becauſe it is ES × SA = FS × SB (658,) and SB = FV + VS = FS + 2VS; therefore $\frac{1}{2}SB = \frac{1}{2}FS + VS = CV$, conſequently 2CV = SB. Therefore $\overline{FH+SH} \times \overline{FH-SH} = FS \times 2CV$.

3682. *The Rectangle of the Sines of two Arches added to the Rectangle of their Co-ſines, make a Sum equal to the Rectangle under the Radius and Co-ſine of their Difference.* (Fig. 7.) Let the two Arches be AC, and CD (= CB); their Sum AD, and Difference AB; let CF and OF be the Sine and Co-ſine of the greater Arch AC; and let *m* D (= *m* B) and O *m* be thoſe of the leſſer Arch CD, or OB. Alſo let BE and OE be thoſe of the Difference AB. Draw *m n* parallel to CF; and *m v* parallel to AO; then it is plain the Triangles OCF, O *m n*, and D *m v* are ſimilar; therefore we have OC : OF :: O *m* : O *n*, whence OC × O *n* = OF × O *m*. Again, OC : CF :: D *m* : *m v*, therefore OC × *m v* = CF × D *m*. Conſequently $OC \times \overline{On+mv} = OC \times OE = \overline{OF \times Om} + \overline{CF \times Dm}$. *Q. E. D.*

3683. Theſe Lemmas premiſed, if we look back on Fig. 4. we ſhall there find the ſame Triangle FHS (as in Fig. 6.) by drawing the Line FH, and letting fall the Perpendicular HV. Put AC = CP = *a*, CB = *b*, CS = CF = *c*, SH = *v*, and the Co-ſine of the Angle HSP = *x*, to the Radius SK = SN = 1. Then $1 : x :: v : vx = SV$, alſo FH = AP − SH (769) = 2*a* − *v*, and FS = 2*c*; wherefore $\overline{FH+SH} \times \overline{FH-SH} = FS \times 2CV$; that is, $2a \times \overline{2a-2v} = 2c \times 2 \times \overline{c+xv}$; from whence we have $v = \frac{a^2-c^2}{a+cx} = \frac{b^2}{a+cx} = SH$, the *Diſtance of the Earth from the Sun.*

3684. And becauſe $SN^2 = SK^2 = ab$ (3657), we ſhall have $\frac{SN^2}{SH^2} \times \dot{y} = \frac{a \times \overline{a + cx}^2}{b^3} \times \dot{y}$, the Fluxion of the true Longitude A H, or Diſtance from the Aphelion A. Having thus obtained an Expreſſion of the Increment of the Longitude for the *Elliptic Anomaly*, and having before found it for the *Obliquity of the Ecliptic* (3647), we can find how far both theſe Cauſes together will affect the Motion of the Earth in *Right Aſcenſion*, ſince in that alone conſiſts the whole Ground and Reaſon of the *Equation of Time*; and if we thus unite both their Effects we ſhall have the whole or *abſolute Equation* in one Theorem, for any Value of x, or Longitude of the Earth from either Apſis A or P.

3685 For the Fluxion of the Longitude or true Anomaly being to that of the Equator or Motion in Right Aſcenſion always as $\overline{cs\,CD}^2 : cs\,E \times R :: \dot{z} : \dot{x}$, (3647.) Therefore ſay, As $\overline{cs\,CD}^2 : cs\,E \times R :: \frac{a \times \overline{a + cx}^2}{b^3} \times \dot{y} : \frac{a \times \overline{a + cx}^2}{b^3} \times \dot{y} \times \frac{cs\,E \times R}{\overline{cs\,CD}^2}$ = the Fluxion of the Motion in the Equator, ariſing from both the Cauſes of Irregularity united.

3686. Now this being compared with the equable Motion in Right Aſcenſion of the imaginary Point in the Circle G L P I, it will eaſily appear when their Difference is the greateſt or a Maximum, becauſe in that Caſe their Fluxions will be equal (3646,) *viz.* $\frac{a \times \overline{a + cx}^2}{b^3} \times \dot{y} \times \frac{cs\,E \times R}{\overline{cs\,CD}^2} = \dot{y}$; and conſequently in that Caſe $\frac{a \times \overline{a + cx}^2}{b^3} = \frac{\overline{cs\,DC}^2}{cs\,E \times R}$.

3687. But as this latter Part of the Equation is not of the ſame Form with the firſt, it muſt be reduced to algebraic Terms; therefore put m = Sine of K S ♋, the Diſtance of the Perihelion P from the Solſtice ♋; its Co-ſine = n; then the Co-ſine of the Angle P S H being = x, its Sine will be = $\sqrt{1 - xx}$; and by the Lemma (3682) we have $nx + m\sqrt{1 - xx}$ = Co-ſine of H S ♋, or C S ♋ = Sine of ♈ C or Longitude of the Earth at H reduced to the Ecliptic G L K I.

3688. Let

3688. Let E N M (Fig. 8.) be the Projection of that half of the Equator which is above the Plane of the Ecliptic, and E Q M of that half which is below it; and C D a Perpendicular thereto; then is the Right-angled Triangle C E D the ſame with that in Fig. 1. Put p = Sine of the conſtant Angle E, and q = Co-ſine, and we have (by *Spherics*,) Radius (1) : *s.*E C $(nx + m\sqrt{1-x^2})$ :: $p : pnx + pm\sqrt{1-x^2}$ = Sine of CD, whoſe Square Co-ſine is therefore $1 - \overline{pnx + pm\sqrt{1-xx}}^2$ = $\overline{cs.\text{C D}}^2$; and the cs E = q; theſe Values ſubſtituted in the Equation above, give $\frac{a \times \overline{a + cx}^2}{b^3} = \frac{1 - \overline{pnx + pm\sqrt{1-xx}}^2}{q}$. This Equation reduced, gives the Value of x, or Quantity of the Angle H S P.

3689. While the Angle H S P is *acute*, x will be affirmative, or $+x$; but when *obtuſe* it will be negative, or $-x$. Alſo while H is in the Semi-ellipſis A L P, m will be affirmative or $+m$; but when H is in the other Half P I A it is negative, or $-m$; and the Equation there becomes $\frac{a \times \overline{a + cx}^2}{b^3} = \frac{1 - \overline{pnx - pm\sqrt{1-xx}}^2}{q}$.

3690. From all which it is evident, that this *Compound Equation of Time*, as it ariſes from the *Sums* and *Differences* of the two ſingle Equations muſt be of a different Value in different Parts of the Orbit; and that there will be *four* of thoſe *Maxima* in all, *viz.* two ariſing from the Sums, while x is poſitive; and two from the Differences of the ſingle Equations, when x is negative, or $-x$.

3691. But to make this Matter yet plainer, let A = the Equation for the Earth's *Elliptic Orbit*, and B = that for the *Obliquity of the Eclipic*; then all the Time the Earth H is deſcending from A to P, the former is negative, or — A (3663.) And in aſcending from P to A, it will be affirmative or + A. Alſo in the 1ſt and 3d Quarters of the Ecliptic we have — B; but in the 2d and 4th it is + B (3653, 3657.)

3692. Now 'tis evident, in the Diagram (Fig. 8.) that the firſt Quadrant of the Ecliptic ♈ ♋ wholly coincides with the

Semi-Ellipſe A L P, and therefore the *Maximum* Equation will here ariſe from A + B, which, as they are both *Negative*, ſhews this Equation is ſo too, or that it muſt be *taken from* the Time by the Dial, to have the mean Time by the Watch or Clock, which is then *too ſlow*. This Equation is 16′ 13″ and is when the Earth is in ♉ : 10°, or the Sun in ♏ : 10°, on *November* 2d.

3693. Again, almoſt the whole of the 2d Quadrant of the Ecliptic is in the other Semi-Ellipſe P I A; conſequently the two ſimple Equations here being both poſitive their Sum A + B will be ſo too, or muſt be *added* to the *Solar Time* to get the Mean by the *Watch*, which is here *too faſt*. This Maximum amounts to 14′ 49″ on *February* 10th. When the Earth is in ♌ : 21° : 35′, or the Sun in ♒ 21° : 35′.

3694. The third Quadrant of the Ecliptic is wholly on the ſame Side with the Semi-Ellipſe A I P, and the Earth will enter it before the third *Maximum* Equation; therefore B will be *Negative* and that Equation will be A — B = 4′ 5″, to be *ſubducted* from the ſolar Time by the Dial, becauſe now B is greater than A, and the Clocks are *too ſlow*, again. This happens on *May* 15th, when the Earth is in ♏ : 24° : 18′ or the Sun in ♉ 24° 18′.

3695. Laſtly, the laſt Quadrant of the Ecliptic is nearly the whole of it on the ſame Side of the Line of the Apſides A P with the Semi-ellipſe A L P; here B will be poſitive, or + B, and A negative, or — A, and therefore, the Equation, when greateſt, will be = B — A = 5′ 55″; and becauſe B is here alſo greater than A, therefore the Equation is to be added to the apparent Time by the Dial to have the true or equal Time by the Clock or Watch which is now *too faſt*, or before the Dial. This *Maximum* happens on *July* 26, the Earth being in 3° of ♒, and the Sun in 3° of ♌.

3696. If the Reader attentively conſiders the ſeveral particular Circumſtances and Relations of theſe two ſimple Equations he will ſee how this Compound Equation will gradually ariſe by *Addition* and *Subtraction*, and increaſe to its various *Maxima*, and then alternately decreaſe to nothing, according as the Numbers appear in the following Table, which is here adjuſted for ready Uſe to the *Signs* and *Degrees of the Ecliptic*.

A Table of the Compound Equation of Time depending on the Sun's Place.

Signs	♈		♉		♊		♋		♌		♍	
	Add.		*Sub.*		*Sub.*		*Add.*		*Add.*		*Add.*	
Deg.	′	″	′	″	′	″	′	″	′	″	′	″
0	7	40	1	9	3	56	1	6	5	49	2	14
1	7	21	1	23	3	52	1	19	5	51	1	58
2	7	2	1	36	3	47	1	33	5	52	1	41
3	6	43	1	49	3	43	1	47	5	53	1	24
4	6	24	2	1	3	38	2	0	5	53	1	7
5	6	5	2	13	3	32	2	13	5	52	0	50
6	5	46	2	23	3	25	2	26	5	51	0	32
7	5	27	2	34	3	18	2	38	5	49	0	13
8	5	8	2	44	3	11	2	51	5	47	—	5
9	4	49	2	53	3	3	3	4	5	44	0	24
10	4	30	3	2	2	54	3	16	5	40	0	44
11	4	11	3	10	2	44	3	27	5	35	1	4
12	3	53	3	17	2	35	3	38	5	30	1	24
13	3	34	3	25	2	24	3	50	5	24	1	44
14	3	16	3	32	2	15	4	1	5	17	2	4
15	2	57	3	39	2	4	4	11	5	10	2	24
16	2	39	3	45	1	53	4	21	5	2	2	45
17	2	21	3	50	1	41	4	30	4	54	3	6
18	2	3	3	54	1	29	4	39	4	45	3	27
19	1	45	3	57	1	18	4	48	4	36	3	48
20	1	27	4	0	1	6	4	57	4	26	4	9
21	1	10	4	2	0	53	5	5	4	15	4	30
22	0	53	4	4	0	41	5	12	4	4	4	51
23	0	37	4	6	0	28	5	18	3	52	5	12
24	0	20	4	6	0	15	5	24	3	39	5	34
25	0	4	4	6	0	2	5	31	3	27	5	55
26	—	11	4	5	+	12	5	36	3	13	6	17
27	0	26	4	4	0	25	5	40	2	59	6	39
28	0	41	4	2	0	39	5	43	2	45	7	0
29	0	56	3	59	0	53	5	46	2	30	7	21
30	1	9	3	56	1	6	5	49	2	14	7	42

The Equation following the Sign + to be added, and the Sign — to be ſubducted from the apparent Time, to get the Mean Time.

A Table of the Compound Equation of Time depending on the Sun's Place.

♎		♏		♐		♑		♒		♓	
Sub.		*Sub.*		*Sub.*		*Sub.*		*Add.*		*Add.*	
′	″	′	″	′	″	′	″	′	″	′	″
7	42	15	32	13	29	1	4	11	38	14	26
8	3	15	40	13	13	0	35	11	55	14	20
8	24	15	47	12	56	0	5	12	12	14	13
8	44	15	53	12	38	+	25	12	27	14	6
9	4	15	58	12	19	0	54	12	41	13	59
9	24	16	2	11	59	1	23	12	54	13	50
9	43	16	5	11	38	1	53	13	7	13	41
10	3	16	8	11	17	2	22	13	20	13	30
10	22	16	10	10	55	2	51	13	32	13	20
10	41	16	11	10	33	3	19	13	43	13	9
11	0	16	11	10	10	3	48	13	53	12	57
11	18	16	11	9	46	4	16	14	2	12	45
11	36	16	10	9	22	4	43	14	11	12	33
11	53	16	8	8	58	5	11	14	19	12	20
12	10	16	5	8	34	5	38	14	25	12	6
12	27	16	1	8	8	6	4	14	31	11	52
12	43	15	57	7	42	6	30	14	36	11	37
12	59	15	52	7	15	6	55	14	40	11	22
13	14	15	46	6	48	7	21	14	43	11	7
13	29	15	39	6	21	7	45	14	46	10	52
13	43	15	32	5	53	8	10	14	48	10	36
12	57	15	24	5	25	8	34	14	49	10	19
14	10	15	14	4	58	8	57	14	50	10	2
14	22	15	3	4	30	9	19	14	50	9	45
14	34	14	51	4	1	9	41	14	49	9	28
14	46	14	39	3	32	10	2	14	47	9	11
14	56	14	27	3	2	10	23	14	45	8	53
15	6	14	14	2	33	10	42	14	41	8	35
15	15	14	0	2	3	11	1	14	37	8	17
15	24	13	45	1	33	11	20	14	32	7	59
15	32	13	29	1	4	11	38	14	27	7	40

The Equations following the Sign + to be added, and the Sign — to be subducted from the apparent Time, to get the Mean Time.

CHAP. XXVIII.

Of the beſt METHODS *of drawing a* MERIDIAN LINE *by* Concentric CIRCLES, *by the* HYPERBOLA, &c. *The Theory of a* New EQUAL ALTITUDE INSTRUMENT *for that Purpoſe.*

3698. AS CLOCKS and DIALS are of a like Nature and Uſe, ſo they are mutually ſubſervient to each other, and ſerve to correct one another, when either ſhall chance to be at Fault; but to this End it muſt be ſuppoſed that they are originally each of them properly conſtructed, and the Dial, particularly, fixed in a *true Poſition.* And as this is ſo important a Point, we here propoſe to give ſuch Directions and Precautions as are neceſſary to be obſerved in that Affair.

3699. The firſt Thing to be obſerved in fixing a *Horizontal Dial* is, that the Surface of the Pedaſtal be *truly level* or *horizontal*, and this muſt be examined and thoroughly rectified by a Plumb-line, or Spirit Level. The Reaſon of this Injunction is becauſe the Gnomon of the Dial cannot be *parallel to the Axis of the Earth* without ſuch an *horizontal Poſition*; as we have ſhewn when we treated of Dialling.

3700. The ſecond Thing neceſſary is to place the Meridian of the Dial truly *North and South*, or exactly in the *Meridian of the Place*; for this Purpoſe it will not be ſufficient to ſet it by a *Magnetical Needle* in a Rectangular Box, neither by a Clock, Watch, or any other Dial; but quite independently of any other Time-Piece, by a true MERIDIAN LINE. This is upon Suppoſition the Dial is a very good one, and intended to go very correctly.

3701. As a MERIDIAN LINE is of the moſt extenſive Uſe, we ſhall here give ſome of the principal Methods for drawing one. The moſt ſimple and practical of theſe is the following. Upon a Point A (properly choſen) as a Center, deſcribe ſeveral concentric Circles as N H E, O I D, P K C; and in the ſaid Center A fix a ſtrong Wire, as truly upright or perpendicular to the Horizon as poſſible, the Height of it ſix or eight Inches; then

then obſerving, in the Forenoon, very nicely where the End of the *Shadow* touches each Circle, and there make fine Marks, as ſuppoſe in the Points C, D, E; and do the ſame for the Afternoon Shadows, as at H, I, K. Then biſect the Arch E H in F, and the Arch C K in G; and through the Points A, F, G, draw the Right Line A F G, and it ſhall be the *Meridian-Line* required.

3702. It has been formerly ſhewn (1766,) that the Curve L R M deſcribed by the End of the Shadow is an *Hyperbola* (in any Latitude leſs than 66½ Degrees,) whoſe Vertex is R and its Axis the *Meridian Line* P R G. Further, it is eaſy to conceive, that the Sun in the Horizon, projects the Shadow to an infinite Diſtance, and there the Curve of the Hyberbola coincides with the *Aſymtote*; and that this is at a Point in the Horizon juſt oppoſite to the Sun; conſequently the infinite Line joining theſe two oppoſite Points will be an *Aſymptote* to the hyperbolic Curve of the Shadow deſcribed on the given Horizontal Plane, and will contain with the Meridian of the Place an Angle equal to the *Co-Amplitude* for that Day.

3703. Therefore from the given LATITUDE of the Place, and the DECLINATION and AMPLITUDE of the Sun, the *Hyperbola* for the given Day may be deſcribed, and the *Meridian Line* thereby found by a ſingle Obſervation of the Shadow; and verified by any Number of them you pleaſe. In order to this Practice, it is to be conſidered, that on any Day in the *Summer* or *Winter* when the Sun's Declination is the ſame, there will be the ſame or equal Hyperbolas deſcribed by the Shadow, whoſe Vertices will be determined by the Shadow of the Meridian Altitude on each Day.

3704. Thus, for Example: Let A B be a Gnomon of a given Height, and let A C B be the Meridian Altitude of the Sun, equal to the Sum of the *Co-Latitude* and *Declination*, whence the Diſtance A C is known. Again, the Angle A D B, on the Winter Day, is the Meridian Altitude, equal to the Difference of the Co-Latitude and Declination; and therefore A D is known; from which take A C, and the Remainder C D is the *Tranſverſe Axis* of the Hyperbola's.

3705. Biſect C D in E, and that will be the Center of the Hyperbola's; thro' which draw two Right Lines G P and L M

making

making with the Line A D Angles G E A and M E D equal to the *Co-Amplitude* and they will be the *Aſymptotes* to the *Hyperbolas*.

3706. On the Vertical Point C erect the Perpendicular C K meeting the Aſymptote E G in K, and C K will be the Semi-Conjugate of theſe Hyperbolas; therefore make E F and E Q equal to E K, and the Points F and Q will be the *Foci* of the *Hyperbolas*, (768.)

3707. Thus having the *Diameters* and *Foci*, the Curves H C I and N D O may be readily deſcribed, either by the Inſtrument in the Plate of the Fuſee, or by finding a Number of Points S in the Curve which is very eaſy to do, becauſe it will be every where Q S — F S = C D (769;) and ſo for every Point S you have Q S = C D + F S, or F S = Q S — C D, by which any Number of thoſe Points are found, and the Curve of the Hyperbola drawn through them as required.

3708. Having thus conſtructed the Hyperbola H C I for the given Day, let the Paper on which it is drawn, be ſo placed on the Plane, that the Point A may exactly coincide with the ſmall Hole in which the Wire or Pin was fixed, and then ſo moved or adjuſted, that the Length of any *obſerved Shadow* A R taken in a Pair of fine Compaſſes may be applied therein, that is, when one Point of the Compaſſes is in A the other may fall preciſely in the Curve of the Hyperbola at R; then will the Axis of the Hyperbolas F E Q be in the true Meridian of the Place. And by obſerving a Number of theſe Shadows, its Poſition may be verified to great Exactneſs.

3709. The Time moſt proper for this Proceſs is at or near the *Solſtices*, and the Meridian determined at or about the *Summer Solſtice*, will be moſt exact, as the Shadows A R are then ſhorteſt and beſt defined at the Ends. The Reaſon of chuſing theſe Times of the Year, is becauſe the *Hyperbolas* have now the greateſt Degree of Curvature, from which they degenerate gradually till, in the Equinoctial Day, they become a *Right Line*, (1764.)

3710. But when the *Meridian Line* is to be determined of a conſiderable Length, other Methods may be more readily applied; that by an *Equal-Altitude-Teleſcope* is certainly a very good one, but the Expence of ſuch an Inſtrument, the Skill of the

Perſon, and Difficulty of the Performance, will confine this Method to very few Hands.

3711. The Method of *Tycho Brahe* was much eſteemed by himſelf and others in his Time; it conſiſted in having an Inſtrument ſo placed on a Plane that he could eaſily thereby obſerve the *greateſt Elongation* of any *circumpolar Star* towards the Eaſt and Weſt; conſequently the End of this Inſtrument would deſcribe an Arch, the Half whereof would meaſure theſe equal Elongations from the *Meridian*, which therefore became known or determined; *viz.* by biſecting that Arch, and drawing a Line through that Biſection and the Center on which the Inſtrument moved. But this Method is more ſuited to *Aſtronomers* than for common Uſe.

3712. Dr. DERHAM's *Meridian Inſtrument* is well known as a very uſeful One for drawing a Meridian Line with tolerable Exactneſs. But I no not think any Method ſo good, (that is, ſo exact,) as thoſe which have their Points and Centers *upon* (and not *above)* the Plane on which the ſaid Line is to be drawn, and which alſo are entirely independent of *Shadows*, becauſe in great Lengths Shadows always become *penumbral*, or ill defined, and therefore not fit for an Affair of ſuch Preciſion.

3713. I ſhall therefore propoſe a new Method of conſtructing an *equal Altitude Inſtrument* that moves immediately on the Plane, or Floor, or Pavement where the Meridian Line is required to be drawn, and which will give as little Trouble, and as great Accuracy, as can be expected in ſuch an Operation. This Inſtrument conſiſts of a Ruler A B C made of Mahogony, or other Wood, not ſubject to warp; At A is a ſmall Hole in a Piece of Braſs in which the Ruler is moveable about a ſlender ſteel Pin. The central Line A C is ſuppoſed to be drawn with great Care and very ſtrait; the Part B C is a fiducial Edge of Ivory, exactly coinciding with the ſaid Line A C; and at D is a ſmall black Line, which, upon the Motion of the Ruler, will deſcribe any Arch of a Circle, as D E F.

3714. Upon this Piece of Wood (or Metal) there is placed an Inſtrument for taking equal Altitudes of the Sun, conſtructed in the following Manner. G H is the Baſis of it, and from the Center N move the two *Indices*, N P and N L, upon two Arches of a Circle, *viz.* I K fixed to the immoveable Baſe

Baſe G H; and the Arch M O N, fixed to the Alidade or Index L N; and conſequently moveable with it. Upon the Index N P, is a Sight-Vane at R, with *Croſs-Hairs* in its Perforation, and a Shadow-Vane N Q is fixed at the Center N, exactly parallel to the former.

3715. The peculiar Artifice of this Conſtruction is, that in a *ſmall Form*, it performs the Office of a *large Quadrant*. For the Arch M O is divided into every 10 Degrees only, and therefore is not required to be large; the Index P N is placed to that Diviſion which is next leſs than the Sun's Altitude; and then it is evident, the Remainder in Degrees and Minutes is meaſured by the Index L N, and its *Vernier Plate* at L, upon the Arch I K, which is divided into 15 or 20 Degrees at moſt, and therefore but of a ſhort Length.

3716. This Inſtrument properly fixed upon the large Alidade A C (as in Fig. 3.) it is plain, that if in the Forenoon it has that Situation A C when the Indices are ſo adjuſted that the Shadows of the Croſs-Hairs, in the Vane R, fall preciſely on the black Lines croſſing in the Center of the Vane Q N, and a fine Point or Mark be made at D; and then, in the Afternoon, the ſaid Alidade A C be brought to ſuch a Situation, that the central Line A C be that of A F, when the Shadows of the Croſs-Hairs exactly coincide again with the Lines on the Vane Q, I ſay then, it follows, that if the Arch D E be biſected in E, and the Line A E be drawn, it ſhall be the *Meridian Line* required.

3717. This is all ſo plain as to want no farther Explication; and I find it ſo eaſy in Practice to take any Altitudes, and to meaſure any Angles in general with this compendious Inſtrument, that I can venture to recommend it to the ingenious practical Geometrician, as worthy of his Notice. It may be furniſhed with a Spirit Level to place it horizontally, and if a Lens of a proper focal Diſtance be placed in the Vane R, it will not only ſhew the Height of the Sun at any Time to a Minute, but alſo that of any other Object by its Picture, on the Vane N Q, properly darkened. As HADLEY's Quadrants are capable of the ſame Improvement and conciſe Form, I ſhall give a further Account of them at another Time; only obſerving, they may be applied in the above Method alſo for drawing a *Meridian Line*.

CHAP. XXIX.

The THEORY *and* CONSTRUCTION *of an* ELLIPTICAL, CIRCULAR, *and* DIAMETRAL DIAL, *which by Means of a common* HORIZONTAL DIAL, *will find the true* MERIDIAN.

3718. WE have ſhewn the beſt Methods of drawing a *Meridian Line*, by which a Dial may be *ſet truly* upon a horizontal Plane. We ſhall next ſhew how Dials may be conſtructed in ſuch a Manner as to *ſet themſelves*, or to *find their own Meridian* by Means of a *common Horizontal Dial* only. One of this Sort is called an ELLIPTIC DIAL, the other a CIRCULAR DIAL, becauſe in the firſt the Hours are the *unequal* Diviſions of an *Ellipſis*, and in the latter they are the equal Diviſions of a Circle; in both which Reſpects they differ from common horizontal Dials, where the Hours are the unequal Diviſion of a Circle, as we have ſhewn (1772).

3719. The Theory of the *Elliptic* DIAL we ſhall here deliver with all the Perſpicuity we can from a New, and (we preſume) moſt natural Projection of the Sphere *orthographically*, in what is uſually called the *Analemma*, explained form 1703 to 1720. Therefore (in Fig. 1.*) let HZOY be the *general Meridian*; EQ, the *Equator*; PX, the *Hour Circle* of *Six*; ZY, the *prime Vertical*; and HO, the *Horizon*. AF, LK, two *Parallels of Declination*.

3720. Then in our INSTITUTES of *Gnomonical Perſpective*, it has been ſhewn (1738, *&c.*) that the Sun having any Declination EG, will, by its Ray GC, deſcribe that Day a Cone ACF, whoſe Baſe is the Parallel AF upon the Surface of the Sphere, whoſe *Axis* is that of the Sphere PC, its *Vertex* C, and *Height* CB. This Cone we ſhall (for Diſtinction's Sake) call the *Radial Cone*.

3721. Then if the Axis ZCY of the prime Vertical be conſidered as an opake Line, it is plain, that Point of it at C, will intercept the Ray GC, and thereby produce a *lineal Shadow* CK in the ſame Direction, which Shadow will therefore deſcribe

* In the Plate entituled the Theory of *Elliptical, Circular*, and *Diametral Dials*.

ſcribe an equal and oppoſite Cone LCK, whoſe Baſe in the Surface of the Sphere will be the oppoſite equal Parallel LK. This therefore we ſhall call the *umbral Cone*.

3722. Then as the 24 Hour-circles divide the Equator EQ, and all its Parallels AF, LK, into ſo many equal Parts, it is evident the Ray GC in applying to thoſe Circles ſeverally, will aſſign the Poſitions of 24 Lines dividing the Surface of the Radial Cone into 24 equal Parts. And alſo, of Neceſſity, the Shadow CK will at the ſame Time give the Poſition of 24 correſponding Lines on the Surface of the umbral Cone; which therefore may be conſidered as *Hour-Lines* on the Surfaces of theſe Cones, interſecting each other in the common Vertex C.

3723. As the Sun's Declination EA decreaſes, the circular Baſes AF and LK of theſe Cones will encreaſe, till at Length the Sun being the Equator at E, both thoſe Baſes coincide, and the Cones degenerate into the Plane of the Equator EQ; where thoſe 24 Hour-Lines alſo ſeverally unite and form the *polar Dial*, deſcribed (1759) whoſe Gnomon is PC the Axis of the Sphere.

3724. It is farther evident, that, in the Caſe of the Sun's being in the Equator, the Hours will be indicated by *two Shadows united into one*, projected from the two opake Gnomons PC and ZC, on the Plane and graduated Perimeter of the Equator. For whatever Hour-Circle the Sun may be upon, the Shadow of the Stile CP (as being its Axis) will be in the Plane of that Circle (1751). For the ſame Reaſon, the vertical Circle paſſing through the Sun at the ſame Time, will have the Shadow of its Axis ZC projected in its own Plane; therefore, becauſe the Planes of both of thoſe Circles interſect each other, and alſo the Plane of the Equator, in one common Line; the Shadows of both thoſe Gnomons will, of Courſe, be projected into that Line, which will be the Shadow pointing out the preſent Moment on the Dial. Indeed, properly ſpeaking, the Shadow in the Caſe now ſpecified, is only that of the Point C in the Interſection of the two Gnomons.

3725. Again, ſuppoſe the Sun has any Degree of Declination EA, the Parallel AF will be its Path for that Day; now ſuppoſe it in any Hour-Circle, as that of Six PCX, in the Point B; and let ZBY be the Vertical paſſing through the Center of the

the Sun: Then it is eaſy to underſtand, that the Ray proceeding from the Sun's Center at B, will deſcribe the Line B C on the Side of the radial Cone A C F, becauſe that is the common Interſection of the two Planes on the Side of that Cone. Here the ſaid Ray is obſtructed by the opake Particle C in the Interſection of the two Gnomons or Wires, Z Y and P X, and by that Means the Shadow of that Point will be a Line on the Umbral Cone L C K, exactly in the ſame Direction C R, and therefore on the contrary Side.

3726. Hence then, if the Baſe L R K of the Umbral Cone were a Circle divided into 24 Hours, and the Cone itſelf were away; then it is plain, the Point R will be the Hour of Six on the Weſtern Side of that Circle, if B be the Hour of Six in the Morning, upon that Point will the Shadow of the Interſection C preciſely fall. If we ſuppoſe not only the Perimeter, but alſo the Plane of the Baſe (or Parallel) L K to be there placed, then would the two Axes P X and Z Y paſs through it in the Points R and V, of which the former is the Center. And on that Plane would be projected two Shadows, one of the Part R C, and the other of the Part C V, of the ſaid two Axes.

3727. And in this Manner the Plane of any particular Parallel L K may be conſidered as a Dial-plane, having *two Stiles or Gnomons* R C, V C, whoſe two Shadows conſtantly interſect each other on the Hour-circle in its graduated Circumference, and thereby ſhew the Hour for that particular Day.

3728. But as the Periphery of a Parallel is a variable Quantity, increaſing or decreaſing daily, it can by no Means anſwer the Intention of an *Hour-circle*, which ought to be a fixed or determinate Thing; and therefore if inſtead of the *Parallel of Declination*, we ſubſtitute a *moveable Equator*, it will anſwer the Purpoſe of an univerſal Dial for finding the Meridian in any Latitude, and conſequently, independent of the Magnetic Needle, will ſet itſelf, or ſhew the true Hour of the Day; as will thus eaſily appear.

3729. Let the given Parallel of Declination be A F, and draw I G and M N, Tangents to the Equator E Q; and then ſuppoſe a moveable Circle, juſt equal to the Equator, and divided into 24 Hours, Minutes, *&c.* were to ſlide up and down upon theſe Tangent Lines of the Sphere, ſo as readily to be placed in a Situ-

Situation ſimilar to that of the Parallel L K in the Umbral Cone, correſponding to the Parallel A F of the Radial Cone. Such a *moveable Equator* will be repreſented by the dotted Line or Circle I W N; for the Lines C L, C A, C K, C F, continued out, will cut the Tangents in I, G, and N, M; therefore E I = C W = Q N, conſequently the Plane of the moveable Equator I N, is parallel to the Plane of the Parallel L K, and the Peripheries of each are parallel Circles of the ſame Cone I C N; therefore they are divided ſimilarly by the Shadows of the two Gnomons W C, and U C, or it will be every where R V : V K :: W U : U N. And thus the Hour will be ſhewn truly, and alike in each.

3730. If therefore a Dial be conſtructed according to this *Analemma*, it will be the Plane of this moveable Equator; and ſhew both the *Hour*, and the *Meridian-Line* at the ſame Time: For the equational Hour-circle being rectified to the Tangent of the Sun's Declination (for any given Day) on the Lines G I, M N, the Hour ſhewn by the Gnomon W C (as being Part of the Axis of the Sphere) can never be true, but when H Z O is in the Plane of the Meridian; but the Hour ſhewn by the Gnomon U C, muſt alſo in that Caſe be true; therefore, when the Interſection of the Shadows of theſe two Gnomons fall preciſely on the Hour-circle, then both the *Hour and Poſition of the Meridian-Line will be given by the Dial.*

3731. The ſame Thing will be effected by a fixed Equator E Q, and a moveable Gnomon Z C; for while the Equator is carried from E Q to I N, the Index or Axis Z I, is moved through the Plane of the Equator from W to U; and therefore, if we take C S = W U, and from the Point S draw S *a* parallel to C Z, and alſo drawing G *b* M, we have the two Gnomons C *b*, and S *b* exactly equal to the former two, W C and U C, and alike ſituated to the Plane of the fixed Equator E Q, as they were to the moveable One I N. And therefore the Hour will be ſhewn in this Caſe, the ſame as in the other.

3732. Laſtly, the ſame Purpoſe will be anſwered by a Gnomon D *b*, moveable upon a horizontal Line H O, through the Space C D, ſince the Point (*b*) in the Interſection, gives the Hour, and is the ſame in both Caſes.

3733.

3733. If therefore upon the Plane of the Horizon HO in any Latitude EZ = PO, the Equator EQ be projected into an *Ellipſis*, and it be properly divided in every 15th Degree by a Table of Sines as directed (1718) theſe Diviſions will be the Hours, and then an upright Stile D *a*, moveable in the Hour-Line of XII, or Meridian HO, will conſtantly ſhew the true Hour on this *Elliptic Equator*, as before it did on the *Circular One*. Becauſe in the Projection there is a perfect Similarity of all the Parts in One, to all the correſponding Parts of the other, as we have ſhewed (1715).

3734. In this Caſe, the Shadows of the two Gnomons D *b*, and C *b*, require two different Hour-circles, and ſo will be no longer *one Dial*, but *two of a diſtinct Nature*. Thus let Fig. 2. be the orthog. Projection on the Plane of the Horizon FOM; and let AQB be the Ellipſis into which the Equator is projected, and divided into its proper Hours; alſo let P be the projected North-pole, Z the Zenith, and D the Foot of the perpendicular Stile S *b*, for any Declination of the Sun EG in Fig. 1. Then ſuppoſe the Shadow of that Stile falls upon the Hour XI, it will be projected into the Line DXI, and this muſt be the Hour-Line of XI on the horizontal Plane, as is evident from the Nature of the Projection; for the Space ZD here, is equal to CD there; and D is the Projection of the Foot S of the ſaid Stile in the Plane of the Equator.

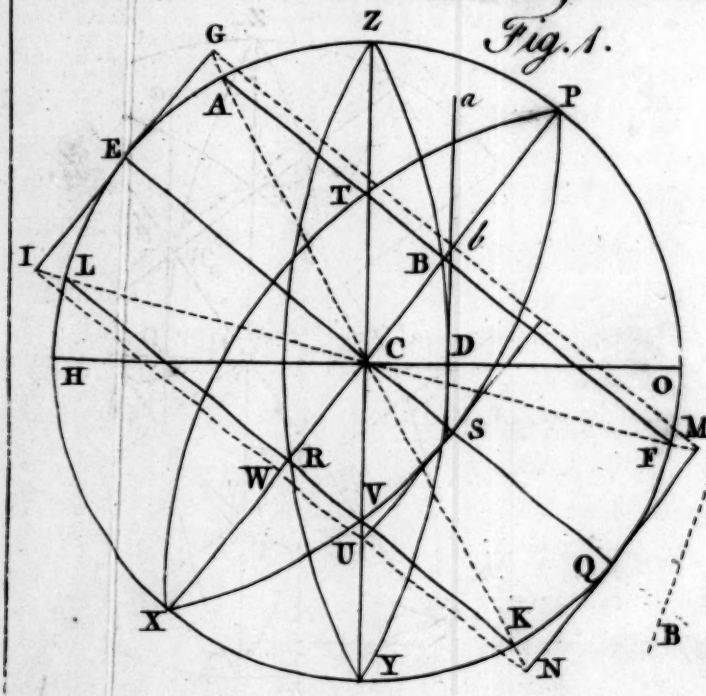
3735. But with Reſpect to the Stile C *b* (in Fig. 1.) its Shadow here cannot paſs through the Point (or Hour of) XI in the Ellipſis AQB; for the Foot C being in the Center Z of the horizontal Plane, and the Hour-circle of XI cutting the Horizon in E, making EO = 11° : 51′ (1772) it is plain, ZE will be the Hour-Line of XI on the horizontal Plane, or the Shadow of the Stile C *b*. And drawing the Line Z XI, there is formed the ſpherical right-angled Triangle XI ZQ, wherein ZQ = the Latitude, and XI Q = the Hour, being given, the Angle at Z will be found = 18° : 55′, which is 7° : 4′ greater than the Angle EZO. Conſequently the two Gnomons will, in this Projection, require two different Curves for the Hours, *viz.* the *Ellipſis* AQB, and the *Circle* FOM, which therefore conſtitute two different Dials.

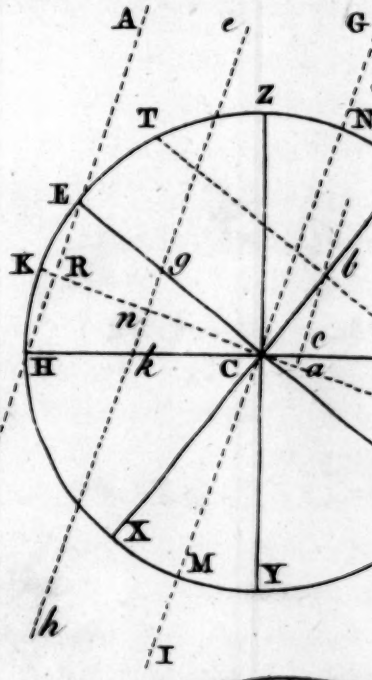
3736.

Plate 20

The THEORY of Elliptical, Circular and Diametral DIALS.

Fig. 1.





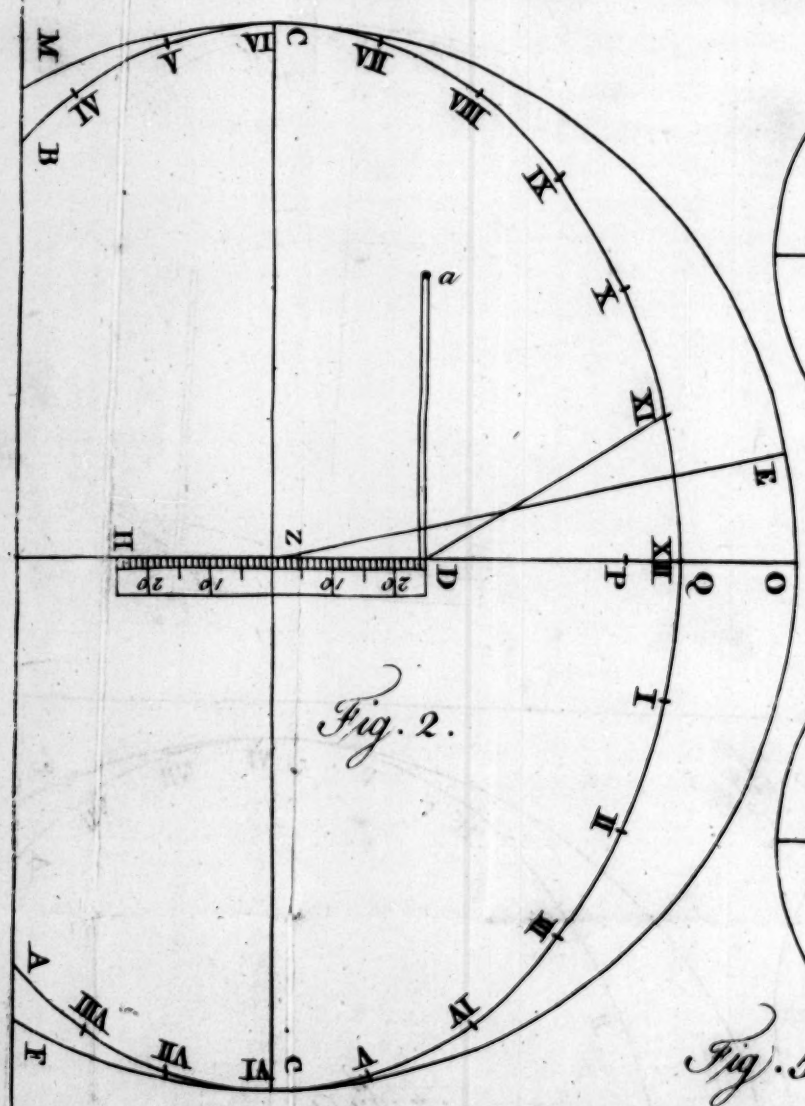


Fig. 2.

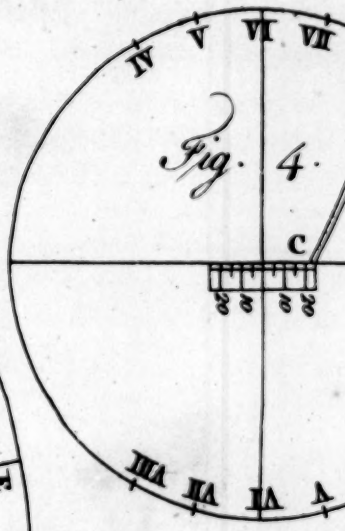


Fig. 4.

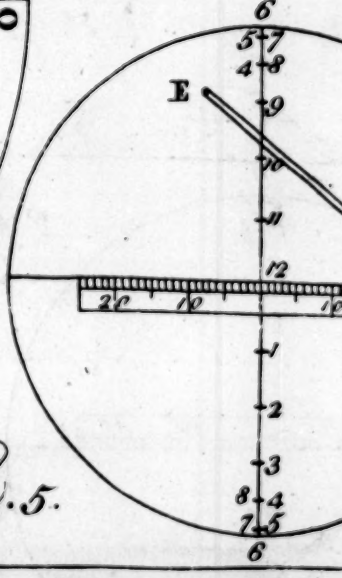


Fig. 5.

3736. When EG (= EI) is the Tangent of the Sun's greater Declination EA, then AF and LK are the two *Tropics*; and CD = ZD (Fig. 2.) is the greateſt Space through which the perpendicular Index D*a* is to be moved in the horizontal or meridian Line HO, which is to be graduated into 23° : 30′, to correſpond to the Sun's Declination in this Manner. In the Triangle *b*CD, we have C*b* = Tangent of 23° : 30′, and the Angle at *b* = ZCP = Co-Latitude. And as Radius : *s.* C*b*D :: C*b* : CD; therefore ſince C*b* is the Tangent of 23° : 30′ to the Radius CP, CD will be the Tangent of the ſame Arch to the Sine of ZP made Radius, or applied in the Sector from 45 to 45, upon the Lines of Tangents. Thus ZD and ZH is graduated for the Summer and Winter Half-Year, by which, to ſet the Index or Gnomon for the Declination of the Sun, proper to the given Day.

3737. Therefore it is evident, that when the ſame Minute is ſhewn by the Elliptic and Horizontal Dial, the Hour Line of XII, *viz.* HO, will be in the Plane of the Meridian of the Place, and conſequently a Right Line drawn parallel to it will be a true *Meridian-Line*. And thus ſuch a double Dial will at all Times *ſet itſelf* without any Aſſiſtance of the *Magnetical Needle*, or any other *Meridian Line* than its *own*.

CHAP. XXX.

The THEORY *and* CONSTRUCTION *of* CIRCULAR *and* DIAMETRAL DIALS, *which, with a Horizontal* DIAL, *find a true* MERIDIAN LINE.

3738. BEſides the foregoing *Elliptic Dial*, there is yet another Sort which by Means of a common *Horizontal Dial*, will *ſet itſelf*, or ſhew the Poſition of the *Meridian Line* without the help of the *Needle*; and by which, of Courſe, the *Variation*

of the Needle is alſo eaſily diſcovered in the given Place. But this Dial is of a *circular Form*, and what may ſeem a little ſtrange, is, that the Hours on this Circle are all *equidiſtant*, as in the Equator itſelf. The whole Artifice or Conſtruction of this Dial depends upon a due Conſideration of the Properties and different Sections of a *Scalenous Cylinder* which will be eaſy to underſtand after what we have premiſed of the *Scalenous Cone* (1508,) and its *Subcontrary Sections* (1510, &c.)

3739. Firſt then let it be conſidered that if Tangent-lines were drawn through the ſeveral Points of the Equator, (or any Circle) and at right Angles to its Plane, they would conſtitute a *right Cylinder*, whoſe Section by a Plane perpendicular to its Axis will be a *Circle*, but in all other Caſes an *Ellipſis*, as we have ſhewn (1715.)

3740. Secondly; if theſe *Tangent Lines* are now ſuppoſed gradually to *incline* or *lean* towards the Plane of the Equator, then they form what we call a *Scalenous Cylinder*, being compreſſed by this Inclination of the Lines into a flattiſh Form; and whoſe Section by a Plane perpendicular to its Axis will be an *Ellipſis*, one Diameter of the right Cylinder being contracted gradually into a leſs Length, while the other continues the ſame; alſo all other Sections will be *Ellipſes*, except two, *viz.* one *Parallel* to the Equator (or original Circle) and the other by a Plane in a *Subcontrary Poſition*.

3741. Thirdly; theſe *Tangent Lines* conſtantly inclining to the Plane of the Equator will at laſt perfectly coincide with it, or the *Scalene Cylinder* now degenerates into a Plane coincident with that of the Equator, which, in this Caſe, is projected by theſe Lines or Rays into a *ſtrait Line*, *viz.* one of its Diameters. And this Conſideration lays the Foundation of another Species of *Dialling*, *viz.* *Rectilineal*, which we ſhall enlarge upon by-and-by.

3742. But to come directly to the Point, and for Illuſtration of what has been ſaid, let A B F D be the *Scalenous Cylinder* whoſe Axis is I G, perpendicular to which let the Line R S be drawn; this will be the ſhorteſt Diameter of the *Elliptic Section* and ſuppoſe the longeſt Diameter be H O, which alſo is conceived to paſs through the common Interſection C of the Axis, and the other Diameter R S (See Fig. 3.) Upon the Point C, with

with the Radius C H deſcribe the Circle H Z O Y, cutting the flatted Sides of the Cylinder in the Points E, H, and O, Q; and draw the Line E C Q.

3742. Then it is evident, that ſince H O is equal to the longeſt Diameter of the Elliptic Section by a Plane through R S, it is therefore equal to the Diameter of the original Circle or Baſe of the inclined Cylinder A B F D, and alſo parallel to the Diameter through whoſe Extremities the Lines A B, F D paſs. Conſequently, the Section of the Cylinder by a Plane (through H O) parallel to the Baſe will be alſo a *Circle*, equal to the *Circular Baſe*.

3743. And becauſe the Angles R E C and S Q C are equal, as alſo R H C = S O C, (by reaſon of parallel Lines A B, D F interſecting the equal Lines H O and E Q) therefore the Section of the Cylinder through E Q will be *ſubcontrary* to that through H O, (1509,) and conſequently alſo a *Circle*, agreeable to (1510.)

3744. Now if F P Q X be conſidered as a *Meridian* of the Sphere, which is *orthographically projected* thereon, and E Q the Equator; we may then look upon H O as the Horizon of ſome Place Z, whoſe Latitude is E Z, and whoſe Co-latitude Z P is biſected by the Axis of the Cylinder I G, becauſe the Section through R S is ſuppoſed to be perpendicular thereto, and conſequently equally divides the Angle E C H in the Point K, or O C Q in the Point L.

3745. Then (from what we demonſtrated in the laſt Chapter) it is plain, that if K L be conſidered as a Plane having a Perpendicular Stile or Gnomon C N, the Projection of the Equator E Q thereon by Rays A B, F D, parallel to C N, will be an *Ellipſes*, whoſe ſhorteſt Diameter is R S, and longeſt K L. And that as the Sun advances to the Parallel T V, ſo the perpendicular Stile being parallelly removed from C to (*a*) will ſhew the Hour on that *Ellipſis*; and if T V be conſidered as the Tropic of *Cancer*, then C *a* will be the Length of the *Summer* Half of the *Zodiac* on the Dial-plate.

3746. Suppoſe now the Plane K L were taken away, but the Gnomon N C to remain; then it is evident the Rays A B, G I, D F, will project the Equator E Q into a *Circle* on the horizontal Plane H O (1510.) And the ſaid Rays will divide the Horizontal Circle in the ſame Manner or Ratio, as they do

the Equator itſelf. For let (*eh*) be any one of thoſe Rays, which paſſing through the Point (*g*) of the Equator, projects it into the Point (*k*) in the horizontal Circle H O; then becauſe *eh* is parallel to A B, we have C E : C *g* :: C H : C *k*. Wherefore the Horizontal Circle will be divided into 24 equal Parts by the Rays which paſs through every 15th Degree of the Equator.

3747. Therefore 'tis evident, that if upon the Horizon H O, a Circle be drawn, and divided into 24 equal Parts; thoſe Diviſions will be the Hours truly marked out by the Shadow of a Stile or Gnomon C N elevated above the Plane thereof in the Angle N C O = P O + P N = *the Sum of the Latitude, and half the Co-Latitude of the Place* Z. So that at *London*, for Example, the Angle N C O = 51° 30′ + 19° 15′ = 70° 45′.

3748. 'Tis farther evident, that this Stile C N is a *moveable One*, as it originally and properly belongs to the *Elliptic Dial* on the Plane K L (3745;) where its parallel Motion, Northward, is C *a*; but this is now to be eſtimated in the Plane of the Horizon from C towards O, where it will be expreſſed by the Line C *c* whoſe Value is thus found. In the Triangle *b* C *a* (right-angled at *a*) we have *Rad.* (C P) : Sine of C *ba* (= O C L) :: C *b* : C *a*. Again, in the ſimilar right angled Triangles *c* C *a*, O C S, and *f* C L, we have C *a* : C *c* :: C L (Rad.) : C *f*; therefore $C\,a = \frac{O\,S \times C\,b}{Rad.} = \frac{C\,c \times Rad.}{C\,f}$; therefore $C\,b : C\,c :: Rad.^2 : O\,S \times C\,f :: Rad. : \frac{O\,S \times C\,f}{Rad.}$ = Tangent of L C *f*. Whence the Value of C *c* is known.

3749. But becauſe it will be moſt convenient to ſet the Index C N by the Sun's Declination, therefore the Semi-zodiac C *c* muſt be conſidered as a Tangent of 23° 30′ to ſome Radius, which from the Analogy above is evidently the Line L *f*, *viz.* *the natural Tangent of the Half Co-Latitude of the Place* z. Therefore by applying the Line L *f* from 45 to 45 in Lines of Tangents on the Sector for Radius; you may then take the parallel Diſtance from 23° 30′ to 23° 30′, and that will give the Length of C *c* from the Center C each Way in the Meridian Line for the Summer and Winter half Year. And thus your *Circular Dial* will be compleated, as in Fig. 4.

3750. There

3750. There are many other Particulars obſervable in this Sort of Dials, but we muſt paſs them by at preſent as not neceſſary to our Purpoſe, which is only to ſhew *how by two Dials of a different Conſtruction, and placed upon one Right Line, a true Meridian Line may be drawn, and the Hour ſhewn, without any other Aſſiſtance.* — This we have ſhewn may be done by a *common horizontal Dial* combined with the *Elliptic*, and *Circular* Ones; we ſhall next ſhew the Nature and Conſtruction of a *Rectilineal* or DIAMETRAL DIAL for the ſame Purpoſe.

3751. Suppoſe (Fig. 3.) that the *Tangent Rays* A B, G I, D F, were (in the firſt Place) all at right Angles to the Plane of the Equator E Q; an infinite Number of them would, in this Caſe, conſtitute a *perfect Cylinder* whoſe Section, perpendicular to the Axis would every where be a *Circle*, as being parallel to the circular Plane of the Equator.

3752. Let the Rays now gradually deviate from that rectangular Poſition to one more and more inclined to the ſaid Plane of the Equator in a Direction from P towards E. Then it is evident, all that Time the *Radial Cylinder* will become more and more flatted or ſcalenous; and every perpendicular Section R S will become of a leſs and leſs Diameter; and, of Courſe, the *Hour-Circle* in the Equator degenerates into an *Ellipſis*, and becomes more and more ſo; 'till at length, the Rays all arrive at the Plane of the Equator, and thence the *ſcalenous Cylinder* itſelf becomes a *Plane*, and coincides with that of the Equator; alſo the horary Ellipſis now Collapſes, and both Sides unite in a Right Line which is the Diameter of a greater Circle P X paſſing through the Poles.

3753. During this ſuppoſed Motion of the Tangent Rays the Dial Plane K L has deſcribed the Quadrant E X, and its Gnomon or Index C N the Quadrant P E, and its Hour-Circle has paſſed thro' every Degree of Elliptic Curvature from a Circle to that of a Right Line; and the Poſition of its Gnomon, which at firſt was in the Axis of the World C P, is now C E in the Plane of the Equator.

3754. This Poſition of the Dial-Plane P X and its Gnomon C E fits it for *An univerſal Right-lined* DIAL *in every Latitude*; for from the Nature of the *Analemma*, it is evident, that the 24 Hour-Points of the Equator are now projected into the *El-*

liptic

liptic Line (as it may be called) or the Eaſt and Weſt Diameter of the Plane D X. Which Line is equally a Diameter in the Plane of any great Circle, as that, for Inſtance, of the Horizon H O, becauſe it is the *Common Section* of them all, and ſo is equally as much in one as in another.

3755. Therefore ſince the Rays which paſs through the Beginning of each 15th Degree of the Equator, will project thoſe Points into the ſaid Common Section or Hour-line of the Dial, this Line will be thereby divided in the Manner of a *Line of Sines* each Way from the Center, as repreſented in Fig. 5.

3756. The Stile of this we have ſhewn (3753) muſt be preciſely elevated to the Plane of the Equator, *viz.* the Angle E C H (Fig. 3.) for any propoſed Horizon H O; and as this Dial is of the *Elliptic Kind*, its Stile or Gnomon C E is a *moveable One*. And therefore if T V be the Parallel of the Sun's greateſt Declination, it will alſo repreſent the Poſition of the ſaid Gnomon for the tropical Day, as being parallel to C E, and interſecting the Axis C P in the Point *(b)* making C *b* = Tangent of 23° 30′. Therefore, ſince in the Triangle C *b* V, it is C *b* : C V :: Radius : Secant of the Angle P C O, the Diſtance C V is known for any Latitude P O or E Z.

3757. And ſince C *b* is the Tangent of 23° 30′ to the Radius C P; therefore C V will be the Tangent of 23° 30′ to a *Radius equal to the Secant of the Latitude*, and hence C V the Semi-Zodiac for this Dial, is eaſily graduated for the duly adjuſting the Index to the Sun's Declination, as directed (3736,) and as ſhewn in Fig. 5.

3758. But this *Rectilineal Dial* is not ſo conveniently combined with the *common horizontal* Dial for finding the true Hour or Meridian Line as the *Elliptical Dial* is; becauſe in this the Hours at the extreme Parts of the Line run ſo near together, as not to admit of ſufficient Accuracy in obſerving and comparing the Time in each. Nor is the *circular* Dial (Fig. 4.) ſo fit for this Purpoſe as the *Elliptic One*, becauſe of the *ſmall Length of the Zodiac* for ſetting the Stile C N (Fig. 3.) not allowing a ſufficient Eſtimation of the Sun's Declination.

3759. 'Tis very obſervable that both the *circular* and *right-lined Dials* are *univerſal Ones*, or will ſerve for *all Latitudes*, becauſe

cauſe the *horary* Diviſions of each always remain the ſame, and the Indices or Stiles only require to be properly rectified, *viz.* that of the *Circular Dial* as directed (3749,) and that of the *Rectilineal One* to the *Co-Latitude* HE, be it what it will.

3760. Laſtly, in the *Right-lined Dial* (Fig. 5.) the Reader will obſerve that the Shadow of the Stile muſt go *backward* and *forward twice a Day* in the Summer Half-year, *viz. Morning* and *Evening*, as is evident by Inſpection. What remains on this Subject of DIALLING will be delivered in the Third Volume; wherein it is propoſed to give the THEORY and CONSTRUCTION of all the uſeful Mathematical and Philoſophical INSTRUMENTS, as alſo all the neceſſary TABLES uſed in thoſe Sciences, with the *Rationale* and particular *Uſes* of each.

END *of the* SECOND VOLUME.